A STUDY ON INTERVAL-VALUED FUZZY GRAPHS

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Abstract

Fuzzy graphs which have revolutionized the analysis of problematic data to arrive at a better decision making power are of different kinds. Among them, the simplest and generalized form is the interval-valued fuzzy graph. The main purpose of this paper is to introduce the notion of an interval-valued p -morphism on interval-valued fuzzy graphs. The action of interval-valued p -morphism on interval-valued strong regular graphs are studied and proved elegant theorems on weak and co weak isomorphism. Also μ -complement of highly irregular product interval-valued fuzzy graphs, α -cut and strength cut graph of an interval-valued fuzzy graphs are discussed.

Keywords: Interval-valued p -morphism, μ -complement of a product interval-valued fuzzy graph, α -cut and strength cut graph of an interval-valued fuzzy graph.

1. Introduction

The moment science is involved in finding solutions to theoretical problems mathematical dependency increases. It has been proved by researchers that frameworks of analysis developed using mathematical models, especially those based on fuzzy logic were capable of handling uncertain data sets. The new mathematical models have shown their superiority over the conventional fuzzy logic based sets. Issues of doubt, data inconsistency and wrong or mismatched data were effectively analyzed with the help of graph theory in the various domains like mechanical engineering, electrical engineering, with special focus on machine systems. The self-learning feature of these new mathematical models has given the possibility of scaling the size of the operations to suit the requirement industrial requirement. In short, mathematical models based on fuzzy graph theory have simplified the handling of problematic to arrive at rational conclusions.


2. Preliminaries

Some definitions and conventions used in this paper are discussed in this section. Literature review is available in [1, 4, 5, 17].

Definition 2.1 A graph \( G=(V,E) \) is an ordered pair consisting of a non-empty vertex set \( V \), an edge set \( E \) and a connection that associates with every edge between two vertices (not as a matter of course particular) called its end points.
Definition 2.2 Let $G = (V, E)$ be a graph. Then $S = (N, L)$ is said to be a subgraph of $G$ if $N \subseteq V$ and $L \subseteq E$.

Definition 2.3 A fuzzy set $A$ on a universal set $X$ is characterized by function $m : X \rightarrow [0, 1]$, which is called the membership function. A fuzzy set is denoted by $A = (X, m)$.

Definition 2.4 A fuzzy graph $\delta = (V, \sigma, \mu)$ is a non-empty set $V$ together with a pair of functions $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$ such that for all $m, n \in V$, $\mu(mn) \leq \min \{\sigma(m), \sigma(n)\}$, where $\sigma(m)$ and $\mu(mn)$ represent the membership values of the vertex $m$ and of the edge $mn$ in $\delta$ respectively. The underlying crisp graph of the fuzzy graph $\delta = (V, \sigma, \mu)$ is denoted as $\delta^* = (V, \sigma^*, \mu^*)$, where $\sigma^* = \{x \in V / \sigma(x) > 0\}$ and $\mu^* = \{xy \in V \times V / \mu(xy) > 0\}$. Thus for underlying fuzzy graph $\sigma^* = V$.

Definition 2.5 A fuzzy graph $\delta = (V, \sigma, \mu)$ is complete if $\mu(xy) = \min \{\sigma(x), \sigma(y)\}$ for all $x, y \in V$, where $xy$ denotes the edge between the vertices $x$ and $y$. The fuzzy graph $\delta_a = (V, \sigma_a, \mu_a)$ is called a fuzzy subgraph of $\delta = (V, \sigma, \mu)$ if $\sigma_a(m) \leq \sigma(m)$ for all $m \in V$ and $\mu_a(mn) \leq \mu(mn)$ for all edges $mn \in E$.

Definition 2.6 The interval-valued fuzzy set $W$ in $V$ is defined by $W = \left\{ x \left[ \left[ \mu_w^-, (x), \mu_w^+ (x) \right] / x \in V \right] \right\}$, where $\mu_w^-(x), \mu_w^+(x)$ are fuzzy subsets of $V$ such that $\mu_w^-(x) \leq \mu_w^+(x)$ for all $x \in V$. If $G = (V, E)$ is a crisp graph, then by an interval-valued fuzzy relation $F$ on a set $E$ we mean an interval-valued fuzzy set such that $\mu_F^-(xy) \leq \min \left\{ \mu_w^-(x), \mu_w^+(y) \right\}$, $\mu_F^+(xy) \leq \min \left\{ \mu_w^+(x), \mu_w^+(y) \right\}$, for all $x, y \in V$.

Definition 2.7 The interval-valued fuzzy graph is a pair $G = (W, F)$ of a graph $G^* = (V, E)$, where $W = \left[ \mu_w^-, \mu_w^+ \right]$ is an interval-valued fuzzy set on $V$ and $F = \left[ \mu_F^-, \mu_F^+ \right]$ is an interval-valued fuzzy relation on $E$ such that $\mu_F^-(xy) \leq \min \left\{ \mu_w^-(x), \mu_w^+(y) \right\}$, $\mu_F^+(xy) \leq \min \left\{ \mu_w^+(x), \mu_w^+(y) \right\}$ for all $xy \in E$.

The underlying crisp graph of an interval-valued graph $G = (W, F)$ is the graph $G^* = (V, E)$ where $V = \left\{ \{w / \left[ \left[ \mu_w^-(w), \mu_w^+(w) \right] \right] \} / \{0, 0 \} \right\}$ is called vertex set and $E = \left\{ \{wf / \left[ \left[ \mu_F^-(wf), \mu_F^+(wf) \right] \right] \} / \{0, 0 \} \right\}$ is called an edge set.

Definition 2.8 Let $G = (W, F)$ be an interval-valued fuzzy graph of $G^* = (V, E)$. Then $G = (W, F)$ is said to be strong if $\mu_F^-(xy) = \min \left\{ \mu_w^-(x), \mu_w^+(y) \right\}$, $\mu_F^+(xy) = \min \left\{ \mu_w^+(x), \mu_w^+(y) \right\}$, for all $xy \in E$.

Definition 2.9 Let $G = (W, F)$ be an interval-valued fuzzy graph of $G^* = (V, E)$. Then $G = (W, F)$ is said to be complete if $\mu_F^-(xy) = \min \left\{ \mu_w^-(x), \mu_w^+(y) \right\}$, $\mu_F^+(xy) = \min \left\{ \mu_w^+(x), \mu_w^+(y) \right\}$, for all $x, y \in V$.

Definition 2.10 The complement of an interval-valued fuzzy graph $G = (W, F)$ of a graph $G^* = (V, E)$ is an interval-valued fuzzy graph $\overline{G} = \left( \overline{W}, F \right)$, where $\overline{W} = W = \left[ \mu_w^-, \mu_w^+ \right]$ and $F = \left[ \mu_F^-, \mu_F^+ \right]$, here
\[\mu_w (xy) = \min \{\mu_w (x), \mu_w (y)\} - \mu_w (xy), \quad \mu_w^+ (xy) = \min \{\mu_w^+ (x), \mu_w^+ (y)\} - \mu_w^+ (xy) \quad \text{for all} \quad x, y \in V.\]

**Definition 2.11** Let \( G_1 = (W_1, F_1) \) and \( G_2 = (W_2, F_2) \) be two interval-valued fuzzy graphs on \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \) respectively, such that a homomorphism \( p \) from \( G_1 \) to \( G_2 \) is a mapping \( p : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\[\mu_0^- (u) \leq \mu_0^- (p(u)), \quad \mu_0^+ (u) \leq \mu_0^+ (p(u)), \quad \forall u \in V_1,\]

\[\mu_0^- (uv) \leq \mu_0^- (p(u) p(v)), \quad \mu_0^+ (uv) \leq \mu_0^+ (p(u) p(v)), \quad \forall uv \in E_1.\]

An isomorphism \( p \) from \( G_1 \) to \( G_2 \) is a bijective mapping \( p : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\[\mu_0^- (u) = \mu_0^- (p(u)), \quad \mu_0^+ (u) = \mu_0^+ (p(u)), \quad \forall u \in V_1,\]

\[\mu_0^- (uv) = \mu_0^- (p(u) p(v)), \quad \mu_0^+ (uv) = \mu_0^+ (p(u) p(v)), \quad \forall uv \in E_1.\]

A weak isomorphism \( p \) from \( G_1 \) to \( G_2 \) is a bijective mapping \( p : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\[p \text{ is homomorphism}, \quad \mu_0^- (u) = \mu_0^- (p(u)), \quad \mu_0^+ (u) = \mu_0^+ (p(u)), \quad \forall u \in V_1.\]

A co weak isomorphism \( p \) from \( G_1 \) to \( G_2 \) is a bijective mapping \( p : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\[p \text{ is homomorphism}, \quad \mu_0^- (uv) = \mu_0^- (p(u) p(v)), \quad \mu_0^+ (uv) = \mu_0^+ (p(u) p(v)), \quad \forall uv \in E_1.\]

**Definition 2.12** Let \( G = (W, F) \) be an interval-valued fuzzy graph, where \( W = \left[\mu_w^-, \mu_w^+\right] \) and \( F = \left[\mu_F, \mu_F^+\right] \). The degree of a vertex is defined as \( d_v = (d_G (w), d^*_G (w)) \) where, \( d_G (w) = \sum \mu_F (wx), \quad wx \in E, \quad w \neq x \) is the negative degree of a vertex \( w \) and \( d^*_G (w) = \sum \mu_F^+ (wx), \quad wx \in E, \quad w \neq x \) is the positive degree of a vertex \( w \).

**Definition 2.13** The order of \( G \) is defined as \( O(G) = \left( \sum_{w \in F} \mu_w^-, \sum_{w \in F} \mu_w^+ \right) \).

**Definition 2.14** The size of \( G \) is defined as \( S(G) = (S_G, S^*_G) = \left( \sum_{xy} \mu_F (xy), \sum_{xy} \mu_F^+ (xy) \right) \).

3. Regularity on isomorphic interval-valued fuzzy graph

In this section, we introduced an interval-valued \( p \)-morphism on interval-valued fuzzy graphs and regular interval-valued fuzzy graph, we proved some important theorems on weak and co weak isomorphism.

**Definition 3.1** Let \( G_1 = (W_1, F_1) \) and \( G_2 = (W_2, F_2) \) be two interval-valued fuzzy graphs on \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \) respectively. A bijective function \( P : V_1 \rightarrow V_2 \) is called interval-valued morphism or interval-valued \( p \)-morphism if there exists two numbers \( l_1 > 0 \) and \( l_2 > 0 \), such that \( \mu_0^- (p(u)) = l_1 \mu_0^- (u), \mu_0^+ (p(u)) = l_2 \mu_0^+ (u), \quad \forall u \in V_1, \)
\( \mu_{F_i}^+(p(u)p(v)) = l_2\mu_{F_i}^-(uv), \mu_{F_i}^-(p(u)p(v)) = l_2\mu_{F_i}^+(uv), \forall uv \in E_i. \) In such a case, \( p \) will be called a \((l_1,l_2)\) interval-valued \( p \)-morphism from \( G_1 \) to \( G_2 \). If \( l_1 = l_2 = l \), we call \( p \), an interval-valued \( p \)-morphism.

**Example 3.1** Consider an interval-valued fuzzy graph \( G_1 = (W_1,F_1) \) shown in Figure 1(a). Then

\[
W_1 = \left\{ \frac{[0.2,0.3]}{J}, \frac{[0.3,0.3]}{K}, \frac{[0.2,0.3]}{L} \right\} \quad \text{and} \quad F_1 = \left\{ \frac{[0.2,0.3]}{JK}, \frac{[0.2,0.25]}{KL}, \frac{[0.1,0.2]}{LJ} \right\}.
\]

Again consider another interval-valued fuzzy graph \( G_2 = (W_2,F_2) \) shown in Figure 1(b). Then

\[
W_2 = \left\{ \frac{[0.6,0.9]}{J'}, \frac{[0.9,0.9]}{K'}, \frac{[0.6,0.9]}{L'} \right\} \quad \text{and} \quad F_2 = \left\{ \frac{[0.4,0.6]}{J'K'}, \frac{[0.4,0.5]}{K'L'}, \frac{[0.2,0.4]}{L'J'} \right\}.
\]

![Figure 1: \( p \)-morphism between interval-valued fuzzy graphs \( G_1 \) and \( G_2 \)](image)

Here, there is an interval-valued \( p \)-morphism such that \( p(J) = J', p(K) = K', p(L) = L', l_1 = 3, l_2 = 2 \).

**Definition 3.2** Let \( G = (W,F) \) be a connected interval-valued fuzzy graph. Then \( G \) is said to be a highly irregular interval-valued fuzzy graph if every vertex of \( G \) is adjacent to vertices with distinct degrees.

**Example 3.2** Consider an interval-valued fuzzy graph \( G = (W,F) \) shown in Figure 2.

\[
W = \left\{ \frac{[0.3,0.4]}{J}, \frac{[0.2,0.5]}{K}, \frac{[0.5,0.5]}{L}, \frac{[0.4,0.7]}{M} \right\} \quad \text{and} \quad F = \left\{ \frac{[0.1,0.3]}{JK}, \frac{[0.1,0.2]}{KL}, \frac{[0.3,0.4]}{LM}, \frac{[0.2,0.3]}{MJ} \right\}.
\]

![Figure 2: A highly irregular interval-valued fuzzy graph](image)
Then \( d_G(J) = (0.3, 0.6), d_G(K) = (0.2, 0.5), d_G(L) = (0.4, 0.6), d_G(M) = (0.5, 0.7) \).

We see that every vertex of \( G \) is adjacent to vertices with distinct degrees. Hence \( G = (W,F) \) is highly irregular interval-valued fuzzy graph.

**Theorem 3.1** The relation \( p \)-morphism is an equivalence relation in the collection of interval-valued fuzzy graphs.

**Proof** Consider the collection of all interval-valued fuzzy graphs. Define the relation \( G_1 \cong G_2 \) if there exists a \((l_1, l_2)\) \( p \)-morphism from \( G_1 \) to \( G_2 \) where both \( l_1 \neq 0 \) and \( l_2 \neq 0 \). Consider the identity morphism \( G_1 \) to \( G_1 \). It is a \((1,1)\) morphism from \( G_1 \) to \( G_1 \) and hence \( \cong \) is reflexive.

Let \( G_1 \cong G_2 \). Then there exists a \((l_1, l_2)\)-morphism from \( G_1 \) to \( G_2 \) for some \( l_1 \neq 0 \) and \( l_2 \neq 0 \). Therefore \( \mu_{l_1}^{-}(p(u)) = l_1 \mu_{l_1}^{-}(u), \mu_{l_1}^{+}(p(u)) = l_1 \mu_{l_1}^{+}(u), \forall u \in V_1 \) and \( \mu_{l_2}^{-}(p(u)p(v)) = l_2 \mu_{l_2}^{-}(uv) \), \( \mu_{l_2}^{+}(p(u)p(v)) = l_2 \mu_{l_2}^{+}(uv), \forall uv \in E_1 \).

Consider \( p^{-1}:G_2 \rightarrow G_1 \). Let \( m, n \in V_2 \). Since \( p^{-1} \) is bijective, we have \( m = p(u), n = p(v) \), for some \( u, v \in V_2 \).

Now \( \mu_{l_1}^{-}(p^{-1}(m)) = \mu_{l_1}^{-}(p^{-1}(p(u))) = \mu_{l_1}^{-}(u) = \frac{1}{l_1} \mu_{l_2}^{-}(p(u)) = \frac{1}{l_1} \mu_{l_2}^{-}(m) \)

\( \mu_{l_1}^{+}(p^{-1}(m)) = \mu_{l_1}^{+}(p^{-1}(p(u))) = \mu_{l_1}^{+}(u) = \frac{1}{l_1} \mu_{l_2}^{+}(p(u)) = \frac{1}{l_1} \mu_{l_2}^{+}(m) \).

\( \mu_{l_2}^{-}(p^{-1}(m)p^{-1}(n)) = \mu_{l_2}^{-}(p^{-1}(p(u))p^{-1}(p(v))) = \mu_{l_2}^{-}(uv) = \frac{1}{l_2} \mu_{l_2}^{-}(p(u)p(v)) = \frac{1}{l_2} \mu_{l_2}^{-}(mm) \)

\( \mu_{l_2}^{+}(p^{-1}(m)p^{-1}(n)) = \mu_{l_2}^{+}(p^{-1}(p(u))p^{-1}(p(v))) = \mu_{l_2}^{+}(uv) = \frac{1}{l_2} \mu_{l_2}^{+}(p(u)p(v)) = \frac{1}{l_2} \mu_{l_2}^{+}(mm) \).

Thus, there exists \( \left( \frac{1}{l_1}, \frac{1}{l_2} \right) \)-morphism from \( G_2 \) to \( G_1 \). Therefore, \( G_2 \cong G_1 \) and hence \( \cong \) is symmetric.

Let \( G_1 \cong G_2 \) and \( G_2 \cong G_3 \). Then, there exists a \((l_1, l_2)\)-morphism from \( G_1 \) to \( G_2 \) say \( p \) for some \( l_1 \neq 0, l_2 \neq 0 \) and there exists \((l_3, l_4)\)-morphism from \( G_2 \) to \( G_3 \) say \( q \) for some \( l_3 \neq 0 \) and \( l_4 \neq 0 \). So \( \mu_{l_1}^{-}(q(x)) = l_3 \mu_{l_2}^{-}(x), \mu_{l_1}^{+}(q(x)) = l_3 \mu_{l_2}^{+}(x), \forall x \in V_2 \), and \( \mu_{l_2}^{-}(q(x)q(y)) = l_4 \mu_{l_3}^{-}(xy), \mu_{l_2}^{+}(q(x)q(y)) = l_4 \mu_{l_3}^{+}(xy), \forall xy \in E_2 \).

Let \( r: q \circ p: G_1 \rightarrow G_3 \) be a mapping.

Now, \( \mu_{l_1}^{-}(r(u)) = \mu_{l_1}^{-}(q \circ p)(u)) = \mu_{l_1}^{-}(q(p(u))) = l_1 \mu_{l_2}^{-}(p(u)) = l_1 \mu_{l_2}^{-}(u) \),

\( \mu_{l_1}^{+}(r(u)) = \mu_{l_1}^{+}(q \circ p)(u)) = \mu_{l_1}^{+}(q(p(u))) = l_1 \mu_{l_2}^{+}(p(u)) = l_1 \mu_{l_2}^{+}(u) \).

\( \mu_{l_2}^{-}(r(u)r(v)) = \mu_{l_2}^{-}(q \circ p)(u)(q \circ p)(v)) = \mu_{l_2}^{-}(q(p(u)))q(p(v)) \)

\( \mu_{l_2}^{+}(r(u)r(v)) = \mu_{l_2}^{+}(q \circ p)(u)(q \circ p)(v)) = \mu_{l_2}^{+}(q(p(u)))q(p(v)) \)
\[ l_1 \mu^-_{r_1} (xy) = \mu^-_{r_2} (p(x) p(y)) = \mu^-_{r_2} (p(x) \land p(y)) = l_1 \mu^-_{r_1} (x) \land l_1 \mu^-_{r_1} (y) = l_1 \mu^-_{r_1} (xy) \]

Hence, \( l_1 \mu^-_{r_1} (xy) = l_1 \mu^-_{r_1} (xy), \ \forall xy \in E_1 \).

\[ l_2 \mu^-_{r_1} (xy) = l_2 \mu^-_{r_1} (xy) \]

The equation holds if and only if \( l_1 = l_2 \).

**Theorem 3.3** If the interval-valued fuzzy graph \( G_1 \) is co weak isomorphic to \( G_2 \) and if \( G_1 \) is regular then \( G_2 \) is also regular.

**Proof** As an interval-valued fuzzy graph \( G_1 \) is co weak isomorphic to \( G_2 \). Then there exists a co-weak isomorphism \( p: G_1 \to G_2 \) which is bijective that satisfies

\[ \mu^-_{r_1} (u) \leq \mu^-_{r_2} (p(u)), \mu^-_{r_1} (u) \leq \mu^-_{r_2} (p(u)), \ \forall u \in V_1, \]

\[ \mu^-_{r_1} (uv) = \mu^-_{r_1} (p(u) p(v)), \mu^-_{r_2} (uv) = \mu^-_{r_2} (p(u) p(v)), \ \forall uv \in E_1. \]

As \( G_1 \) is regular, for \( u \in V_1, \sum_{u \neq y, y \in V_1} \mu^-_{r_1} (uv) \) is constant and \( \sum_{u \neq y, y \in V_1} \mu^-_{r_2} (uv) \) is constant.
Now \( \sum_{\{u:v(1)\}} \mu_{F_i}^-(p(u)p(v)) = \sum_{u:v(1)} \mu_{F_i}^-(uv) = \text{constant and} \) \( \sum_{\{u:v(1)\}} \mu_{F_i}^+(p(u)p(v)) = \sum_{u:v(1)} \mu_{F_i}^+(uv) = \text{constant. Therefore } G_2 \text{ is regular.} \)

**Theorem 3.4** Let \( G_i \) and \( G_2 \) be two interval-valued fuzzy graphs. If \( G_i \) is weak isomorphic to \( G_2 \) and if \( G_i \) is strong then \( G_2 \) is also strong.

**Proof** As \( G_i \) is an interval-valued fuzzy graph is weak isomorphic with \( G_2 \). Then there exists a weak isomorphism \( p : G_i \rightarrow G_2 \) which is bijective that satisfies

\[
\mu_{F_1}^- (u) = \mu_{F_2}^- (p(u)), \mu_{F_1}^+ (u) = \mu_{F_2}^+ (p(u)), \forall u \in V_i \text{ and} \\
\mu_{F_i}^- (uv) \leq \mu_{F_2}^- (p(u)p(v)), \mu_{F_i}^+ (uv) \leq \mu_{F_2}^+ (p(u)p(v)), \forall uv \in E_i. \quad \text{As } G_i \text{ is strong, we have } \mu_{F_i}^- (uv) = \min \left( \mu_{F_1}^- (u), \mu_{F_1}^- (v) \right) \text{ and } \mu_{F_i}^+ (uv) = \min \left( \mu_{F_1}^+ (u), \mu_{F_1}^+ (v) \right). \quad \text{Now, we get} \\
\mu_{F_2}^- (p(u)p(v)) \geq \mu_{F_2}^- (uv) = \min \left( \mu_{F_2}^- (u), \mu_{F_2}^- (v) \right) = \min \left( \mu_{F_1}^- (p(u)), \mu_{F_1}^- (p(v)) \right). \\
\text{By the definition, } \mu_{F_2}^- (p(u)p(v)) \leq \min \left( \mu_{F_2}^- (p(u)), \mu_{F_2}^- (p(v)) \right). \\
\text{Similarly, } \mu_{F_2}^+ (p(u)p(v)) \geq \mu_{F_2}^+ (uv) = \min \left( \mu_{F_2}^+ (u), \mu_{F_2}^+ (v) \right) = \min \left( \mu_{F_1}^+ (p(u)), \mu_{F_1}^+ (p(v)) \right) \\
\text{By the definition, } \mu_{F_2}^+ (p(u)p(v)) \leq \min \left( \mu_{F_2}^+ (p(u)), \mu_{F_2}^+ (p(v)) \right). \\
\text{Therefore } \mu_{F_2}^+ (p(u)p(v)) = \min \left( \mu_{F_2}^+ (p(u)), \mu_{F_2}^+ (p(v)) \right). \text{ So } G_2 \text{ is strong.} \)

**Theorem 3.5** If the interval-valued fuzzy graph \( G_i \) is co weak isomorphic with a strong regular interval-valued fuzzy graph \( G_2 \), then \( G_i \) is strong regular interval-valued fuzzy graph.

**Proof** As an interval-valued fuzzy graph \( G_i \) is co weak isomorphic to \( G_2 \), there exists a co-weak isomorphism \( p : G_i \rightarrow G_2 \) which is bijective that satisfies

\[
\mu_{F_1}^- (u) \leq \mu_{F_2}^- (p(u)), \mu_{F_1}^+ (u) \leq \mu_{F_2}^+ (p(u)), \forall u \in V_i \text{ and} \\
\mu_{F_i}^- (uv) = \mu_{F_2}^- (p(u)p(v)), \mu_{F_i}^+ (uv) = \mu_{F_2}^+ (p(u)p(v)), \forall uv \in E_i. \\
\text{Now, we get} \\
\mu_{F_2}^- (uv) = \mu_{F_2}^- (p(u)p(v)) \leq \min \left( \mu_{F_2}^- (p(u)), \mu_{F_2}^- (p(v)) \right) \geq \min \left( \mu_{F_1}^- (u), \mu_{F_1}^- (v) \right) \\
\mu_{F_2}^+ (uv) = \mu_{F_2}^+ (p(u)p(v)) \geq \min \left( \mu_{F_2}^+ (p(u)), \mu_{F_2}^+ (p(v)) \right) \geq \min \left( \mu_{F_1}^+ (u), \mu_{F_1}^+ (v) \right) \\
\text{But by the definition, } \mu_{F_2}^- (uv) \leq \min \left( \mu_{F_1}^- (u), \mu_{F_1}^- (v) \right), \\
\mu_{F_2}^+ (uv) \leq \min \left( \mu_{F_1}^+ (u), \mu_{F_1}^+ (v) \right). \)
Hence $\mu_{F_1}^-(uv) = \min(\mu_{v_1}^-(u), \mu_{v_2}^-(v))$ and $\mu_{F_1}^+(uv) = \min(\mu_{v_1}^+(u), \mu_{v_2}^+(v))$. Therefore $G_1$ is strong. 

Also for $u \in V_1$, $\sum_{u \in V_1} \mu_{F_1}^+(uv) = \sum_{u \in V_1} \mu_{F_1}^-(p(u)p(v)) = \text{constant}$ as $G_2$ is regular and $\mu_{F_2}^+(uv) = \mu_{F_2}^-(p(u)p(v))$ is constant as $G_2$ is regular. Therefore $G_1$ is regular.

**Theorem 3.6** Let $G_1$ and $G_2$ be two isomorphic interval-valued fuzzy graphs then $G_1$ is strong regular if and only if $G_2$ is strong regular.

**Proof** As an interval-valued fuzzy graph $G_1$ is isomorphic with an interval-valued fuzzy graph $G_2$, there exists an isomorphism $p : G_1 \rightarrow G_2$ which is bijective and satisfies $\mu_{v_1}^-(u) = \mu_{v_2}^-(p(u)), \mu_{v_1}^+(u) = \mu_{v_2}^+(p(u)), \forall u \in V_1$, $\mu_{F_1}^-(uv) = \mu_{F_2}^-(p(u)p(v)), \mu_{F_1}^+(uv) = \mu_{F_2}^-(p(u)p(v)), \forall uv \in E_1$. Now, $G_1$ is strong if and only if $\mu_{F_1}^+(uv) = \min(\mu_{v_1}^-(u), \mu_{v_2}^-(v))$ and $\mu_{F_1}^+(uv) = \min(\mu_{v_1}^+(u), \mu_{v_2}^+(v))$ if and only if $\mu_{F_2}^-(p(u)p(v)) = \min(\mu_{v_2}^-(p(u)), \mu_{v_2}^+(p(v))), \mu_{F_2}^-(p(u)p(v)) = \min(\mu_{v_2}^-(p(u)), \mu_{v_2}^+(p(v)))$ if and only if $G_2$ is strong. $G_1$ is regular if and only if for $u \in V_1$, $\sum_{u \in V_1} \mu_{F_1}^+(uv) = \text{constant}$ and $\sum_{p(u)p(v) \in E_1} \mu_{F_2}^+(p(u)p(v)) = \text{constant}$, for all $p(u) \in E_2$ if and only if $G_2$ is regular.

**Theorem 3.7** An interval-valued fuzzy graph $G$ is strong regular if and only if its complement $\overline{G}$ is strong regular interval-valued fuzzy graph also.

**Proof.** The complement of an interval-valued fuzzy graph is defined as $\overline{\mu}^+_w = \overline{\mu}^-_w, \overline{\mu}^+_w = \overline{\mu}^+_w, \overline{\mu}_w^- (xy) = \min\{\mu_w^-(x), \mu_w^+(y)\} - \mu_w^+(xy), \overline{\mu}_w^+(xy) = \min\{\mu_w^-(x), \mu_w^+(y)\} - \mu_w^+(xy)$ . As $G$ is strong regular if and only if $\overline{\mu}_w^+(xy) = \min\{\mu_w^-(x), \mu_w^+(y)\} - \mu_w^+(xy) = \mu_w^+(xy) - \mu_w^+(xy) = 0$, $\overline{\mu}_w^-(xy) = \min\{\mu_w^-(x), \mu_w^+(y)\} - \mu_w^-(xy) = \mu_w^+(xy) - \mu_w^-(xy) = 0$ if and only if $\overline{\mu}_w^-(xy) = 0$ and $\overline{\mu}_w^-(xy) = 0$ if and only if $\overline{G}$ is strong regular interval-valued fuzzy graph.

**Theorem 3.8** For any two isomorphic highly irregular interval-valued fuzzy graphs, their order and size are same.

**Proof** If $p : G_1 \rightarrow G_2$ is an isomorphism between the two highly irregular interval-valued fuzzy graphs $G_1$ and $G_2$ with the underlying sets $V_1$ and $V_2$ respectively. Then $\mu_{v_1}^-(u) = \mu_{v_2}^-(p(u)), \mu_{v_1}^+(u) = \mu_{v_2}^+(p(u)), \forall u \in V_1$, $\mu_{F_1}^-(uv) = \mu_{F_2}^-(p(u)p(v)), \mu_{F_1}^+(uv) = \mu_{F_2}^-(p(u)p(v)), \forall uv \in E_1$. 

So, we get
\[ O(G_1) = \left( \sum_{x_1 \in V_1} \mu_{w_1}^v (x_1) \right) \cdot \left( \sum_{x_1 \in V_1} \mu_{w_1}^v (x_1) \right) = \left( \sum_{x_1 \in V_1} \mu_{w_2}^v (p(x_1)) \right) \cdot \left( \sum_{x_1 \in V_1} \mu_{w_2}^v (p(x_1)) \right) = \left( \sum_{x_2 \in V_2} \mu_{w_2}^v (x_2) \right) \cdot \left( \sum_{x_2 \in V_2} \mu_{w_2}^v (x_2) \right) = O(G_2). \]

\[ S(G_1) = \left( \sum_{(x_1, y_1) \in E_1} \mu_{F_1}^v (x_1, y_1) \right) \cdot \left( \sum_{(x_1, y_1) \in E_1} \mu_{F_1}^v (x_1, y_1) \right) = \left( \sum_{(x_1, y_1) \in E_1} \mu_{F_2}^v (p(x_1) p(y_1)) \right) \cdot \left( \sum_{(x_1, y_1) \in E_1} \mu_{F_2}^v (p(x_1) p(y_1)) \right) = \left( \sum_{x_2 \in V_2} \mu_{F_2}^v (x_2, y_2) \right) \cdot \left( \sum_{x_2 \in V_2} \mu_{F_2}^v (x_2, y_2) \right) = S(G_2). \]

**Theorem 3.9** If \( G_1 \) and \( G_2 \) are isomorphic highly irregular interval-valued fuzzy graphs. Then, the degrees of the corresponding vertices \( u \) and \( p(u) \) are preserved.

**Proof** If \( p: G_1 \rightarrow G_2 \) is an isomorphism between the highly irregular interval-valued fuzzy graphs \( G_1 \) and \( G_2 \) with the underlying sets \( V_1 \) and \( V_2 \) respectively.

Then \( \mu_{F_1}^v (x, y_1) = \mu_{F_2}^v (p(x) p(y_1)) \) and \( \mu_{F_1}^v (x, y_1) = \mu_{F_2}^v (p(x) p(y_1)) \) \( \forall x, y_1 \in V_1 \).

Therefore \( d_{G_1}^*(x_1) = \sum_{x_1, y_1 \in V_1} \mu_{F_1}^v (x_1, y_1) = \sum_{p(x) p(y_1) \in V_2} \mu_{F_2}^v (p(x) p(y_1)) = d_{G_2}^*(p(x_1)) \),

\[ d_{G_1}^*(x_1) = \sum_{x_1, y_1 \in V_1} \mu_{F_1}^v (x_1, y_1) = \sum_{p(x) p(y_1) \in V_2} \mu_{F_2}^v (p(x) p(y_1)) = d_{G_2}^*(p(x_1)) \forall x_1 \in V_1. \]

i.e. the degrees of the corresponding vertices of \( G_1 \) and \( G_2 \) are same.

### 4. \( \mu \)-complement of a product interval-valued fuzzy graph

**Definition 4.1** The product interval-valued fuzzy graph is a pair \( G=(W, F) \) of a graph

\[ G^*=(V, E) \] where \( W=[\mu_W, \mu_W^v] \) is an interval-valued fuzzy set on \( V \) and \( F=[\mu_F, \mu_F^v] \) is an interval-valued fuzzy relation on \( E \) such that \( \mu_F(x) \leq \mu_W(x) \times \mu_W(y), \mu_F^v(x) \leq \mu_W^v(x) \times \mu_W^v(y), \)

for all \( xy \in E \).

Every product interval-valued fuzzy graph is also an interval-valued fuzzy graph.

**Definition 4.2** Let \( G=(W, F) \) be a product interval-valued fuzzy graph. Then \( \mu \)-complement of \( G \) is defined as \( G^\mu=(V, W, F^\mu) \), where \( F^\mu=[\mu_F^\mu, \mu_F^\mu^v] \) and

\[ \mu_F^\mu(xy) = \begin{cases} \mu_W(x) \times \mu_W(y) - \mu_F(xy) & \text{if } \mu_F(xy) > 0 \\ 0 & \text{if } \mu_F(xy) = 0 \end{cases} \]

\[ \mu_F^\mu^v(xy) = \begin{cases} \mu_W^v(x) \times \mu_W^v(y) - \mu_F^v(xy) & \text{if } \mu_F^v(xy) > 0 \\ 0 & \text{if } \mu_F^v(xy) = 0 \end{cases} \]
**Example 4.1** Consider a product interval-valued fuzzy graph $G = (W, F)$ as shown in Figure 3(a), where

$$W = \{ [0.2, 0.4]_J, [0.2, 0.38]_K, [0.3, 0.4]_L \}$$

and

$$F = \{ [0.1, 0.15]_{JK}, [0.1, 0.2]_{KL}, [0.2, 0.4]_{LJ} \}.$$

![Figure 3: \(\mu\)-complement of a product interval-valued fuzzy graph $G$](image)

Then $\mu$-Complement of a product interval-valued fuzzy graph $G$ is shown in Figure 3 (b).

**Theorem 4.1** Let $G$ be a highly irregular product interval-valued fuzzy graph then its $\mu$-complement need not be highly irregular.

**Proof** Let $G$ be a highly irregular product interval-valued fuzzy graph. In $G$, for every vertex, the adjacent vertices with distinct degrees or the non-adjacent vertices with distinct degrees may happen to be adjacent vertices with same degrees in its $\mu$-complement. This contradicts the definition of highly irregular product interval-valued fuzzy graph.

**Theorem 4.2** Let $G_1$ and $G_2$ be two highly irregular product interval-valued fuzzy graphs. If $G_1$ and $G_2$ are isomorphic, then $\mu$-complements of $G_1$ and $G_2$ are isomorphic and vice versa.

**Proof** Suppose that $G_1$ and $G_2$ are isomorphic, there exists a bijective map $p : V_1 \rightarrow V_2$ which satisfies $\mu_{w_1}^-(u) = \mu_{w_2}^-(p(u))$, $\mu_{w_1}^+(u) = \mu_{w_2}^+(p(u))$, $\forall u \in V_1$ and $\mu_{E_1}^-((uv)) = \mu_{E_2}^-((p(u)p(v)))$, $\mu_{E_1}^+(((uv))) = \mu_{E_2}^+((p(u)p(v)))$, $\forall uv \in E_1$.

By the definition of $\mu$-complement, we have

$$\mu_{E_1}^-(xy) = \mu_{E_1}^-(x) \times \mu_{E_1}^-(y) - \mu_{E_1}^-(xy) = \mu_{E_1}^-(p(x)) \times \mu_{E_1}^-(p(y)) - \mu_{E_1}^-(p(x)p(y))),$$

$$\mu_{E_1}^+(xy) = \mu_{E_1}^+(x) \times \mu_{E_1}^+(y) - \mu_{E_1}^+(xy) = \mu_{E_1}^+(p(x)) \times \mu_{E_1}^+(p(y)) - \mu_{E_1}^+(p(x)p(y)))$$

for all $xy \in E_1$. Hence, $G_1^\mu \cong G_2^\mu$.

Similarly, we can prove the converse.

5. **Strength cut graph of an interval-valued fuzzy graph.**

In this part $\alpha$-strength cut graph of an interval-valued fuzzy graph $G$ is defined with an example.

For an interval-valued fuzzy graph $G = (W, F)$, an edge $mn \in E$ is said to be effective if $(0.5)\text{min}\{\mu_w^+(m), \mu_w^-(n)\} \leq \mu_{mn}^+ (mn)$ and $(0.5)\text{min}\{\mu_w^+(m), \mu_w^-(n)\} \leq \mu_{mn}^- (mn)$ and it is non effective otherwise. The strength of an edge $pq$ in an interval-valued fuzzy graph $G = (W, F)$ is denoted by...
\( \tau_{pq} \) and is defined as \( \tau_{pq} = \left[ \tau_{pq}^-, \tau_{pq}^+ \right] \) where \( \tau_{(p,q)} = \frac{\mu_p(pq)}{\min\{\mu_p(p), \mu_q(q)\}} \) and
\( \tau_{pq}^+ = \frac{\mu_p^+(pq)}{\min\{\mu_p^+(p), \mu_q^+(q)\}} \). Again strength of a vertex \( w \) is denoted by \( \tau_w \) and is defined as
\( \tau_w = \left[ \tau_w^-, \tau_w^+ \right] \) where \( \tau_w^- \) is the greatest value along its membership value \( \mu_w(w) \) and the strengths \( \tau_{wx}^- \) of edges \( wx \) incident to \( w \) and \( \tau_w^+ \) is the greatest value along its membership value \( \mu_w^+(w) \) and the strengths \( \tau_{wx}^+ \) of edges \( wx \) incident to \( w \).

**Definition 5.1** Suppose \( G = (W,F) \) is an interval-valued fuzzy graph. For any \( 0 \leq \alpha \leq 1 \),
\( \alpha \)-strength cut graph of \( G \) is defined to be the crisp graph \( G^\alpha = (V^\alpha, E^\alpha) \) such that
\( V^\alpha = \{ p \in V \mid \tau_p \geq [\alpha, \alpha] \} \) and \( E^\alpha = \{ pq \in E \mid \tau_{pq} \geq [\alpha, \alpha] \} \).

**Definition 5.2** For \( 0 \leq \alpha \leq 1 \), \( \alpha \) – cut graph of an interval-valued fuzzy graph \( G = (W,F) \), where \( W = [\mu_w, \mu_w^+] \), \( F = [\mu^-, \mu^+] \) is a crisp graph \( G^\alpha = (V^\alpha, E^\alpha) \) such that
\( V^\alpha = \{ x \in V \mid \mu_w^-(x), \mu_w^+(x) \geq [\alpha, \alpha] \} \) and \( E^\alpha = \{ xy \in E \mid \mu_f^-(xy), \mu_f^+(xy) \geq [\alpha, \alpha] \} \).

**Example 5.1** Let \( G = (W, F) \) be an interval-valued fuzzy graph of \( G^\alpha = (V, E) \), where
\[
W = \begin{bmatrix}
0.6 & 0.7 \\
0.29 & 0.3 \\
0.9 & 0.9 \\
0.7 & 0.8 \\
0.75 & 0.9
\end{bmatrix}
\]
and
\[
F = \begin{bmatrix}
u & v & u & w & x & y \\
0.2 & 0.25 & 0.5 & 0.5 & 0.55 & 0.69 \\
0.22 & 0.09 & 0.26 & 0.2 & 0.04 & 0.69 & 0.89 & 0.69 & 0.79
\end{bmatrix}
\]

Then, the resultant 0.5 – cut graph and 0.5 – strength cut graph are shown in Figure 4(b) and Figure 4(c) respectively.

(a) Interval-valued fuzzy graph \( G \)

(b) 0.5–cut graph of \( G \)

(c) 0.5–strength cut graph of \( G \)

Figure 4: Example of a 0.5–strength cut graph
Theorem 5.1 Let \( G \) be an interval-valued fuzzy graph. If \( 0 \leq \alpha \leq \beta \leq 1 \), then \( G^\beta \subseteq G^\alpha \).

Proof Suppose \( G = (W, F) \) is an interval-valued fuzzy graph and \( 0 \leq \alpha \leq \beta \leq 1 \).

Then \( G^\alpha = (V^\alpha, E^\alpha) \) such that \( V^\alpha = \{ p \in V / \tau_p \geq \alpha \} \) and \( E^\alpha = \{ pq \in E, p, q \in V / \tau_{pq} \geq \alpha \} \).

Also, \( G^\beta = (V^\beta, E^\beta) \) such that \( V^\beta = \{ p \in V / \tau_p \geq \beta \} \) and \( E^\beta = \{ pq \in E, p, q \in V / \tau_{pq} \geq \beta \} \).

Let \( m \in V^\beta \). Then \( \tau_m \geq \beta \geq \alpha \). Therefore, \( m \in V^\alpha \). In the same way, for any element \( mn \in E^\beta \) gives that \( mn \in E^\alpha \). Therefore, \( G^\beta \subseteq G^\alpha \).

Theorem 5.2 Let \( G \) be an interval-valued fuzzy graph. If \( 0 \leq \alpha \leq 1 \), then \( G_\alpha \subseteq G^\alpha \).

Proof Suppose \( G = (W, F) \) is an interval-valued fuzzy graph. Then

\( \alpha \)-cut of interval-valued fuzzy graph, we have discussed \( p \)-morphisms, \( \mu \) -complement, \( \alpha \) -cut of an interval-valued fuzzy graph and \( \mu \) -complement of a product interval valued fuzzy graph. In future we are extending our research work to interval-valued fuzzy planar graphs, interval–valued fuzzy hyper graphs.

Conclusions

Fuzzy graph theory concepts are used in computer science, decision making, mining data. An interval-valued fuzzy model is generalisation of the fuzzy model, we have discussed

\[ \mu_r (pq) \]

\[ \min \{ \mu_r (p), \mu_r (q) \} \]

\[ \mu (pq) \]

\[ \min \{ \mu (p), \mu (q) \} \]

Hence \( \tau_{pq} \geq [\alpha, \alpha] \) and \( pq \in E^\alpha \). Thus for every edge of \( G_\alpha \), there exists an edge in \( G^\alpha \). By the strength of vertices definition, \( V_\alpha \subseteq V^\alpha \). So the result \( G_\alpha \subseteq G^\alpha \) is true.
References

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