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RELATIVITY PROBLEMS FROM STANDPOINT OF DIFFERENTIAL EQUATIONS

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Abstract
In this paper the geometry of a space is investigated using not the logic of motion of a classical particle, but the properties of motion of a field. This appears to be sufficient for the algebraic theory of differential equations to bring us unambiguously to a qualitatively new mathematical space and field theory. It turns out that each differential equation describing some process constructs its own geometry – field geometry. With that, the arising corresponding space represents a union of three: space-time, inner space and dynamical space. The principles of relativity are qualitatively broadened, an explanation is found for the existence of unitary symmetry that commutes with the Lorentz group but is generated by its representation.

Keywords:
Test field, Relativism, Unified field theory, Processes and anti-processes, Field quantization

Introduction

The starting point of our investigation is the conjecture that differential equations with their actually inherent laws of process motion are a subtler and more reliable mathematical tool to be used in studying the properties of field and space and their interrelationship. One can hardly deny the fact that general algebraic properties of differential equations are the properties of motion of processes described by these equations. That is why we carry out our investigation of the field theory by means of differential equations, while the geometrical constructions are based exclusively on the algebro-geometric properties of these equations.

According to [1, 2, 3], each equation defines its own calculus (differential and integral) which is isomorphic to the standard calculus. It is shown that the equation in its system of calculus becomes linear. The equation introduces invariantly a space of independent variables (space and time) and defines inertial frames of reference. This approach compels us to cast aside the notion of a classical mathematical point and to introduce instead the notion of a process as an elementary object. It is the processes described by autonomous partial differential equations that become the main tools of the investigation of the world that surrounds us. As we have seen in [1], using one process it is possible to describe all other processes of the same class. Moreover, within each class we can pass over from the description of one process to the description of another process. This means that differential equations bring us to the necessity to interpret the principles of relativity in a new light.

The theory is constructed from the standpoint of the observer who is inside some given process. As has been mentioned, in its calculi and frames of reference this process is described by a linear equation. The process itself can be interpreted as free motion of some field that can be in that or another state admitted by the equation. Changes from one state to another bring about a transformation that leaves the equation of the process invariant. It is required of the process that the group acting on the reference system be the Lorentz group. Using this field and its various states the observer probes the surrounding world. We call this field a test field.

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When giving a differential equation, we in fact automatically give three spaces: a space-time, a group representation space and a tangent space. A tangent space arises because of the presence of derivatives in the equation. These three spaces are united by a differential equation into a whole geometric complex. In the tangent space there exists a group of nonintegrable transformations which is connected through a differential equation with the Lorentz group representation. We call this group dynamic. The dynamic group together with the Lorentz group completely defines the arisen geometry of the space. We use the Dirac equation as an example to show that the found dynamic group commutes with the Lorentz group and is isomorphic to a noncompact group $SU(3, 3)$. It turns out that this geometry contains a mechanism that violates the arisen unitary group.

Field Geometry

In this chapter we investigate the geometry which arises on the basis of free motion of a field.

1. *Fundamental principles*

In [1] we have studied with sufficient completeness the algebraic properties of differential equations. Based on these studies, we formulate here the starting standpoints which underlie the entire further theory.

The simplest object of the investigated theory is a process that plays the same fundamental role as a material point and an event in classical physics. By a process we mean a physical phenomenon that evolves in space-time in the absence of external disturbances. We postulate that any process in some frame of reference is always described by an autonomous quasilinear system of first order partial differential equations.

Proceeding from the algebraic theory of differential equations [1–3], we believe that this postulate evidently tells us if not everything but at lest almost everything about the properties of real processes and all their possible states. Note that we do not consider here processes which are not described by differential equations.

For convenience, let us introduce some terms and notation. We call the space-time the external space, while the space, where the sought functions of differential equations undergo changes, the internal space. The topological product of the external and the internal space is called the total space. The dimension of a system of equations describing a given process should be interpreted as the dimension of the process itself or, which is the same, the dimension of the external space. The set of all kinds of processes having the same dimension is called the set of processes of the same class [1].

Furthermore, the Greek letters $\nu, \tau, \sigma, \ldots$ denote the tensor indexes of the external space, while the Latin letters $k, n, m, \ldots$ denote the tensor indexes of the internal space. The tensor indexes corresponding to the coordinates of the conjugate internal space are sometimes overlined. The capital Latin letters $A, B, C, \ldots$ denote the indexes running through the values $\nu, k, \overline{n}$.

Let us discuss in more detail the properties of processes and their corollaries arising from the algebraic properties of differential equations.

(1) As we have seen in [1], the differential equation of each process defines its own frame of reference of the external space and the double numerical field acting over the elements of the internal space. This double numerical field brings us in turn to a double frame of reference with a simultaneous appearance of the double numerical field acting in the external space [1, 2]. In terms of this frame of

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reference we can write differential equations of other processes of the same class. In other words, we can recognize all other processes from the standpoint of the initial process.

Note that the double numerical field generated by the differential equation has an alternative character [1, 2]. The evolving process uses only one of the fields, totally ignoring the other field. This topic will be discussed later in follow-up paper.

(2) From the theorem on the existence and uniqueness of a solution of a differential equation it follows that a real process always evolves while being in a concrete state. On the other hand, any process may be in various states admitted by a differential equation. If there exists a transformation of the considered equation that changes one state to another one, then such states are called equivalent. Otherwise, they are called irreducible. It is obvious that transformations acting in equivalent states form a group. Hence we immediately conclude that under the action of the formed group the differential equation of the process remains invariant in view of the fact that in the absence of external disturbances the equation describes the process with all its possible states.

(3) For various but equivalent states, the reference frames of the external space, which are defined by the prescribed process, can be different. As has been said in (2), there exists a transformation that changes one state of the process to another state, which results in the transformation of one reference frame into another frame. On the other hand, as shown in [1], in the proper calculus the invariance of the differential equation of the process leads to a linear transformation of the frame of reference, which means that there arise inertial frames.

We next assume that the group of transformations, which is generated by the transformation of equivalent states of considered process and acts in the external space, is the Lorentz group. Along with this assumption, it is required that the differential equations of other processes written in the system of calculus of the considered process be invariant with respect to this group.

(4) Suppose we are given some process with the observer inside. The process is the motion of some field with its all possible states. From the standpoint of his process the observer defines the frame of reference and system of calculus. As shown in [1], the differential equation in the proper calculus is written in the linear form. Therefore we postulate that the equation of this process in the proper calculus has the form

$$a^\nu \frac{\partial u}{\partial x^\nu} = mu, \tag{1.1}$$

where $a^\nu (\nu = 1, 2, 3, 4)$ are square N-matrices, the mass $m \neq 0$ and is measured in the inverse units of length.

For equation (1.1) to describe real particles it must satisfy two conditions: the equation must be invariant with respect to the Lorentz group and, besides, the equality

$$(a^\nu)^+ = -a^\nu \tag{1.2}$$

must be fulfilled for the Lorentz conjugation.

In the sequel we will consider only equations like (1.1) which satisfy these conditions. In that case, as different from other equations, we will call (1.1) an admissible equation.

(5) As has been shown in Section *10* from [1], the algebraic theory brings us to a conclusion that in order that the description of processes be orderly, solutions of differential equations should be extended to the matrix algebra. Thus we can assume that a solution $u(x)$ of equation (1.1) is an N-dimensional square matrix. An equation of plane waves for (1.1) and the Dirac conjugate equation have the form

$$a_\alpha \frac{du_\alpha}{dz_\alpha} = mu_\alpha, \tag{1.3}$$

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$$\frac{du^+_\alpha}{dz_\alpha} a_\alpha = -m u^+_\alpha,$$

where (1.2) is taken into account. In (1.3) we have introduced the following notation: $z_\alpha = l_\alpha^{-1}\alpha_\nu x^\nu$, $a_\alpha = l_\alpha^{-1}\alpha_\nu a^\nu,\ l_\alpha = \alpha_\nu l^\nu$, and $\alpha \in \Omega$ [1].

According to Section *10* from [1], we can introduce the metric in the space $\Gamma^1_\alpha \times \Gamma^{N\times N}_\alpha$:

$$ds^2_\alpha = \overset{\circ}{g}_\alpha dz^2_\alpha + 2H_\alpha sp[(w^+_\alpha w_\alpha)^{-1} dw^+_\alpha dw_\alpha], \tag{1.4}$$

where ${w_\alpha}^k_n$ are the coordinates of elements of the space $\Gamma^{N\times N}_\alpha$. Note that, as different from [1] where we denote by w_α the solutions of linear equations, here and in the sequel ${w_\alpha}^k_n$ are assumed to be independent values. Along with $\Gamma^{N\times N}_\alpha$, the equation of plane waves introduces the real one-dimensional space Γ^1_Ω through which the variable z_α runs.

From the construction of (1.4) we see that g_α and H_α from equation (1.3) cannot be defined in a straightforward manner.

(6) By virtue of the results of [1], the consideration of the algebraic properties of differential equations like (1.1) results in obtaining a trivial fiber space $P(\Gamma_4, \Gamma^{N\times N}_\alpha, \pi)$ with base space Γ_4, fibers $\Gamma^{N\times N}_\alpha$ and projection $\pi : P \to \Gamma_4$. Γ_4 is the space with elements $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

According to [1], an arbitrary solution $u(x)$ of equation (1.1) is a χ -mapping of the trivial fiber space P into the space $\Gamma^{N\times N}$ when the fiber elements are plane waves u_α. In particular, one of χ-mappings is a sum

$$\chi_{\Omega}{}^k_n = \sum_{\alpha\in\Omega} q_\alpha\, {w_\alpha}^k_n, \tag{1.5}$$

$$(k, n = 1, \ldots, N).$$

As in [1], the following condition is imposed on the numerical coefficients q_α:

$$\sum_{\alpha\in\Omega} q_\alpha = 1.$$

Without going into details we wish to note that equation (1.1) implies the existence of the following algebraic operations $\Gamma^{N\times N}_\alpha$ [1]:

$$(\overset{1}{w} \dot{\oplus} \overset{2}{w})^k_n = \sum_{\alpha\in\Omega} q_\alpha ({\overset{1}{w}_\alpha}{}^k_n + {\overset{2}{w}_\alpha}{}^k_n), \tag{1.6}$$

$$(\overset{1}{w} \dot{\otimes} \overset{2}{w})^k_n = \sum_{\alpha\in\Omega} \sum_{m=1}^{n} q_\alpha {\overset{1}{w}_\alpha}{}^k_m {\overset{2}{w}_\alpha}{}^m_n,$$

where $\overset{1}{w}, \overset{2}{w} \in \Gamma^{N\times N}$ and $\Gamma^{N\times N}$ is the internal space of equation (1.1). The neutral elements in operations (1.6) are

$$0 = \sum_{\alpha\in\Omega} 0, \tag{1.7}$$

where 0 is a zero N-matrix. The unit element is the matrix

$$E_\Omega{}^k_n = \sum_{\alpha\in\Omega} q_\alpha \delta^k_n = \delta^k_n, \tag{1.8}$$

where δ^k_n is the Kronecker symbol. The elements w and w^{-1} of the space $\Gamma^{N\times N}$ are called reciprocal if the equality

$$w \dot{\otimes} w^{-1} = E_\Omega \tag{1.9}$$

is fulfilled.

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For metric (1.4) to be realized, in the fibers $\Gamma_\alpha^{N\times N}$ we have to make the following requirement: if the matrix $w_\alpha \neq 0$, then it must be nonsingular. Since in (1.6) the matrix multiplication is used, as shown in [1], this requirement brings us to a conclusion that (1.5–9) forms an associative algebraic field [5]. Along with this, there exist alternative operations of matrix addition and multiplication [1, 2]. In other words, in the space $\Gamma^{N\times N}$ there acts the double algebraic body. We remark incidentally that the conjugate neutral elements arisen in the double body are represented by the zero N-matrix O and the N-matrix ∞, where all elements are infinities. In the role of the conjugate unit element we have the matrix

$$\widehat{E}_{\Omega}{}_n^k = \ddot{\sum_{\alpha\in\Omega}} \frac{1}{q_\alpha}\hat{\delta}_n^k = \widehat{\delta}_n^k,$$

where $\ddot{\sum}$ is an alternative sum to the standard sum [1], and

$$\hat{\delta}_n^k = \begin{cases} 1, & k = n, \\ \infty, & k \neq n. \end{cases}$$

(7) In the space $\Gamma^{N\times N}$ we introduce the metric

$$ds^2 = 2HSp[(w^+ \dot{\otimes} w)^{-1} \dot{\otimes} dw^+ \dot{\otimes} dw] \tag{1.10}$$

where $SpA = \underset{k}{\dot{\oplus}} A_k^k$, and H is some constant of length square dimension. Using the above-mentioned algebra, from (6) we obtain

$$d\sigma^2 = 2H \sum_{\alpha\in\Omega} q_\alpha sp[(w_\alpha^+ w_\alpha)^{-1} dw_\alpha^+ dw_\alpha]. \tag{1.11}$$

Let us return to (1.4) and sum it with respect to the set $\Omega \subset \Gamma_4$. According to [1], we obtain

$$ds^2 = \overset{\circ}{g}_{v\sigma} dx^v dx^\sigma + 2\sum_{\alpha\in\Omega} q_\alpha H_\alpha sp[(w_\alpha^+ w_\alpha)^{-1} dw_\alpha^+ dw_\alpha], \tag{1.12}$$

It is assumed that the coefficients $\overset{\circ}{g}_\alpha$ in (1.4) are chosen so that in (1.12)

$$\overset{\circ}{g}_{\nu\tau} dx^v dx^\tau \tag{1.13}$$

coincides with the metric of the Minkowski space Γ_4.

Comparing the second summand from (1.12) with (1.11), we find that $H_\alpha = H$ for all $\alpha \in \Omega$. Speaking in general, this is quite logical, since all fibers $\Gamma_\alpha^{N\times N}$ in the space P are equal. Recall that in the general solution

$$u(x) = \sum_{\alpha\in\Omega} q_\alpha u_\alpha \tag{1.14}$$

of equation (1.1) the plane waves u_α as summands are equal. Therefore H plays the role of a universal constant.

Thus the total metric of the space $\Gamma^4 \times P$ takes the final form

$$ds^2 = \overset{\circ}{g}_{\nu\tau} dx^v dx^r + 2H\sum_{\alpha\in\Omega} q_\alpha sp[(w_\alpha^+ w_\alpha)^{-1} dw_\alpha^+ dw_\alpha]. \tag{1.15}$$

From (1.10–11) it follows that (1.15) coincides with the metric of the space $\Gamma^4 \times \Gamma^{N\times N}$, where Γ^4 is the Minkowski space and $\Gamma^{N\times N}$ is the internal space.

(8) From equality (1.2) we immediately conclude that for the Dirac conjugation we have $a_\alpha^+ = -a_\alpha$. Then the commutator $[a_\alpha^+, a_\alpha]$ is identically equal to zero. In that case, as shown in Section *10* from [1], solutions u_α^+, u_α of equations (1.3) are geodesic in the space $\Gamma_\alpha^{N\times N}$ with metric (1.4).

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Let us consider two states u_α and u_β for $\beta \neq \alpha$. There exists no transformation leaving equation (1.1) invariant and, simultaneously, changing u_α to u_β. Therefore u_α and u_β are irreducible states of process (1.1). Then, taking into account algebra (1.5–9), it can be shown that solution (1.14) of equation (1.1) is also geodesic in the space $\Gamma^4 \times \Gamma^{N\times N}$ with metric (1.15).

Thus the observer, who is inside the considered process, applies in his system of calculus, the above arguments and constructions. Using the field which is described by the admissible equation (1.1) and is simultaneously geodesic, he can probe the surrounding world through changes occurring in the states of the field. We call this field the test field for process (1.1), within which the observer is enclosed.

(9) The definition of the admissibility of equation (1.1) immediately implies that the internal space must be first of all the space of representation of the Lorentz group [4]. In addition to the Lorentz group, equation (1.1) may be invariant with respect to some group acting only in the internal space. Hence we come to the conclusion that, as different from the external space, the algebraic structure of the internal space and its dimension may undergo changes in passing from one process to another one.

We have thus described the algebro-geometric structure of the total space constructed by the logic of differential equations.

(10) To understand better the outlines of the arisen geometry at this stage of the investigation, we will consider the case where the internal space is a spinor space. For this, among admissible equations we choose the Dirac equation

$$\gamma^\nu \frac{\partial u}{\partial x^\nu} = -imu, \tag{1.16}$$

where u is a 4×4 square matrix and the mass $m \neq 0$. The Dirac matrices γ^v satisfy the well known relations

$$\gamma_\nu\gamma_\tau + \gamma_\tau\gamma_\nu = 2\overset{\circ}{g}_{\nu\tau}, \tag{1.17}$$

where $\overset{\circ}{g}_{\nu\tau} = diag(-1,-1,-1,1)$ is the metric tensor of the Minkowski space and

$$\gamma_\nu = \overset{\circ}{g}_{\nu\sigma}\gamma^\sigma.$$

The base Dirac matrices γ_ν are chosen so that for the Hermitian conjugation there hold the equalities

$$(\gamma_4)^{(H)} = \gamma_4, \ \ (\gamma_a)^{(H)} = -\gamma_a, \ (a = 1,2,3), \ (\gamma_5)^{(H)} = -\gamma_5,$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \ \ \gamma_5^2 = -1,$$

where is the Hermitian conjugation. Then the Dirac conjugation is written as

$$(\gamma_\nu)^+ = \gamma_4(\gamma_\nu)^{(H)}\gamma_4 = \gamma_\nu,$$
$$(\gamma_5)^+ = \gamma_4(\gamma_5)^{(H)}\gamma_4 = \gamma_5,$$
$$u^+ = u^{(H)}\gamma_4.$$

For the Dirac conjugation, equation (1.16) is written in the form

$$\frac{\partial u^+}{\partial x^\nu}\gamma^\nu = imu^+. \tag{1.18}$$

From (1.16) and (1.18) we easily obtain the plane wave equation

$$\gamma_\alpha \frac{du_\alpha}{dz_\alpha} = -imu_\alpha, \tag{1.19}$$

$$\frac{du_\alpha^+}{dz_\alpha}\gamma_\alpha = imu_\alpha^+,$$

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where $\gamma_\alpha = l_\alpha^{-1}\alpha_\nu\gamma^\nu,\ l_\alpha = \alpha_\nu l^\nu$. Then, by virtue of the reasoning of (5)–(7), we can write

$$ds^2 = \overset{\circ}{g}_{\nu\tau}dx^\nu dx^\tau + 2H\sum_{\alpha\in\Omega} q_\alpha sp[(w_\alpha^+ w_\alpha)^{-1}dw_\alpha^+ dw_\alpha],$$

where w_α^+, w_α are 4×4-matrix spinors. Due to the fulfillment of equalities (1.10–11), the metric of the total space can be written in the form

$$ds^2 = \overset{\circ}{g}_{\nu\tau}dx^\nu dx^\tau + 2HSp[w^+ \dot{\otimes} w)^{-1} \dot{\otimes} dw^+ \dot{\otimes} dw]. \tag{1.20}$$

Analogously to (1.20), let us consider the metric

$$ds^2 = \overset{\circ}{g}_{\nu\tau}dx^\nu dx^\tau + 2H\frac{1}{\rho}dw_k^+ dw^k, \tag{1.21}$$

which will underlie the construction of our further theory. Here $\rho = w_k^+ w_k$, and w^+, w are standard Dirac 4-spinors.

It is obvious that (1.21) is a somewhat distorted matrix of (1.20). However such an approach facilitates the construction of geometry by using the available apparatus of modern geometry [6] and also makes it possible to discern the main contours of the theory hidden behind metric (1.20) and therefore behind metric (1.15).

2. Dynamic space

Let us discuss the geometry of a total space more thoroughly. In the standard approach, a total space with the metric introduced above looks like a curved space. On the other hand, in Section *1* it is shown that free motion of fields that are described by linear equations is geodesic. This can be explained by the fact that the interpretation of geometry is always given on the basis of the logic of motion of a classical particle, whereas in the considered theory we use free motion of a field. We have already mentioned that in the theory based on the algebraic theory of differential equations there are no classical particles – their place is taken up by test fields. Recall that the observer constructs his calculi and frames of reference with the aid of the process within which he is confined. He does not know about the existence of classical particles, since they cannot be described by means of the considered process (1.1). He can describe the motion of a test field which can be in various states admitted by the equation. Thus the probing of the space can be done only by using a test field and therefore the interpretation of geometry must be brought in harmony with the properties of field motion.

(1) Let us find a group acting in the total space and not violating the invariance of metric (1.21). With end in view, we are to define the transformation of the total space coordinates in the infinitesimal form

$$\overline{y}^A = y^A + \lambda^A(y), \tag{2.1}$$

where $\lambda^A(y)$ are the infinitesimal functions we want to define. Here and in the sequel we denote by y^A the total space coordinates, i.e.

$$y^\nu = x^\nu,$$
$$y^k = w^k,$$
$$y\overline{k} = w_k^+.$$

After some simple calculations, by the metric invariance condition we find

$$\overline{x}^\nu = x^\nu + \xi^\nu_\sigma x^\sigma + \xi^\nu,$$

$$\overline{w} = w + \xi^a\tau_a w + (\widetilde{\xi} + i\xi)w, \tag{2.2}$$

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$$\overline{w}^{+} = w^{+} - \xi^{a} w^{+} \tau_a + (\widetilde{\xi} - i\xi) w^{+},$$

where ξ are the parameters of the group. It is obvious that ξ^{ν}_{σ}, ξ^{ν} are the parameters of the Lorentz group. For the Dirac conjugation we have $\tau_a^{+} = -\tau_a$. Note that the matrices are extended to the total algebra of Dirac particles.

Group (2.2) is the broadest infinitesimal group acting in the total space and leaving metric (1.21) invariant. From (2.1–2) we make an important conclusion: in a total space there exists no transformation that changes the positions of the coordinates of the external and the internal space and preserves the invariance of the metric. Hence we can state that the external space is absolutely separated from the internal one. Moreover, the internal space and the conjugate internal space are also absolutely separated from each other.

(2) As is known, the Dirac equation (1.16) is invariant with respect to the group

$$\overline{x}^{\nu} = x^{\nu} + \xi^{\nu}_{\sigma} x^{\sigma} + \xi^{\nu},$$

$$\overline{w} = w + \frac{1}{8}\xi^{\nu}_{\sigma}\left[\gamma_{\nu}, \gamma^{\sigma}\right] w + i\xi w, \tag{2.3}$$
$$\overline{w}^{+} = w^{+} - \frac{1}{8}\xi^{\nu}_{\sigma} w^{+}\left[\gamma_{\nu}, \gamma^{\sigma}\right] - i\xi w^{+}.$$

To have a complete picture, we note that any admissible equation (1.1) and the corresponding metric are invariant with respect to the extension group acting in the internal space. We have in the infinitesimal form

$$\overline{w} = w + \widetilde{\xi} w, \tag{2.4}$$
$$\overline{w}^{+} = w^{+} + \widetilde{\xi} w^{+},$$

where $\widetilde{\xi}$ is the extension group parameter.

It is obvious that group (2.3-4) is a subgroup of group (2.2). An analogous situation is formed for the symmetry groups of equation (1.1) as well as for group preserving the invariance of metric (1.15).

In Felix Klein's "Erlangen Program" [7] it is stated that the group of transformations which acts in the space defines the geometry of this space. Then we come to a conclusion that the geometry defined by group (2.3–4) is not adequate to the geometry constructed on the basis of group (2.2).

On the other hand, as is known from A. Einstein's general theory of relativity, the matter being in the space distorts this space, i.e. the matter defines the geometry of the space. This gives us the right to assume that it is not the geometry that defines the properties of the field, but vice versa it is the field that must define the geometry of the space.

Thus the emergence of group (2.2), with respect to which equation (1.16) is not invariant and which is therefore a strange one for the Dirac field, indicates that the construction of the geometry by means of the field has not been completed. To write the metric of the space generated by the field is not sufficient.

(3) Let us return to the admissible equation (1.1). If $I_{\nu\sigma}$ is the generator of the Lie algebra of the Lorentz group representation, then we have [4]

$$\left[a_{\mu}, I_{\nu\sigma}\right] = \mathring{g}_{\mu\sigma} a_{\nu} - \mathring{g}_{\mu\nu} a_{\sigma}, \tag{2.5}$$

where $a_{\mu} = \mathring{g}_{\mu\sigma} a^{\sigma}$. Hence it follows that the representations of the Lorentz group and the matrix a^{ν} contained in equation (1.1) are algebraically interrelated. This relation makes it possible to declassify equation (1.1) with respect to representations of the Lorentz group [4].

In the constructed geometry of the total space the Lie generator $I_{\nu\sigma}$ is explicitly present through the group of transformations of the internal space. At the same time, the matrices a^{ν} do not figure explicitly anywhere in the geometry despite the existence of (2.5). This rather tells us that the geometry

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constructed on the basis of test fields is not complete. Therefore we need to discuss these questions in more detail.

Equation (1.1) contains the product $a^\nu\left(\partial/\partial x^\nu\right)$. As is known, $\partial/\partial x^\nu$ is the tangent vector of the external space. This means that for the representation of equation (1.1), we must include in the theory, in addition to the total space, the tangent space as well. This conclusion is not unexpected. After all, the emergence of elements of the tangent space is explained by the presence, in differential equations, of the derivatives of the unknown functions characterizing the dynamics of the evolution of a solution in the total space. Also, we want to remind that when studying the group properties of differential equations, the equation is considered as an invariant manifold of the jet-space.

Thus, along with external and internal spaces, the differential equation of the field also involves the tangent space in the process description. Therefore the geometry arisen on the basis of process evolution should be constructed as a single object of three spaces. Otherwise the geometry generated by process (1.1) will not be complete. In the sequel, the union of these three spaces will be called the complex of spaces or, simply, the complex.

(4) The presence in equation (1.1) of the matrix a^ν combined with $\partial/\partial x^\nu$ indicates that it is a^ν's that are the carriers of the dynamic characteristic of field motion. In order to construct the tangent space geometry corresponding to the differential equation (for a certain representation of the Lorentz group), we need to find the group acting in this space and explicitly containing a^ν. In the sequel, tangent and cotangent spaces with such a group will be called the dynamic space, while the group itself will be referred to as the dynamic group.

So, let us consider the cotangent space with metric (1.21) and introduce the linear transformation

$$\overline{\delta}y^A = M^A_B(y)\delta y^B, \tag{2.6}$$

where δy^A are the base elements of the cotangent space. The conditions of invariance of the metric with respect to (2.6) lead to the condition of orthogonality of the matrix $M(y)$:

$$\overset{\circ}{g}_{AB}M^A_C M^B_E = \overset{\circ}{g}_{CE}, \tag{2.7}$$

where $\overset{\circ}{g}_{AB}$ is the metric tensor (1.21). From the orthogonal transformations (2.6) we choose a set of transformations, where the matrix $M(y)$ can be written in the form

$$M^B_A(y) = \frac{\partial f^B(y)}{\partial y^A}. \tag{2.8}$$

It is obvious that if (2.6-8) is fulfilled, then transformation (2.6) will be generated by the transformation

$$\overline{y}^A = f^A(y), \tag{2.9}$$

acting in the total space. As shown in (1), from (2.9) and (2.7) it follows that in infinitesimal terms we come to group (2.2).

Let us perform factorization of group (2.6) with respect to subgroup (2.9). After factorization, from (2.6) we obtain the group whose elements are nonintegrable transformations. This means that in the total space there exists no transformation whose tangent image would coincide with nonintegrable (2.6). In other words, for a nonintegrable transformation the points of the total space remain fixed, while the elements of the cotangent space transform by rule (2.6).

(5) To realize the program under discussion, we will study a transformation in the dynamic space of the form

$$\overline{\delta x}^\nu = \delta x^\nu + qi(w^+\gamma^\nu\delta w - \delta w^+\gamma^\nu w),$$

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$$\overline{\delta w} = \delta w - pi\gamma_\nu w\delta x^\nu, \qquad (2.10)$$
$$\overline{\delta w}^+ = \delta w^+ + piw^+\gamma_\nu\delta x^\nu,$$

where p, q are some real functions of y^A and w, w^+ are 4-spinors.

Along with (2.10), we can also write a transformation of the form

$$\overline{\delta x}^\nu = \delta x^\nu + q_1(w^+\gamma^\nu\delta w + \delta w^+\gamma^\nu w),$$

$$\overline{\delta w} = \delta w + p_1\gamma_\nu w\delta x^\nu, \qquad (2.11)$$
$$\overline{\delta w}^+ = \delta w^+ + p_1 w^+\gamma_\nu\delta x^\nu.$$

But since the algebra of Dirac matrices also contains the pseudovector combination $\gamma_\nu\gamma_5$, we can write

$$\overline{\delta x}^\nu = \delta x^\nu + q_2(w^+\gamma^\nu\gamma_5\delta w + \delta w^+\gamma_5\gamma^\nu w),$$

$$\overline{\delta w} = \delta w + p_2\gamma_5\gamma_\nu w\delta x^\nu, \qquad (2.12)$$
$$\overline{\delta w}^+ = \delta w^+ - p_2 w^+\gamma_\nu\gamma_5\delta x^\nu.$$

For the combination $\gamma_\nu\gamma_5$, an analog to (2.11) is the transformation

$$\overline{\delta x}^\nu = \delta x^\nu + iq_3(w^+\gamma^\nu\gamma_5\delta w + \delta w^+\gamma_5\gamma^\nu w),$$

$$\overline{\delta w} = \delta w + ip_3\gamma_5\gamma_\nu w\delta x^\nu, \qquad (2.13)$$
$$\overline{\delta w}^+ = \delta w^+ + ip_3 w^+\gamma_\nu\gamma_5\delta x^\nu.$$

(6) Let us return to transformation (2.10) and study it more carefully. We write the matrix $\widetilde{A}$ that forms the second summands in (2.10):

$$\widetilde{A}^\nu_k = iqw\gamma^\nu_k,$$
$$\widetilde{A}^k_\nu = -ip\gamma^k_\nu w,$$
$$\widetilde{A}^\nu_{\bar{k}} = -iq\gamma^{\nu k}w,$$
$$\widetilde{A}^{\bar{k}}_\nu = ipw^+\gamma_{\nu k}.$$

The other elements of the matrix $\widetilde{A}$ are equal to zero. Here we use the abbreviated notation $w^+\gamma^\nu_k = w^+_n\gamma^{\nu n}{}_k$, $\gamma_\nu{}^k w = \gamma_\nu{}^k{}_n w^n$.

By simple calculations we make sure that the following matrix equality is valid:

$$\widetilde{A}^3 = 2qp\rho\widetilde{A}, \qquad (2.14)$$

where $\rho = w^+_k w^k$.

In order that the base transformation (2.10) form the group, we have to rewrite it taking into account (2.14) as follows:

$$\overline{\delta y}^A = \widetilde{a}^A_B\delta y^B, \qquad (2.15)$$

where

$$\widetilde{a} = 1 + \widetilde{A} + \varepsilon\widetilde{A}^2. \qquad (2.16)$$

Transformation (2.15) preserves the invariance of metric (1.21) of the total space, which in turn is

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equivalent to condition (2.7). Let us substitute (2.16) into (2.7). After simple calculations we obtain

$$\begin{aligned} q &= \sqrt{\frac{H}{2}}\frac{1}{\rho}, \\ p &= \frac{1}{\sqrt{2H}}, \\ \varepsilon &= -1. \end{aligned} \tag{2.17}$$

Along with (2.17), there also exists yet another solution

$$\begin{aligned} q &= -\sqrt{\frac{H}{2}}\frac{1}{\rho}\sin\Theta, \\ p &= \frac{1}{\sqrt{2H}}\sin\Theta, \\ q\varepsilon\rho &= -\frac{1}{\sqrt{2H}}(1-\cos\Theta), \end{aligned} \tag{2.18}$$

where Θ is an arbitrary function of y^A. In (3) of Section *5*, we will see that $\Theta(y)$ is the group parameter not depending on the points of the total space, i.e.

$$\Theta = \xi. \tag{2.19}$$

3. Dynamic group

Let us discuss the dynamic group that arises on the basis of transformations (2.10–13) and preserves the invariance of metric (1.21).

(1) After substituting (2.17) into (2.15–16), we obtain

$$\begin{aligned} \overline{\delta x^\nu} &= \delta X_1^\nu, \\ \overline{\delta w} &= \delta w - i\frac{1}{\sqrt{2H}}(\delta x^\nu - \delta X_1^\nu)\gamma_\nu w, \\ \overline{\delta w}^+ &= \delta w + i\frac{1}{\sqrt{2H}}(\delta x^\nu - \delta X_1^\nu)w^+\gamma_\nu, \end{aligned} \tag{3.1}$$

where we have introduced the notation

$$\delta X_1^\nu = i\sqrt{\frac{H}{2}}\frac{1}{\rho}(w^+\gamma^\nu\delta w - \delta w^+\gamma^\nu w). \tag{3.2}$$

It is obvious that the right-hand part of (3.2) is a linear form of δw and δw^+, but is not a total differential. Hence it follows that (3.1) is a nonintegrable transformation.

Denote the matrix of transformation (3.1) by a_B^A. One can easily verify that the following equality is fulfilled:

$$a^2 = 1, \tag{3.3}$$

where 1 is the unit matrix.

By analogous calculations, from (2.11) we obtain

$$\begin{aligned} \overline{\delta x^\nu} &= \delta X_2^\nu, \\ \overline{\delta w} &= \delta w + \frac{1}{\sqrt{2H}}(\delta x^\nu - \delta X_2^\nu)\gamma_\nu w, \end{aligned} \tag{3.4}$$

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$$\overline{\delta w}^{+} = \delta w^{+} + \frac{1}{\sqrt{2H}}(\delta x^{\nu} - \delta X_2^{\nu})w^{+}\gamma_{\nu},$$

where

$$\delta X_2^{\nu} = \sqrt{\frac{H}{2}}\frac{1}{\rho}(w^{+}\gamma^{\nu}\delta w + \delta w^{+}\gamma^{\nu}w). \quad (3.5)$$

If b_B^A is the matrix of transformation (3.4), then it is not difficult to verify that

$$b^2 = 1.$$

Let us now consider transformations (2.12) and (2.13). As shown by calculations, using these transformations we can construct transformations of form (3.1) and (3.4) if and only if the metric has the form

$$ds^2 = \overset{\circ}{g}_{\nu\tau}dx^{\nu}dx^{\tau} - 2H\frac{1}{\rho}dw_k^{+}dw^{k}. \quad (3.6)$$

Thus when the metric has form (1.21), on the basis of (2.10–13) there arise only transformations (3.1) and (3.4) which do not contain the group parameters.

(2) Let us now return to solution (2.18–19). From (2.15–16) we find

$$\overline{\delta x^{\nu}} = \delta x^{\nu}\cos\xi_1 + \delta X_1^{\nu}\sin\xi_1,$$

$$\overline{\delta w} = \delta w + \frac{i}{\sqrt{2H}}(\delta x^{\nu}\sin\xi_1 + \delta X_1^{\nu}(1-\cos\xi_1))\gamma_{\nu}w, \quad (3.7)$$

$$\overline{\delta w^{+}} = \delta w^{+} - \frac{i}{\sqrt{2H}}(\delta x^{\nu}\sin\xi_1 + \delta X_1^{\nu}(1-\cos\xi_1))w^{+}\gamma_{\nu},$$

where δX_1^{ν} is (3.2) and ξ is the group parameter.

Analogous calculations for (2.12) give

$$\overline{\delta x^{\nu}} = \delta x^{\nu}ch\xi_2 + \delta X_3^{\nu}sh\xi_2,$$

$$\overline{\delta w} = \delta w - \frac{i}{\sqrt{2H}}(\delta x^{\nu}sh\xi_2 + \delta X_3^{\nu}(ch\xi_2 - 1))\gamma_5\gamma_{\nu}w, \quad (3.8)$$

$$\overline{\delta w^{+}} = \delta w^{+} - \frac{i}{\sqrt{2H}}(\delta x^{\nu}sh\xi_2 + \delta X_3^{\nu}(ch\xi_2 - 1))w^{+}\gamma_{\nu}\gamma_5,$$

where

$$\delta X_3^{\nu} = \sqrt{\frac{H}{2}}\frac{1}{\rho}(w^{+}\gamma^{\nu}\gamma_5\delta w + \delta w^{+}\gamma_5\gamma^{\nu}w). \quad (3.9)$$

Like (3.7), this transformation also preserves metric (1.21).

It is obvious that (3.7) and (3.8) form one-parameter groups. Note that (3.7) is the compact group and (3.8) is the noncompact one. If (1.21) is replaced by metric (3.6), then groups (3.7) and (3.8) exchange their compactness.

(3) Let us rewrite (3.7–8) in the infinitesimal form

$$\overline{\delta y^{A}} = \delta y^{A} + \xi_1 A_B^A\delta y^{B}, \quad (3.10)$$

and

$$\overline{\delta y^{A}} = \delta y^{A} + \xi_2 B_B^A\delta y^{B}, \quad (3.11)$$

where the nonzero elements of the matrix A are written in the form

$$A_k^{\nu} = i\sqrt{\frac{H}{2}}\rho^{-1}w^{+}\gamma_k^{\nu},$$

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$$A_{\bar{k}}^{\nu} = -i\sqrt{\frac{H}{2}}\rho^{-1}\gamma^{\nu k}w, \tag{3.12}$$
$$A_{\nu}^{k} = i\frac{1}{\sqrt{2H}}\gamma_{\nu}^{k}w,$$
$$A_{\nu}^{\bar{k}} = -i\frac{1}{\sqrt{2H}}w^{+}\gamma_{\nu k,}$$

while for the matrix B we obtain

$$B_{k}^{\nu} = \sqrt{\frac{H}{2}}\rho^{-1}w^{+}\gamma^{\nu}\gamma_{5k},$$
$$B_{\bar{k}}^{\nu} = \sqrt{\frac{H}{2}}\rho^{-1}\gamma_{5}^{k}\gamma^{\nu}w, \tag{3.13}$$
$$B_{\nu}^{k} = \frac{1}{\sqrt{2H}}\gamma_{5}^{k}\gamma_{\nu}w,$$
$$B_{\nu}^{\bar{k}} = -\frac{1}{\sqrt{2H}}w^{+}\gamma_{\nu}\gamma_{5k}.$$

Using transformations (2.11) and (2.13), we find the one-parameter groups preserving metric (1.21). In the infinitesimal form these transformations look like

$$\overline{\delta y^{A}} = \delta y^{A} + \xi_{3}E_{B}^{A}\delta y^{B}, \tag{3.14}$$

and

$$\overline{\delta y^{A}} = \delta y^{A} + \xi_{4}F_{B}^{A}\delta y^{B}. \tag{3.15}$$

The matrices E and F have the following nonzero elements:

$$E_{k}^{\nu} = i\sqrt{\frac{H}{2}}\rho^{-1}w^{+}\gamma_{k}^{\nu},$$
$$E_{\bar{k}}^{\nu} = \sqrt{\frac{H}{2}}\rho^{-1}\gamma^{\nu k}w, \tag{3.16}$$
$$E_{\nu}^{k} = -\frac{1}{\sqrt{2H}}\gamma_{\nu}^{k}w,$$
$$E_{\nu}^{\bar{k}} = -\frac{1}{\sqrt{2H}}w^{+}\gamma_{\nu k,}$$

and

$$F_{k}^{\nu} = i\sqrt{\frac{H}{2}}\rho^{-1}w^{+}\gamma^{\nu}\gamma_{5k},$$
$$F_{\bar{k}}^{\nu} = -i\sqrt{\frac{H}{2}}\rho^{-1}\gamma_{5}^{k}\gamma^{\nu}w, \tag{3.17}$$
$$F_{\nu}^{k} = i\frac{1}{\sqrt{2H}}\gamma_{5}^{k}\gamma_{\nu}w,$$
$$F_{\nu}^{\bar{k}} = -i\frac{1}{\sqrt{2H}}w^{+}\gamma_{\nu}\gamma_{5k}.$$

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From (3.12–13) and (3.16–17) we immediately obtain

$$SpA = SpB = SpE = SpF = 0.$$

Note that (3.14), like (3.10), acts in the dynamic space compactly, while (3.15) noncompactly.

It is not difficult to verify the following equalities:

$$\begin{aligned} A^{\nu}_{\xi}A^{\xi}_{\tau} &= -\delta^{\nu}_{\tau}, \\ B^{\nu}_{\xi}B^{\xi}_{\tau} &= \delta^{\nu}_{\tau}, \\ E^{\nu}_{\xi}E^{\xi}_{\tau} &= -\delta^{\nu}_{\tau}, \\ F^{\nu}_{\xi}F^{\xi}_{\tau} &= \delta^{\nu}_{\tau}, \end{aligned} \tag{3.18}$$

where the summation is performed over the index ξ: $A^{\nu}_{\xi}A^{\xi}_{\tau} = A^{\nu}_{k}A^{k}_{\tau} + A^{\nu}_{\overline{k}}A^{\overline{k}}_{\tau}$, and δ^{ν}_{τ} is the Kronecker symbol.

Let us consider the commutator $[A, B]$ which we denote by L. Nonzero elements of the matrix L^{A}_{B} have the form

$$\begin{aligned} L^{\nu}_{\tau} &= -2\widetilde{C}^{\nu}_{\varepsilon}, \\ L^{\xi}_{\eta} &= A^{\xi}_{\sigma}B^{\sigma}_{\eta} - B^{\xi}_{\sigma}A^{\sigma}_{\eta}, \end{aligned} \tag{3.19}$$

where ξ, η run through the indexes and overlined indexes of the internal space. In (3.19) there has appeared the matrix $\widetilde{C}$ with nonzero elements

$$\widetilde{C}^{\nu}_{\tau} = \frac{i}{2}\rho^{-1}w^{+}[\gamma_{\tau}, \gamma^{\nu}]\gamma^{5}w. \tag{3.20}$$

Since $\widetilde{C}_{\tau\nu} = \overset{\circ}{g}_{\tau\sigma}\widetilde{C}^{\sigma}_{\nu}$ is antisymmetric, it is obvious that the elements of the 4-matrix $\widetilde{C}$ satisfy the equality

$$\widetilde{C}^{4} = \mu_{0}1 + \mu_{1}\widetilde{C}^{2}, \tag{3.21}$$

where

$$\begin{aligned} \mu_{0} &= -\det \widetilde{C}, \\ \mu_{1} &= sp(\widetilde{C}^{2}). \end{aligned} \tag{3.22}$$

By virtue of (3.21), it can be assumed that 1, $\widetilde{C}$, $\widetilde{C}^{2}$, $\widetilde{C}^{3}$ are independent matrices, since the rest of the matrices $\widetilde{C}^{m}$ $(m \geq 4)$ are linearly expressed through them. It should be noted that in (3.21) the higher degree of the polynomial of $\widetilde{C}$ coincides with the dimension of the external space.

Further, forming all possible commutators of A, B, L and taking into account (3.12–16), we obtain a complete Lie algebra. In that case, due to equality (3.18) the arisen new algebras are expressed either linearly through A and B or quadratically. Moreover, some of them contain the matrices $\widetilde{C}$, $\widetilde{C}^{2}$, $\widetilde{C}^{3}$, and they convolute with the matrices A and B by means of the tensor indexes of the external space. For instance, instead of the elements A^{ν}_{k} there appear elements of the form $(\widetilde{C}^{2})^{\nu}_{\tau}A^{\tau}_{k}$.

Now let us form all possible commutators of the matrices A and E. In the matrix algebra, instead of the matrix $\widetilde{C}$ (3.20) there appears the matrix

$$C^{\nu}_{\tau} = \frac{i}{2}\rho^{-1}w^{+}[\gamma_{\tau}, \gamma^{\nu}]w. \tag{3.23}$$

It is not difficult to verify that the matrices $\widetilde{C}$ and C are related by the equalities

$$\widetilde{C}^{\nu}_{\tau}C^{\tau}_{\sigma} = \mu_{2}\delta^{\nu}_{\sigma}, \tag{3.24}$$

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where δ^ν_τ is the Kronecker symbol and

$$\mu_2 = C^1_2 C^3_4 + C^2_3 C^1_4 + C^2_4 C^3_1. \qquad (3.25)$$

From (3.21) and (3.24) it immediately follows that C is linearly expressed through $\widetilde{C}$ and $\widetilde{C}^3$.

The fulfillment of equalities (3.18), (3.21) and (3.24) indicates that the Lie algebra of the complete continuous dynamic group formed by all base matrices A, B, E, F and the concurrently arisen matrix $\widetilde{C}$ is closed. As preliminary calculations show, the dimension of the complete continuous group is equal to 35. We have mentioned above that the spurs of the base matrices are equal to zero. Then the spurs of the matrices forming the complete Lie algebra are also equal to zero. Since the arisen Lie algebra is a matrix one, from the known classification of classical Lie algebras it follows that the complete continuous group is isomorphic to noncompact $SU(3.3)$.

(4) As has been noted in (1), the dynamic transformation (3.1) is nonintegrable. It is not difficult to verify that all other base transformations are also nonintegrable. Thus we conclude that dynamic transformations act only in the dynamic space and, at that, leave the points of the total space fixed.

(5) Let us consider group (2.3) whose subgroup is a Lorentz group. In the infinitesimal form it is written as

$$\overline{y}^A = y^A + \overset{\circ}{\Lambda}{}^A_B y^B,$$
$$\overline{dy}^A = dy^A + \overset{\circ}{\Lambda}{}^A_B dy^B,$$

where the nonzero elements of the matrix $\overset{\circ}{\Lambda}$ have the form

$$\overset{\circ}{\Lambda}{}^\nu_\tau = \xi^\nu_\tau,$$

$$\overset{\circ}{\Lambda}{}^k_n = \frac{1}{8}\xi^\nu_\sigma[\gamma_\nu, \gamma^\sigma]^k_n + i\xi\delta^k_n, \qquad (3.26)$$
$$\overset{\circ}{\Lambda}{}^{\overline{k}}_{\overline{n}} = -\overset{\circ}{\Lambda}{}^n_k,$$

ξ^ν_τ are the Lorentz group parameters, and ξ is the phase group parameter. Then the generator of group (2.3) can be written as

$$\Lambda = \overset{\circ}{\Lambda} - \overset{\circ}{\Lambda}{}^n_k w^n \frac{\partial}{\partial w^k} + \overset{\circ}{\Lambda}{}^n_k w^+_n \frac{\partial}{\partial w^+_k}, \qquad (3.27)$$

where the matrix $\overset{\circ}{\Lambda}$ is (3.26).

Let us consider group (3.7), whose infinitesimal form is (3.10). The elements of this group are functions of w and W^+. In view of this fact and taking into account that dynamic transformations leave the points of the total space fixed, the commutator of the algebras of these two groups takes the form $[A, \Lambda]$. After elementary calculations we obtain

$$[A, \Lambda] = 0. \qquad (3.28)$$

Analogously, it can be shown that that the other base groups (3.11), (3.19) and (3.20), too, commute with the Lorentz group. Hence we conclude that the dynamic continuous group commutes with group (2.3).

Simple calculations show that for the discrete groups (3.1) and (3.4), too, we have the equalities

$$[a, \Lambda] = 0, \qquad (3.29)$$
$$[b, \Lambda] = 0.$$

(6) Now let us return to the admissible equation (1.1) and metric (1.15) generated by it. From the definition of admissibility it follows that (1.1) is invariant with respect to the Lorentz group and

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equalities (1.2) are fulfilled. In that case, the internal space is the space of representation of Lorentz groups and, simultaneously with this fact, we see that equality (2.5) is fulfilled.

In the dynamic space we introduce the following transformations in the infinitesimal form:

$$
\begin{aligned}
\delta x^{\nu} &= \delta x^{\nu} + \xi_1 i \sqrt{\frac{H}{2}} \sum_{\alpha \in \Omega} q_{\alpha} sp[(w_{\alpha}^{+} w_{\alpha})^{-1} \left(w_{\alpha}^{+} a^{\nu} \delta w_{\alpha} - \delta w_{\alpha}^{+} a^{\nu} w_{\alpha} \right)], \\
\overline{\delta w_{\alpha}} &= \delta w_{\alpha} + \xi_1 i \frac{1}{\sqrt{2H}} a_{\nu} w_{\alpha} \delta x^{\nu}, \\
\delta w_{\alpha}^{+} &= \delta w_{\alpha}^{+} - \xi_1 i \frac{1}{\sqrt{2H}} w_{\alpha} a_{\nu} \delta x^{\nu},
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
\delta x^{\nu} &= \delta x^{\nu} + \xi_2 \sqrt{\frac{H}{2}} \sum_{\alpha \in \Omega} q_{\alpha} sp[(w_{\alpha}^{+} w_{\alpha})^{-1} (w_{\alpha}^{+} a^{\nu} \delta w_{\alpha} + \delta w_{\alpha}^{+} a^{\nu} w_{\alpha})], \\
\delta w_{\alpha} &= \delta w_{\alpha} - \xi_2 \frac{1}{\sqrt{2H}} a_{\nu} w_{\alpha} \delta x^{\nu}, \\
\delta w_{\alpha}^{+} &= \delta w_{\alpha}^{+} - \xi_2 \frac{1}{\sqrt{2H}} w_{\alpha} a_{\nu} \delta x^{\nu},
\end{aligned}
\tag{3.31}
$$

where ξ_1, ξ_2 are the group parameters and $a_{\nu} = \overset{\circ}{g}_{\nu\sigma} a^{\sigma}$. These transformations are nonintegrable and therefore they leave the points of the total space fixed. By direct calculations we can verify that (3.30) and (3.31) preserve the invariance of metric (1.15). These one-parameter groups can be treated as base groups by means of which we can construct a complete dynamic group.

We have thus shown that for any irreducible representation of the Lorentz group there exists its own dynamic group.

(7) Analyzing the results of (3), we come to the conclusion that the dimension of the dynamic continuous group depends on the algebraic properties of Dirac matrices (i.e. on the representation of the Lorentz group) and – through equality (3.21) – on the dimension of the external space, but does not depend on the dimension of the internal space. Indeed, if we extend the dimension of the internal space of the Dirac equation and, when doing so, do not violate the algebraic properties of the matrix γ^{ν}, then the dynamic group $SU(3.3)$ will not change.

An analogous picture is observed when we consider arbitrary representations of the Lorentz group. From (3.30–31) it immediately follows that the matrices a^{ν} contained in the admissible equation (1.1) play the key role in the dynamic group construction. These matrices dictate not only representations of the Lorentz group, but also the formation of the corresponding dynamic group.

(8) Let us consider group (2.2). We perform calculations analogous to those done in (5) for base admissible subgroups and group (2.2). It is easy to verify that these groups do not commute with each other.

There arises a question why (2.3) commutes with a dynamic group and (2.2) does not. The most important point in the case of (2.3) is that the internal space is the space of representations of the Lorentz group and, simultaneously with this, equality (2.5) is fulfilled for the matrices γ^{ν} and $[\gamma_{\sigma}, \gamma^{\tau}]$. The nonfulfillment of even one of these two conditions leads to the noncommutativity with the dynamic group.

From (1) and (2) of Section *2*, it immediately follows that group (2.2) not only violates equality (2.5), but, simultaneously with this, the internal space ceases to be the space of representations of the Lorentz group. Obviously, the violation of (2.5) brings about the violation of the invariance of the

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Dirac equation. Therefore group (2.2) should be rejected.

Let us proceed now to equation (1.1). If (1.1) is an admissible equation, then the internal space is the space of representations of the Lorentz group and, simultaneously with this, equality (2.5) is fulfilled. Obviously, the Lorentz group and the dynamic group (3.30–3.31) commute with each other.

4. On the dynamic group representation

If some group acts in the tangent space, then, as is known, the tensor field, if it exists, is transformed according to the tensor laws. In the theory presented here, in a dynamic space there acts a dynamic group. Then any considered tensor field transforms by the rule of representation of this group. On the other hand, the internal field is the field generated by test fields. This means that the internal space is, generally speaking, the space of representations not only for a Lorentz group, but also for a dynamic group.

Justify this reasoning, we consider transformations (3.11) and (3.14) and investigate the Lie algebra of this group. Calculations show that the following commutative relations are valid:

$$\begin{aligned} [B, E] &= P, \\ [P, B] &= -E + \frac{\widetilde{\rho}}{\rho} B, \\ [P, E] &= -B - \frac{\widetilde{\rho}}{\rho} E, \end{aligned} \tag{4.1}$$

where the nonzero elements of the matrix P have the form

$$\begin{aligned} P^{\nu}_{\sigma} &= 0, \\ P^{\xi}_{\eta} &= B^{\xi}_{\nu} E^{\nu}_{\eta} - E^{\xi}_{\nu} B^{\nu}_{\eta}, \end{aligned} \tag{4.2}$$

$\rho = w^{+}w,\ \widehat{\rho} = w^{+}\gamma_5 w$. Here ξ, η runs through the indexes and the overlined indexes of the internal space.

Let us show that there exists a two-dimensional representation of algebra (4.1). For this, we introduce the matrices

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} i\gamma_5 & 0 \\ 0 & -i\gamma_5 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}. \end{aligned} \tag{4.3}$$

Let these matrices act on $v = (w_1, w_2)$, where $w_1,\ w_2$ are two copies of 4-spinors. Extending the Dirac conjugation and multiplying by the matrix $-i\sigma_2$, we obtain

$$v \to v^{+} = (v)^{+} = -iv^{(H)}\gamma_4\sigma_2, \tag{4.4}$$

where $v^{(H)}$ is the Hermitian conjugation to v. For such an extension of the external space, metric (1.21) takes the form

$$ds^2 = \overset{\circ}{g}_{\nu\tau} dx^{\nu} dx^{\tau} + 2\frac{H}{\rho_2} dv^{+} dv, \tag{4.5}$$

where ρ_2 is a scalar value of the form

$$\rho_2 = v^{+}v.$$

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Along with this, let us modify groups (3.11) and (3.14). For this, in the matrices B, E, P we make the replacement: $w \to v$, $w^+ \to v^+$, $\rho = w^+w \to \rho_2 = v^+v$,

$$\gamma^\nu \to \begin{pmatrix} \gamma^\nu & 0 \\ 0 & \gamma^\nu \end{pmatrix} = \gamma^\nu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \gamma^\nu 1.$$

Then we obtain $B \to B_2$, $E \to E_2$, $P \to P_2$. These matrices satisfy the commutative relations (4.1). Thus we confirm the conclusion made in (8) of Section *3* that the algebraic structure of a dynamic group does not change when the dimension of the external space increases.

For conjugation (4.4) we obviously have

$$\sigma_1^+ = -\sigma_1,$$
$$\sigma_2^+ = -\sigma_2,$$
$$\sigma_3^+ = -\sigma_3.$$

Let us introduce new matrices which by means of (4.3) can be written as follows:

$$\widetilde{B} = \frac{1}{2}\sigma_1,$$
$$\widetilde{E} = \frac{\widetilde{\rho}_2}{2\rho_2}\sigma_1 - \frac{1}{2\rho_2}(\rho_2^2 + \widetilde{\rho}_2^2)^{\frac{1}{2}}\sigma_2, \tag{4.6}$$
$$\widetilde{P} = \frac{1}{2\rho_2}(\rho_2^2 + \widetilde{\rho}_2^2)^{\frac{1}{2}}\sigma_3,$$

where $\rho_2 = v^+v$, $\widetilde{\rho}_2 = v^+\gamma_5 v$. By direct calculations it can be verified that matrices (4.6) satisfy the commutative relations (4.1).

From equality (4.6) we express σ_1, σ_2, σ_3 through $\widetilde{B}$, $\widetilde{E}$, $\widetilde{P}$ as follows:

$$\sigma_1 = 2\widetilde{B},$$
$$\sigma_2 = (\rho_2^2 + \widetilde{\rho}_2^2)^{-\frac{1}{2}}(2\widetilde{\rho}_2\widetilde{B} - 2\rho_2\widetilde{E}), \tag{4.7}$$
$$\sigma_3 = 2\rho(\rho_2^2 + \widetilde{\rho}_2^2)^{-\frac{1}{2}}\widetilde{P}.$$

Now, in (4.7) we replace the matrices $\widetilde{B}$, $\widetilde{E}$, $\widetilde{P}$ by B_2, E_2 P_2, respectively. The arisen matrices are denoted by h_1, h_2, h_3. Next, in the modified dynamic space we introduce the group

$$\overline{\delta y_2^A} = \delta y_2^A + t^a h_{a\,B}^{\,A}(v, v^+)\delta y_2^B, \tag{4.8}$$

where

$$y_2^\nu = x^\nu,$$
$$y_2^{(k)} = v^{(k)},$$
$$y_2^{(\bar{k})} = v_k^+.$$

It is obvious that (4.8) is a dynamic group arisen on the basis of the modified one-parameter groups (3.11), (3.14), but with generators h_a, where t^a are the group parameters. It is not difficult to write an exact representation of group (4.8), acting in the internal space:

$$\overline{v} = v + t^a\sigma_a v, \tag{4.9}$$
$$\overline{v}^+ = v^+ - t^a v^+\sigma_a.$$

The differentiation of these equalities gives

$$\overline{dv} = dv + t^a\sigma_a dv, \tag{4.10}$$
$$\overline{dv}^+ = dv^+ - t^a dv^+\sigma_a.$$

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Combining (4.8) and (4.10), in the dynamic space we finally obtain

$$\overline{\delta y_2^A} = \delta y_2^A + t^a[\sigma_a + h_a(v, v^+)]^A_B \delta y_2^B, \tag{4.11}$$

where ${\sigma_a}^\nu_\tau = 0,$ while the other elements coincide with (4.3).

As expected, if the internal space is a representation of the dynamic group, then, in addition to (4.8), there appears a summand with matrices σ_a.

Since the matrices h_a are the functions of v and v^+, by analogy with (3.27), for the elements of the Lie algebra of group (4.9–11) we introduce the generator

$$\varkappa_a = h_a(v, v^+) + \sigma_a - \left(\frac{\partial}{\partial v}\sigma_a v\right) + \left(v^+ \sigma_a \frac{\partial}{\partial v^+}\right), \tag{4.12}$$

where the term

$$v^{(q)} {\sigma_a}^{(p)}_{(q)} \frac{\partial}{\partial v^{(p)}}$$

is formally represented in the form $[(\partial/\partial v)\, \sigma_a v]$. By simple calculations we obtain

$$[\varkappa_a, \varkappa_b] = C^c_{ab} \varkappa_c, \tag{4.13}$$

where C^c_{ab} are the structural constants of a noncompact group $SU\ (1.1)$.

So, we come to the following conclusions:

a) We have shown that the internal space is the space of representations of the Lorentz group and the dynamic group. This result is in a fairly good agreement with our understanding of the set of admissible test fields.

b) As follows from (2.3) and (4.9), the Lorentz and dynamic groups act unitarily in the internal space. The phase group, contained in (2.3) (the parameter ξ) also acts unitarily. As to the extension group (2.4), it falls out of this pattern and does not act unitarily.

c) Under the action of the dynamic group the coordinates v, v^+ of the internal space acquire a certain specific feature: they do not react to the action of the matrices h_a, but get transformed by the representation matrices σ_a. Thus under the action of the Lorentz, phase and dynamic groups we have

$$\overline{v} = v + \frac{1}{8}\xi^\nu_\sigma[\gamma_\nu, \gamma^\sigma]v + \xi^a \sigma_a v + i\xi v, \tag{4.14}$$

$$\overline{v}^+ = v^+ - \frac{1}{8}\xi^\nu_\sigma v^+[\gamma_\nu, \gamma^\sigma] - \xi^a v^+ \sigma_a - i\xi v^+.$$

5. Geometry of a complex

Summarizing the results of our studies, we can say that by giving a differential equation we automatically mean that we give a complex of three spaces: external, internal and dynamic. Let us discuss once more some questions connected with the geometry of a complex based on the field motion logic. As has been said above, our starting point is Felix Klein's statement that the geometry of a space is defined by the group acting in it.

(1) Let us consider the continuous groups (2.3), (4.9) and (4.11). Combining these groups, we obtain the transformation rule for the elements of a complex in the infinitesimal form

$$\overline{x}^\nu = x^\nu + \xi^\nu_\sigma x^\sigma + \xi^\nu,$$

$$\overline{w} = w + \overset{\circ}{T} w, \tag{5.1}$$

$$\overline{w}^+ = w^+ - w^+ \overset{\circ}{T},$$

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$$\overline{\delta y}^A = \delta y^A + (\overset{\circ}{T} + \xi^a h_a)^A_B \delta y^B,$$

where δy^A are the base element of the cotangent space. The nonzero elements of the matrix $\overset{\circ}{T}{}^A_B$ have the form:

$$\overset{\circ}{T}{}^\nu_\tau = \xi^\nu_\tau,$$

$$\overset{\circ}{T}{}^k_n = \frac{1}{8}\xi^\nu_\sigma[\gamma_\nu, \gamma^\sigma]^k_n + \xi^a \sigma_{a}{}^k_n + i\xi\delta^k_n, \tag{5.2}$$

$$\overset{\circ}{T}{}^{\bar{k}}_{\bar{n}} = -\overset{\circ}{T}{}^n_k\ .$$

Denote group (5.1) by $\overset{\circ}{G}$. In (5.1–2) we again use the notation introduced in Sections *1, 2,* but now w denotes the coordinates of the modified internal space, and δy^A the coordinates of the corresponding dynamic space.

(2) From (5.1) it immediately follows that the Lorentz group (parameters ξ^ν_σ, ξ^ν) acts in the external space, while the dynamic group (matrices h_a) acts in the dynamic space. These groups act independent of each other.

As seen from (5.1), the internal space is the space of representations of the Lorentz group (spinor space) and the dynamic group (generators σ_a).

In Section *3* we have shown that the Lorentz group and the dynamic group commute with each other. With this property agrees the commutativity of representations of these groups, which follows from the equalities

$$[[\gamma_\nu, \gamma^\tau], \sigma_a] = 0, \tag{5.3}$$
$$[\gamma_\nu \sigma_2, \sigma_a] = 0.$$

As seen immediately from Sections *2, 3,* base dynamic groups are constructed from the dynamic part of the differential equation of the test field (matrices a^ν and, in the case of the Dirac equation, matrices γ^ν) and from the elements of the internal space. This means that the dynamics of the considered test field is imprinted on the dynamic group and, thereby, on the geometry of the complex.

(3) In the total space let us take two infinitely close points M and M_1 with coordinates y^A and $y^A + dy^A$. As indicated in Section *2*, the dynamic group being a nonintegrable transformation does not act on the total space and leaves the points of this space fixed. This means that the group $\overset{\circ}{G}$ acts on the coordinates dy^A through the matrix $\overset{\circ}{T}$. Then we find

$$\overline{dy}^A = dy^A + \overset{\circ}{T}{}^A_B dy^B. \tag{5.4}$$

The tangent vector $\partial/\partial y^A$ and the cotangent vector dy^A are self-conjugate. Hence (5.4) immediately yields

$$\frac{\overline{\partial}}{\partial y^A} = \frac{\partial}{\partial y^A} - \overset{\circ}{T}{}^B_A \frac{\partial}{\partial y^B}. \tag{5.5}$$

Thus we have come to the conclusion that under the action of the group $\overset{\circ}{G}$, along with the vector Φ^A transformed by rule (5.1_4) as follows

$$\overline{\Phi}^A = \Phi^A + (\overset{\circ}{T} + \xi^a h_a)^A_B \Phi^B, \tag{5.6}$$

there exists a vector F^A whose transformation gives

$$\overline{F}^A = F^A + \overset{\circ}{T}{}^A_B F^B. \tag{5.7}$$

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Following these transformation rules, in the sequel we will call F^A a standard vector, and Φ^A a dynamic vector.

From (5.6–7) it immediately follows that the dynamic group forbids to perform the operation of addition of vectors Φ and F.

By virtue of (5.4), under the action of group (4.11) metric (4.5) remains invariant. The invariance of the metric does not change if instead of (4.9) we take transformation (4.14).

Let us now return to the questions arisen when deriving (2.18–19). If instead of ξ we consider the function $\Theta(y)$, then $\Theta(y)$ will appear in (4.14) as a localized group parameter for σ_a. In that case, for (4.14) to preserve the invariance of metric (4.5), we come to result (2.19). This means that the obtained total geometry does not admit arbitrariness and in the dynamic group the group parameters must be independent.

(4) Let us go back to equation (1.16). The internal space of this equation is extended so that it becomes also the space of representation of the dynamic group. If $w = u(x)$ is an element of the modified internal space, then the equality

$$dw = \frac{\partial u}{\partial x^\nu} dx^\nu$$

and transformations (5.1–2), (5.4–5) readily imply

$$\frac{\overline{\partial u}^k}{\partial x^\nu} = \frac{\partial u^k}{\partial x^\nu} x - \xi^\sigma_\nu \frac{\partial u^k}{\partial x^\sigma} + \overset{\circ}{T}{}^k_n \frac{\partial u^n}{\partial x^\nu}. \tag{5.8}$$

In order to preserve the invariance of equation (1.16) with respect to transformations (5.1–3) and (5.8), it must be reduced to the equation

$$-i\gamma^\nu \sigma_2 \frac{\partial u}{\partial x^\nu} = mu. \tag{5.9}$$

As said in Section *1*, the observer is enclosed within process (1.16). His entire reasoning and all constructions are in the frame of reference and the system of calculus of process (1.16). However, when the internal space becomes the space of representation of the dynamic group, the observer must pass from (1.16) to a process described by the modernized equation (5.9). Recall that here we deal with the representation of subgroup (4.1) of the dynamic group. In all constructions within the framework of process (1.16), we replace the matrices γ^ν by the matrices $-i\gamma^\nu\sigma_2$. Along with this, by virtue of (4.4) the Dirac conjugation for the coordinates of the internal space w is defined in the form

$$w^+ = -iw^{(H)}\gamma_4\sigma_2,$$

where H is the Hermitian conjugation.

(5) Let us investigate the geometric properties of the matrices a and b which form transformations (3.1) and (3.4) in the dynamic space.

a) Since (3.3) holds and (3.1) acts only in the dynamic space, we can interpret transformation (3.1) as a reflection in this space. Hence, under the action of the group $\overset{\circ}{G}$ the equalities

$$[a, h_a] = 0, \tag{5.10}$$

are fulfilled, while the matrix a^A_B is transformed by the rule

$$\overline{a} = a + [\overset{\circ}{T}, a], \tag{5.11}$$

where [,] is the commutator.

b) Let us consider the cotangent vector dy^A which under the action of the group $\overset{\circ}{G}$ transforms

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by rule (5.4). Taking into account (3.3), we write

$$\widetilde{dy}^A = a^A_B dy^B, \qquad (5.12)$$
$$dy^A = a^A_B \widetilde{dy}^B.$$

From (3.1) it immediately follows that, as different from dy^A, $\widetilde{dy}^A$ is not an integrable object. This leads to a conjecture that the matrix a transforms standard vectors to dynamic ones, and dynamic vectors to standard ones. Then this conjecture should be analyzed in terms of the action of the group $\overset{\circ}{G}$. For this, we assume that some standard vector F^A and some dynamic vector Φ^A are interrelated by the equality

$$\Phi^A = a^A_B F^B, \qquad (5.13)$$
$$F^A = a^A_B \Phi^B.$$

As has been said, (5.6–7) are fulfilled under the action of the group $\overset{\circ}{G}$. Then (5.13) can be rewritten in the form

$$\overline{\Phi}^A = \overline{a}^A_B \overline{F}^B, \qquad (5.14)$$
$$\overline{F}^A = \overline{a}^A_B \overline{\Phi}^B.$$

Using the infinitesimal transformations (5.6–7), from the first equality (5.14) with (5.13) taken into account we obtain

$$\overline{a} = a + [\overset{\circ}{T}, a] + \xi^a h_a a. \qquad (5.15)$$

The second equality (5.14) leads to the transformation

$$\overline{a} = a + [\overset{\circ}{T}, a] - \xi^a a h_a. \qquad (5.16)$$

From (5.15) and (5.16) we immediately have

$$\{a, h_a\} = 0, \qquad (5.17)$$

where is { , } an anticommutator.

The above reasoning can be extended to the matrix b as well.

Simple calculations show that the following equalities are fulfilled:

$$\{a, A\} = 0,$$
$$[a, B] = 0,$$
$$[a, E] = 0,$$
$$\{a, F\} = 0, \qquad (5.18)$$
$$[b, A] = 0,$$
$$\{b, B\} = 0,$$
$$\{b, E\} = 0,$$
$$[b, F] = 0,$$

where A, B, E, F are generators of transformations (3.10–11) and (3.14–15). Using equalities (5.18), we can find the commutative relations of the matrices a and b with all commutators made up of matrices A, B, E, F. Then, by analogy with (5.18), either commutation or anticommutation takes place.

As has been said in Section *3*, the Lie algebra of the total dynamic group is formed by the commutation of generators A, B, E, F. From (5.18) it follows that equalities (5.12) and (5.17) are not fulfilled for the whole dynamic group. More exactly, a certain partitioning of the dynamic group takes place. If, for example, the one-parameter groups (3.10) and (3.15) regard transformations (5.13) as a change of standard vectors to dynamic vectors and vice versa, groups (3.11) and (3.14) regard these

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transformations as reflections, i.e. the vector dynamics does not change.

Hence we come to the conclusion that the matrices a and b are a certain mechanism that violates the action of a continuous dynamic group in a dynamic space.

6. Differentiation in a complex of spaces

After we have established the metric of a complex of spaces and found the groups defining the geometry of these spaces, we see that the metric tensor and elements of a dynamic group depend on the points of the internal space. In view of this fact we should construct a method of covariant differentiation. Moreover, since there are no arbitrary functions in dynamic groups and the metric tensor is a concrete function of the points of the internal space, the connectedness elements must also be concrete functions.

For simplicity, we will consider a complex of spaces where there acts group (5.1) and the metric has form (4.5).

Let a covariant derivative of the dynamic vector field Φ^A have the form

$$\overset{\circ}{\nabla}_A \Phi^B = \frac{\partial \Phi^B}{\partial y^A} + m_{A}{}^{B}_{C} \Phi^C. \tag{6.1}$$

Taking into account the results obtained in Section 5, we conclude that under the action of the group $\overset{\circ}{G}$ the tensor value $\overset{\circ}{\nabla}_A \Phi^B$ transforms with respect to the indexes A by rule (5.5), and with respect to B – by rule (5.6):

$$\overline{\overset{\circ}{\nabla}_A} \overline{\Phi}^B = \overset{\circ}{\nabla}_A \Phi^B - \overset{\circ}{T}{}^C_A \overset{\circ}{\nabla}_C \Phi^B + (\overset{\circ}{T} + \xi^a h_a)^B_C \overset{\circ}{\nabla}_A \Phi^C, \tag{6.2}$$

where the matrix $\overset{\circ}{T}$ is (5.2), h are generators of the dynamic group acting in the dynamic space. From (6.2) we immediately obtain

$$\overset{\circ}{\nabla}_A h_{a}{}^{B}_{C} = 0. \tag{6.3}$$

(1) For (6.1) to agree with metric (4.5), the following equality must be fulfilled [6]:

$$\overset{\circ}{\nabla}_A \overset{\circ}{g}_{BC} = 0, \tag{6.4}$$

where $\overset{\circ}{g}_{BC}$ is the metric tensor. The connectedness coefficients are written in the form

$$m_{A}{}^{B}_{C} = \overset{\circ}{m}_{A}{}^{B}_{C} + \widetilde{m}_{A}{}^{B}_{C}, \tag{6.5}$$

where $\overset{\circ}{m}_{A}{}^{B}_{C}$ are Christoffel coefficients

$$\overset{\circ}{m}_{A}{}^{B}_{C} = \frac{1}{2} \overset{\circ}{g}{}^{BE} \left(\frac{\partial \overset{\circ}{g}_{AE}}{\partial y^C} + \frac{\partial \overset{\circ}{g}_{EC}}{\partial y^A} - \frac{\partial \overset{\circ}{g}_{AC}}{\partial y^E} \right). \tag{6.6}$$

From (6.4) we find

$$\widetilde{m}_{A}{}^{E}_{C} \overset{\circ}{g}_{EB} + \widetilde{m}_{A}{}^{E}_{B} \overset{\circ}{g}_{CE} = 0. \tag{6.7}$$

For the convenience of our further calculations we introduce the notation

$$\overset{\circ}{D}_A = \frac{\partial}{\partial y^A} + \overset{\circ}{m}_A, \tag{6.8}$$

where the matrix $\overset{\circ}{m}_A$ $(= m_{A}{}^{B}_{C})$ is formed from (6.6).

(2) In this stage of the investigation we are interested in fundamental issues of the arisen theory. Therefore, for the sake of simplicity, we will consider the dynamic group (3.10). From condition (2.7)

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we easily find

$$A^E_A\overset{\circ}{g}_{EB} + A^E_B\overset{\circ}{g}_{EB} = 0. \tag{6.9}$$

Using the conclusions of (1), for (3.10) condition (6.3) takes the form

$$\overset{\circ}{D}_A A^B_C + [\widetilde{m}_A, A]^B_C = 0, \tag{6.10}$$

where $\overset{\circ}{D}_A$ is (6.8).

By 2.14), it is not difficult to verify that the following matrix equality is valid:

$$A^3 = -A. \tag{6.11}$$

Let us solve equation (6.10) for $\widetilde{m}_A$. Taking into account (6.11), we find

$$\widetilde{m}_A = [\overset{\circ}{D}_A A, A] - \frac{3}{4}A[\overset{\circ}{D}_A A, A]A, \tag{6.12}$$

where

$$\overset{\circ}{D}_A A = \frac{\partial A}{\partial y^A} + [\overset{\circ}{m}_A, A].$$

After substituting (6.12) and (6.5) into (6.1), we obtain the covariant derivative corresponding to the base dynamic group (3.10). Simple calculations using (6.9) show that (6.12) satisfies equality (6.7). This means that the found covariant derivative (6.1) agrees with metric (4.5).

We have thus seen that for the dynamic group there exist connectedness coefficients $\widetilde{m}_A$ and they are expressed through generators of the group. For the total continuous dynamic group, matrices $\widetilde{m}_A$ have a cumbersome structure and therefore we will limit our discussion to the above-given results.

(3) It is obvious that for a standard vector F^A that transforms by rule (5.7) the covariant derivative (6.1) takes the form

$$\overset{\circ}{\nabla}_A F^B = \overset{\circ}{D}_A F^B.$$

We conclude by giving the rule of transformation of $m_{A}{}^B_C$ under the action of the group $\overset{\circ}{G}$. Using (5.6), from (6.1–2) we easily find

$$\overline{m_{A}}{}^B_C = m_{A}{}^B_C - \overset{\circ}{T}{}^E_A m_{E}{}^B_C + [\overset{\circ}{T} + \xi^a h_a, m_A]^B_C - \xi^a \frac{\partial h_{a}{}^B_C}{\partial y^A}. \tag{6.13}$$

In view of (6.3), transformation (6.13) can be rewritten in the form

$$\overline{m_{A}}{}^B_C = m_{A}{}^B_C - \overset{\circ}{T}{}^E_A m_{E}{}^B_C + [\overset{\circ}{T}, m_A]^B_C. \tag{6.14}$$

It appears that under the action of the group $\overset{\circ}{G}$ the connectedness coefficients $m_{A}{}^B_C$ transform like components of a standard tensor of third rank. This agrees with the fact that $m_{A}{}^B_C$ are formed from w, w^+ and γ^ν which transform by rule (5.1) and, at that, $m_{A}{}^B_C$ are concrete functions of w, w^+ and γ^ν.

On the strength of this reasoning, under the action of the group $\overset{\circ}{G}$, matrices transform by the rule

$$\overline{h_a} = h_a + [\overset{\circ}{T}, h_a]. \tag{6.15}$$

Conclusion

Let us summarize the results of our investigation. For this, we will consider a freely evolving process with the observer inside. He studies his process. We remind that the considered process does allow the observer to use any instruments or external processes not to violate the dynamics of his

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process. He can manipulate only with various states of his process and also with its dynamics. The investigation of the algebraic and geometric properties of the process leads to the following conclusions:

a) The process contains its own numerical field (or algebraic body) on the basis of which the corresponding system of calculus is constructed. In this system the differential equation of the process is always linear.

b) The process always evolves in a complex of spaces which consists of the external space (space-time), the internal space and the dynamic space.

c) In the external space, the process defines its inertial frames of reference. The Lorentz group acts in this space.

d) In the dynamic space, the process motion laws inherent in the differential equation of the process define the dynamic group.

e) The internal space, where solutions of the equation undergo changes, is the space of representation of the Lorentz group and the dynamic space.

This is the internal algebro-geometrical structure of any process.

Remark. Representations of the Lorentz group and the dynamic group in the internal space can be both reducible and irreducible. We however do not discuss this issue in detail in this stage of our investigation. In principle, in the internal space there may also exist an incidental group that commutes with representations of the Lorentz group and the dynamic group. But since we are interested in the algebro-geometric structure of processes, we do not consider incidental groups connected with the properties of external and dynamic spaces.

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