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GAME-THEORETIC MODELS OF DYNAMIC DUOPOLY

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Abstract

One Dynamic problem of Mathematical Economics - competition with two sellers and one buyer - is considered. The process consists of stages and the resources of the producers are limited. On each stage the producers deliver to the market some quantity of goods and gain appropriate profit. The total profit is defined by the sum of the profits on each stage. The competition is described in the frames of the Game Theory. Two approaches are considered – antagonistic (the buyer is the "essential player") and cooperative (with two or three players). Antagonistic game is a multistage game with complete information and the solutions are obtained by means of pure strategies. In the cooperative game we seek for Nash and Shapley solutions. Numerical realization is mainly based on the method of Dynamic Programming.

Keywords: *competition, model, resources, multistage game, cooperation, strategy, Dynamic Programming.*

The paper deals with an important problem of Mathematical Economics - Dynamic Duopoly, that is the market served stage wise by two producers (vendors). It's assumed that the goods are homogenous and the price on every stage may be known beforehand or may depend on the volume of the goods delivered to the market.

Assume that each producer posses respectively X and Y quantities of goods and the activity horizon consists of N stages. Let for each stage $t=1,2,...,N$ they deliver to the market x_t and y_t quantity of goods and gain the profit $h_t^{'}(x_t, y_t)$ and $h_t^{''}(x_t, y_t)$ respectively. The total profit is defined by the sum of the profits on each stage $H^{'} = \sum_t h_t^{'}$, $H^{''} = \sum_t h_t^{''}$.

We state the problem as the game of two persons (players). Let's assume first that the buyer (or a large number of minor consumers) is a fictitious participant and doesn't influence nor price neither the selling. Later we will consider the case, when the buyer is an "essential player".

We consider two approaches - antagonistic and cooperative. In the latter case we will basically deal with the Shapley solutions.

1. <u>Antagonistic case</u>. Our dynamic problem will be represented as the zero-sum multistage game. The game is with complete information, thus the optimal (maximin) solutions are obtained by means of pure strategies [1]. The scheme of finding of optimal strategies is based on the principle of dynamic programming [2].

If the profit function of one player (producer, vendor) is $H(X,Y)$, then the profit function of another player will be $-H(X,Y)$. First player chooses $x_1, x_2, \dots, x_N$ quantities, another - $y_1, y_2, \dots, y_N$ quantities. Two vectors $(x_1, x_2, \dots, x_N)$ and $(y_1, y_2, \dots, y_N)$ are respectively the pure strategies of the players.

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Secured profits of the first and the second players are respectively $\max\limits_{(x_1,x_2,\ldots,x_N)}\min\limits_{(y_1,y_2,\ldots,y_N)} H(X,Y)$ and $\max\limits_{(x_1,x_2,\ldots,x_N)}\min\limits_{(y_1,y_2,\ldots,y_N)}\left(-H(X,Y)\right) = -\min\limits_{(y_1,y_2,\ldots,y_N)}\max\limits_{(x_1,x_2,\ldots,x_N)} H(X,Y)$. These two quantities exist and their sum equals zero. Hence $\max\limits_{(x_1,x_2,\ldots,x_N)}\min\limits_{(y_1,y_2,\ldots,y_N)} H(X,Y) = \min\limits_{(y_1,y_2,\ldots,y_N)}\max\limits_{(x_1,x_2,\ldots,x_N)} H(X,Y)$. To find this quantity and corresponding optimal strategy we apply the optimality principle extended by Bellman for the dynamic problem of double extremum. If we introduce the function $H_n(X,Y)$, the profit of the first player in the n-stage process, when both players play optimally and in the beginning of the process they posses the resources respectively equal to X and Y, then we can write the following recurrent equations:

$$\begin{aligned} &H_n(X,Y) = \max_{x_{N-n+1}\le X}\min_{y_{N-n+1}\le Y}\left(h_{N-n+1}\left(x_{N-n+1},y_{N-n+1}\right) + H_{N-n+1}\left(X - x_{N-n+1}, Y - y_{N-n+1}\right)\right),\ n = 2,\ldots,N, \\ &H_1(X,Y) = \max_{x_N\le X}\min_{y_N\le Y} h_N\left(x_N,y_N\right) \end{aligned} \tag{1}$$

Analogous equations can be also written for the case of “minimax”. Thus we obtain optimal, or equilibrium solution: $\overline{X} = \left(\overline{x_1},\overline{x_2},\ldots,\overline{x_N}\right)$, $\overline{Y} = \left(\overline{y_1},\overline{y_2},\ldots,\overline{y_N}\right)$ and the corresponding value of game - $\overline{H_N}$.

Obviously these solutions depend on the functions h_t. Let’s consider some hypothetic variants of these functions. Let r_t be the demand on goods on the stage t, k_t be the penalty for the unit of deficiency and let l_t be the waists for the unit of excess of goods. Then h_t can be represented as

$$h_t\left(x_t,y_t\right) = \begin{cases} l_t\cdot\left(r_t - x_t - y_t\right)\cdot sign\left(x_t - y_t\right), & x_t + y_t \ge r_t\ , \\ k_t\cdot\left(r_t - x_t - y_t\right)\cdot sign\left(x_t - y_t\right), & x_t + y_t \le r_t\ . \end{cases}$$

The content of this expression is the following: appearance of excessive goods at the market is the fault of the importer and hence the loss should be covered by the importer. On the contrary in the case of deficiency the importer of insufficient goods sustains a loss. This is a very simple variant and, in the most of cases, the solution is trivial (depending on the variability of parameters l and k on the stages).

If we take into account the revenues of the players we get the case, where the profit can be expressed by means of their differences

$$h_t\left(x_t,y_t\right) = \begin{cases} \dfrac{r_t}{x_t + y_t}\cdot\left(x_t - y_t\right)\cdot p_t + l_t\cdot\left(r_t - x_t - y_t\right)\cdot sign\left(x_t - y_t\right), & x_t + y_t \ge r_t\ , \\ \left(x_t - y_t\right)\cdot p_t + k_t\cdot\left(r_t - x_t - y_t\right)\cdot sign\left(x_t - y_t\right), & x_t + y_t < r_t\ , \end{cases}$$

where p_t is the price of unit good.

If demand is random and its density of distribution is $\varphi_t\left(r_t\right)$, then (1) gets the form

$$\begin{aligned} &H_n(X,Y) = \max_{x_{N-n+1}\le X}\min_{y_{N-n+1}\le Y}\left(\int_0^\infty h_{N-n+1}\left(x_{N-n+1},y_{N-n+1}\right)\varphi_{N-n+1}(r)\,dr + H_{n-1}\left(X - x_{N-n+1}, Y - y_{N-n+1}\right)\right), \\ &\qquad n = 2,\ldots,N, \\ &H_1(X,Y) = \max_{x_N\le X}\min_{y_N\le Y}\int_0^\infty h_N\left(x_N,y_N\right)\varphi_N(r)\,dr\ . \end{aligned} \tag{2}$$

2. <u>Cooperative case.</u> In this case the players can cooperate and agree their activities. The execution of agreements is obligatory and they do not alter the estimation of the outcome of the game. It should be noted, that in cooperative games the principle of equilibrium is not justified any more. The players should receive maximum of the profit and the latter should be shared in accordance of “common sense” and the principle of “fairness”. These principles should be

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implemented objectively and fairly by the arbiter - the third party. That's why such an approach is called the arbitrage scheme. Below we will consider two approaches of this kind - the arbitrage scheme of Nash and the Shapley vector [3]. Both of them are based on the principles of fairness - axioms and imply unique (generally different) solutions [3]. Similarly, as in the above model, the players of duopoly market posses X and Y quantities of homogenous goods and at every stage t they deliver to the market x_t and y_t quantities of goods. The last influences the prices. Assume that this relation is linear and, as is commonly accepted, can be expressed by the formula $p_t = a - b\cdot\left(x_t + y_t\right)$, where $a >> b > 0$. If the costs of the production are linear, then the costs for the players are respectively $c_1 x_t + d_1$ and $c_2 x_t + d_2$, where c_i are the limit costs while d_i are fixed costs (overheads). We should also, as above, take into account the wastes in the case of deficiency. Naturally the case $x_t + y_t > r_t$ is excluded in the case of fixed r_t. Consequently on the stage t each of the players will receive the profit

$$h_t^{'} = \left(a - b\cdot\left(x_t + y_t\right)\right)\cdot x_t - c_1 x_t - d_1 - k\cdot\frac{r_t - x_t - y_t}{x_t + y_t}\cdot y_t \ ,$$

$$h_t^{"} = \left(a - b\cdot\left(x_t + y_t\right)\right)\cdot y_t - c_2 y_t - d_2 - k\cdot\frac{r_t - x_t - y_t}{x_t + y_t}\cdot x_t \ .$$

Here by means of last summands the penalty for deficit is distributed anti proportionally to the goods delivered to the market.

In the N stage process the summary profits equal respectively $H^{'} = \sum_{t=1}^{N} h_t^{'}\cdot\beta^{t-1}$, $H^{"} = \sum_{t=1}^{N} h_t^{"}\cdot\beta^{t-1}$, where β is a discount coefficient.

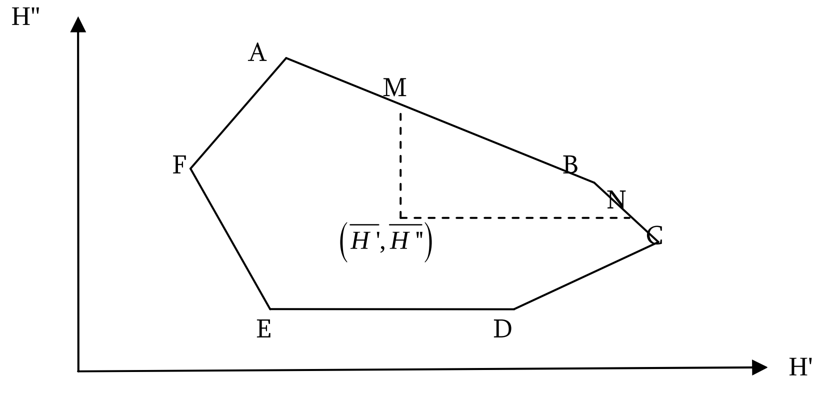
Guaranteed (maximin) profits for each player $\overline{H_N^{'}}\left(X,Y\right)$ and $\overline{H_N^{"}}\left(X,Y\right)$ are elaborated by the following recurrent equations

$$\begin{aligned}
\overline{H_n^{'}}\left(X,Y\right) &= \max_{x_{N-n+1}\le X}\ \min_{y_{N-n+1}\le Y}\left(h_{N-n+1}^{'}\left(x_{N-n+1}, y_{N-n+1}\right) + \overline{H_{n-1}^{'}}\left(X - x_{N-n+1}, Y - y_{N-n+1}\right)\right),\\
\overline{H_n^{"}}\left(X,Y\right) &= \max_{y_{N-n+1}\le Y}\ \min_{x_{N-n+1}\le X}\left(h_{N-n+1}^{"}\left(x_{N-n+1}, y_{N-n+1}\right) + \overline{H_{n-1}^{"}}\left(X - x_{N-n+1}, Y - y_{N-n+1}\right)\right),
\end{aligned} \tag{3}$$

$n = 1,\ldots,N$.

The quantities $\overline{H_N^{'}}$ and $\overline{H_N^{"}}$ are often called the threat-wages of the players, while the pair $\left(\overline{H_N^{'}}, \overline{H_N^{"}}\right)$ is called the threat point, or the status-quo point.

Assume, that the set of admissible pairs $\left(H^{'}, H^{"}\right)$ is a closed polyhedron ABCDEF on the plain



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Let's conduct our argument as it is accepted in the static model: the first player tries to increase $H^{'}$, equivalently - the pair $\left(H^{'}, H^{''}\right)$ should be as far on the right-hand as possible. Another player prefers this point to be situated as higher as possible. Consider the points of the polygon ABC such, that there are no admissible points nor above, neither on the right-hand side to them. The set of such points is called the Pareto solution set. As the players have guaranteed profits $\overline{H^{'}}$ and $\overline{H^{''}}$ respectively, the Pareto set can be reduced to the set of points of the polygon MN. This set is called as Neumann-Morgenstern (shortly N-M) solution. Obviously the players should choose the point from this set. There are various approaches in this regard. As we said above we will consider Nash and Shapley arbitrage solutions. In fact the arbitrage decision or arbitrage scheme is a function (rule), where to the player of any antagonistic conflict (game) it corresponds one and only one wage. This solution is regarded as fair, compromise solution.

The Nash solution is a collection of values of variables $x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N$, for which the function $F = \left(H^{'} - \overline{H^{'}}\right)\left(H^{''} - \overline{H^{''}}\right)$ attains its maximum. This problem is the problem of mathematical programming. In practice it's more expedient to deal with discreet data and to find maximum by the method of Dynamic Programming.

Let's now consider the arbitrage approach, which belongs to Shapley. In this case the players jointly attempt to get maximum of total profit $v(I) = max\sum\left(H^{'} + H^{''}\right)$ and, according to the principle of "fairness", share the profit. The principles are stated in the form of axioms, which imply the existence of unique sharing. Actually the Shapley sharing (vector) expresses the strength of each player. Each component is the average value of additional profits $\left(v(S \cup i) - v(S)\right)$, which can be brought to any coalition S by the corresponding player. Generally in the case of n players the components of Shapley function have the form:

$$\varphi_i(v) = \sum_{S \subset I} \frac{(s-1)!(n-s)!}{n!}\left(v(S) - v\left(S \setminus i\right)\right), \quad i = 1, \ldots, n, \tag{4}$$

where I is the set of players (two elements set in our case), s is the number of elements of the subset $S \subset I$. Here $v\left(S\right)$ is the characteristic function of the game defined on the set of all subsets S of I, representing the value of antagonistic game when one player is the coalition S, while another is the coalition $I \setminus S$.

Generally the problem of calculation of Shapley vector is difficult. In our case this difficulty is overcome. As we mentioned above, we should differ two cases with respect to the consumers. When we deal with the case of large number of minor consumers they are regarded as one, and also as a fictitious participant, which doesn't influence the buy-sell process. Another case considers one essential player – the dynamic participant of the game.

$2^{'}$. When the buyer is a fictitious player we have the cooperative game. We consider two approaches - antagonistic and cooperative. In the latter case we will basically deal with the Shapley solutions. If the price of the unit of goods is constant and equals γ, then the characteristic function for the united coalition and any X, Y, R has the form $v\left(1,2\right) = \gamma \cdot \min\left(X + Y; R\right)$, while for each player separately $v\left(1\right) = \gamma \cdot \min\left(X; \max(0, R - Y)\right)$, $v\left(2\right) = \gamma \cdot \min\left(Y; \max(0, R - X)\right)$.

Let us consider the following cases:

a) If $X \geq R, Y \geq R$, then $v\left(1,2\right) = \gamma \cdot R$, $v(1) = v(2) = 0$, $\varphi_1(v) = \varphi_2(v) = \gamma \cdot \dfrac{R}{2}$,

i.e. when each of the players can cover the demand, then the profit is shared evenly.

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b) If $X < R, Y < R$, $X + Y \ge R$, then $v(1,2) = \gamma \cdot R$,
$v(1) = \gamma \cdot \min(X, R - Y)$, $v(2) = \gamma \cdot \min(Y, R - X)$.

In the case of this kind of characteristic function the components of the Shapley vector has the form

$$\varphi_1(v) = \frac{1}{2}\gamma \cdot (R + \min(X, R - Y) - \min(Y, R - X)),$$

$$\varphi_2(v) = \frac{1}{2}\gamma \cdot (R + \min(Y, R - X) - \min(X, R - Y)).$$

c) If $R > X + Y$, then $\varphi_1(v) = \gamma \cdot X$, $\varphi_2(v) = \gamma \cdot Y$.

The result is trivial – in the case of deficiency the goods are completely consumed.

$2''$. When the consumer is the so called essential player the agreement is achieved only in the case, when the buyer enters the coalition. Thus we deal with the coalition game of three persons and $v(S) = 0$, if $3 \notin S$.

We obtain:

$$v(I) = v(1,2,3) = \gamma \cdot \min(X + Y; R),\ v(1) = v(2) = v(3) = v(1,2) = 0,$$

$$v(1,3) = \gamma \cdot \min(X; R),\ v(2,3) = \min(Y; R).$$

Let us consider the following cases:

a) If $X \ge R$, $Y \ge R$, then $\varphi_1(v) = \varphi_2(v) = \gamma \cdot \frac{R}{6}$, $\varphi_3(v) = \gamma \cdot \frac{4}{6} R$.

Thus if the demand doesn't exceed the offer of a producer, then the two-third of the total profit $\gamma \cdot R$ goes to the buyer and the rest one-third is evenly shared between the producers.

b) If $R \ge X + Y$, then $v(1,3) = \gamma \cdot X$, $v(2,3) = \gamma \cdot Y$, $v(1,2,3) = \gamma \cdot (X + Y)$,

$$\varphi_1(v) = \gamma \cdot \frac{X}{2},\quad \varphi_2(v) = \gamma \cdot \frac{Y}{2},\quad \varphi_3(v) = \gamma \cdot \frac{X + Y}{2}.$$

The goods are totally consumed and each vendor gets the half of the profit due to their own goods respectively. The other half goes to the buyer.

c) If $X < R, Y < R$, $X + Y \ge R$, then $v(1,3) = \gamma \cdot X$, $v(2,3) = \gamma \cdot Y$, $v(1,2,3) = \gamma \cdot R$,

$$\varphi_1(v) = \gamma \cdot \left(\frac{X}{2} - \frac{X + Y - R}{3}\right),\ \varphi_2(v) = \gamma \cdot \left(\frac{Y}{2} - \frac{X + Y - R}{3}\right),\ \varphi_3(v) = \gamma \cdot \left(\frac{X + Y}{2} - \frac{X + Y - R}{3}\right).$$

In comparison with the previous case the players have equal losses because of the superfluous offer.

It is worth to note, that if at the stages the demands are assumed to be different, but fixed, then it wouldn't be difficult to derive the components of the Shapley vector. It seems more interesting to consider the cases, when the demands are random, or they depend on the volume of goods at the market.

References

1. Dresher M. Games of Strategy Theory and Applications. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961.
2. Bellman R. Dynamic Programming. Princeton, New Jersey, Princeton University Press, 1957.
3. Luce R., Raiffa H., Games and Decisions. New York: John Wiley and Sons, 1957.

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