Contributions on s-Edge Regular Bipolar Fuzzy Graphs

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Abstract

Bipolar fuzzy graphs are revolutionized the analysis of probabilistic data to arrive at a judicious decision making power. In this paper we introduce the notation of s-edge regular bipolar fuzzy graph, strongly regular bipolar fuzzy graph and Bi regular bipolar fuzzy graphs, describes various methods of their construction and discuss some of their important theorems related to these graphs and also investigate on equivalence theorem of these graphs.

Key words: s-edge regular bipolar fuzzy graph, strongly edge regular bipolar fuzzy graph, Bi regular bipolar fuzzy graphs, bipartite bipolar fuzzy graph.

1. Introduction

The moment science is involved in finding solutions to theoretical problems mathematical dependency increases. It has been proved by researchers that frameworks of analysis developed using mathematical models, especially those based on fuzzy logic were capable of handling uncertain data sets. The new mathematical models have shown their superiority over the conventional fuzzy logic based sets. Issues of doubt, data inconsistency and wrong or mismatched data were effectively analyzed with the help of Graph theory in the various domains like medical, life sciences, management sciences, Mechanical engineering, electrical engineering, with special focus on machine learning systems. The self-learning feature of these new mathematical models have given the possibility of scaling the size of the operations to suit the requirement industrial requirement. In short, mathematical models based on fuzzy graph theory have simplified the handling of probabilistic to arrive at rational conclusions.

The problem of Koenig’s berg bridge, in eighteenth century laid the foundation to graph theory, where Euler’s strongly suggested that there is a solution using graph theory. In 1965, Zadeh developed a fuzzy set theory using the real numbers between 0 and 1. In 1994, Zhang [16 -18] extended the fuzzy set theory to bipolar fuzzy sets in which the first point predicts the reality condition while the second point represents a decimal opposite to the first point.

2. Preliminaries

Some definitions and conventions used in this paper are discussed in this section. Literature review is available in [1, 6, 13-14].

**Definition 2.1.** A Graph \( G = (V, E) \) is an ordered pair consisting of a non-empty vertex set \( V \), an edge set \( E \) and a connection that associates with every edge between two edges (not as a matter of course particular) called its end points.

**Definition 2.2.** Let \( G = (V, E) \) be a graph. Then \( S = (N, L) \) is said to be a sub graph of \( G \) if \( N \subseteq V \) and \( L \subseteq E \).

**Definition 2.3.** A fuzzy set \( A \) on a universal set \( X \) is characterized by function \( m : [0, 1] \rightarrow [0, 1] \) which is called the membership function. A fuzzy set is denoted by \( (A, m) \).

**Definition 2.4** A fuzzy graph \( (V, \sigma, \mu) \) is a non-empty set \( V \) together with a pair of functions \( \sigma : V \rightarrow [0, 1] \) and \( \mu : V \times V \rightarrow [0, 1] \) such that for all \( m, n \in V, \mu(mn) \leq \min \{\sigma(m), \sigma(n)\} \) where \( \sigma(m) \) and \( \mu(mn) \) represent the membership values of the vertex \( m \) and of the edge \( mn \) in \( \delta \) respectively. The underlying crisp graph of the fuzzy graph \( \delta = (V, \sigma, \mu) \) is denoted as \( \delta^* = (V^*, \sigma^*, \mu^*) \) where \( \sigma^* = \{ x \in V / \sigma(x) > 0 \} \) and \( \mu^* = \{ xy \in V \times V / \mu(xy) > 0 \} \). Thus for underlying fuzzy graph, \( \sigma^* = V \).

**Definition 2.5.** A fuzzy graph \( \delta = (V, \sigma, \mu) \) is called complete if \( \mu(xy) = \min \{\sigma(x), \sigma(y)\} \) \( \forall x, y \in V \) where \( xy \) denotes the edge between the edges \( x \) and \( y \). The fuzzy graph \( \delta_a = (V, \sigma_a, \mu_a) \) is called a fuzzy sub graph of \( \delta = (V, \sigma, \mu) \) if \( \sigma_a(m) \leq \sigma(m) \) for all \( m \) and \( \mu_a(mn) \leq \mu(mn) \) for all edges \( mn, m, n \in V \).

**Definition 2.6.** The bipolar fuzzy set \( W \) in \( V \) is defined by \( W = (\mu^+, \mu^-) : X \times X \rightarrow [0, 1] \times [-1, 0] \) a bipolar fuzzy relation on \( X \) such that \( \mu^+(x, y) \in [0, 1] \) and \( \mu^-(x, y) \in [-1, 0] \).

**Definition 2.7.** Let \( W = (\mu^+, \mu^-) \) and \( F = (\mu^+, \mu^-) \) be a bipolar fuzzy sets on set \( X \). If \( W = (\mu^+, \mu^-) \) is a bipolar fuzzy relation on \( X \), then \( W = (\mu^+, \mu^-) \) is called a bipolar fuzzy relation on \( F = (\mu^+, \mu^-) \) if \( \mu^+(xy) \leq \min \{\mu^+(x, y), \mu^-(y)\}, \mu^-(xy) \geq \max \{\mu^+(x), \mu^-(y)\} \) for all \( xy \in E \).

**Definition 2.8.** The bipolar fuzzy graph is a pair \( G = (W, F) \) of a graph \( G^* = (V, E) \), where \( W = (\mu^+, \mu^-) \) is a bipolar fuzzy set on \( V \) and \( F = (\mu^+, \mu^-) \) is a bipolar fuzzy relation on \( E \) such that \( \mu^+(xy) \leq \min \{\mu^+(x, y), \mu^-(y)\}, \mu^-(xy) \geq \max \{\mu^+(x), \mu^-(y)\} \) for all \( x, y \in V \).

The underlying crisp graph of \( G = (W, F) \) is the crisp graph \( G^* = (V, E) \) where \( V = \{ v / \mu^+(v) > 0 \text{ or } \mu^-(v) < 0 \} \) and \( E = \{ uv / \mu^+(uv) > 0 \text{ or } \mu^-(uv) < 0 \} \).
Definition 2.9. Let \( G = (W, F) \) be a bipolar fuzzy graph of \( G^* = (V, E) \). Then \( G = (W, F) \) is said to be strong if \( \mu^+_F (xy) = \min \{ \mu^+_W (x), \mu^+_W (y) \} \), \( \mu^-_F (xy) = \max \{ \mu^-_W (x), \mu^-_W (y) \} \), for all \( xy \in E \).

Definition 2.10. Let \( G = (W, F) \) be a bipolar fuzzy graph of \( G^* = (V, E) \). Then \( G = (W, F) \) is said to be complete if \( \mu^+_F (xy) = \min \{ \mu^+_W (x), \mu^+_W (y) \} \), \( \mu^-_F (xy) = \max \{ \mu^-_W (x), \mu^-_W (y) \} \), for all \( xy \in E \).

Definition 2.11. The complement of a bipolar fuzzy graph \( G = (W, F) \) of a graph \( G^* = (V, E) \) is a bipolar fuzzy graph \( \bar{G} = \left( \bar{W}, \bar{F} \right) \), where \( \bar{W} = W = \left( \mu^+_W (x), \mu^-_W (x) \right) \) and \( \bar{F} = \left( \mu^+_F (x), \mu^-_F (x) \right) \) here
\[
\mu^+_F (xy) = \min \{ \mu^+_W (x), \mu^+_W (y) \} - \mu^-_F (xy), \quad \mu^-_F (xy) = \max \{ \mu^-_W (x), \mu^-_W (y) \} - \mu^-_F (xy) \text{ for all } xy \in E.
\]

Definition 2.12. Let \( G = (W, F) \) be a bipolar fuzzy graph, where \( W = \left( \mu^+_W, \mu^-_W \right) \) and \( F = \left( \mu^+_F, \mu^-_F \right) \). The degree of a vertex is defined as \( d_G = \left( d^+_G (w), d^-_G (w) \right) \), where \( d^+_G (w) = \sum \mu^+_F (wx) \ {w \neq x, w \in E} \) is the positive degree of a vertex \( w \) and \( d^-_G (w) = \sum \mu^-_F (wx) \ {w \neq x, w \in E} \) is the negative degree of a vertex.

Definition 2.13. Let \( G^* = (V, E) \) be a crisp graph and let \( e = w_r w_s \) be an edge in \( G^* \). Then, the degree of an edge \( e = w_r w_s \in E \) is defined as \( d^-_G (w_r w_s) = d^+_G (w_r) + d^+_G (w_s) - 2 \).

Definition 2.14. The minimum open neighborhood degree of an edge is defined as \( \delta_E (G) = \min \{ d^-_G (xy) / xy \in E \} \).

The maximum open neighborhood degree of an edge is defined as \( \Delta_E (G) = \max \{ d^-_G (xy) / xy \in E \} \).

Definition 2.15. The total open neighborhood degree of an edge \( xy \in E \) in a bipolar fuzzy graph \( G = (W, F) \) defined as \( td^-_G (xy) = \left( td^+_G (xy), td^-_G (xy) \right) \), where \( td^+_G (xy) = d^+_G (x) + d^-_G (y) - \mu^-_F (xy) \) is the total open neighborhood positive degree of an edge and \( td^-_G (xy) = d^-_G (x) + d^+_G (y) - \mu^+_F (xy) \) is the total open neighborhood negative degree of an edge.

The minimum total open neighborhood degree of an edge is defined as \( \delta_{td} (G) = \min \{ td^-_G (xy) / xy \in E \} \).

The maximum total open neighborhood degree of an edge is defined as \( \Delta_{td} (G) = \max \{ td^-_G (xy) / xy \in E \} \).

2.16. Let \( G = (W, F) \) be a bipolar fuzzy graph. The neighborhood of a degree of a vertex \( y \) is defined as \( d^-_N (y) = \left( d^+_N (y), d^-_N (y) \right) \), where \( d^+_N (w) = \sum_{x \in N(y)} \mu^+_W (x) \) and \( d^-_N (w) = \sum_{x \in N(y)} \mu^-_W (x) \).
2.17. Bipartite bipolar fuzzy graph is a bipolar fuzzy graph \( G = (W, F) \) in which the vertex set \( W \) can be partitioned in to two sets \( V_1 \) and \( V_2 \) where \( V_1 \neq \emptyset \) and \( V_2 \neq \emptyset \). Such that

(a) \( \mu^+_F (w, w') = 0 \) and \( \mu^-_F (w, w') = 0 \), if \( w, w' \in V_1 \) or \( w, w' \in V_2 \).

(b) \( \mu^+_F (w, w') = 0 \), \( \mu^-_F (w, w') < 0 \), if \( w \in V_1 \) or \( w \in V_2 \).

(c) \( \mu^+_F (w, w') > 0 \), \( \mu^-_F (w, w') = 0 \) if \( w \in V_1 \) or \( w \in V_2 \), for some \( r \) and \( s \).

3. Contributions on s-edge regular bipolar fuzzy graphs, strongly regular bipolar fuzzy graph, biregular bipolar fuzzy graph

In this section we investigate a new s-edge regular bipolar fuzzy graphs, strongly regular bipolar fuzzy graph, biregular bipolar fuzzy graph.

**Definition 3.1.** Let \( G = (W, F) \) be a bipolar fuzzy graph on \( G^* \). If all the edges have the same neighborhood degree \( s \), then \( G \) is called an s-edge regular bipolar fuzzy graph. The open neighborhood degree of an edge \( xy \in E \) in a bipolar fuzzy graph \( G = (W, F) \) is defined as \( d_G(x, y) = (d^+_G(x, y), d^-_G(x, y)) \), where \( d^+_G(x, y) = d^+_G(x) + d^+_G(y) - 2 \mu^-_F(x, y) \) is the positive open neighborhood degree of an edge and \( d^-_G(x, y) = d^-_G(x) + d^-_G(y) - 2 \mu^-_F(x, y) \) is the negative open neighborhood degree of an edge.

**Example 3.1.**

![Bipolar fuzzy graph](image)

\[ d_G(K) = (0.6, -1) \]
\[ d_G(L) = (0.3, -0.7) \]
\[ d_G(M) = (0.5, -0.9) \]
\[ d_G(N) = (0.7, -1.1) \]
\[ d_G(O) = (0.9, -1.3) \]
\[ d^+_G(KL) = d^+_G(K) + d^+_G(L) - 2 \mu^-_F(KL) = 0.6 + 0.3 - 2(0.1) = 0.7 \]
\[ d^-_G(KL) = d^-_G(K) + d^-_G(L) - 2 \mu^-_F(KL) = -1.0 - 0.7 - 2(-0.3) = -1.1 \]
\[ d_G(KL) = (d^+_G(KL), d^-_G(KL)) = (0.7, -1.1) \]
\[ d^+_G(LM) = d^+_G(L) + d^+_G(M) - 2 \mu^-_F(LM) = 0.3 + 0.5 - 2(0.2) = 0.4 \]
\[ d^-_G(LM) = d^-_G(L) + d^-_G(M) - 2 \mu^-_F(LM) = -0.7 - 0.9 - 2(-0.4) = -0.8 \]
\[ d_G(LM) = (d^+_G(LM), d^-_G(LM)) = (0.4, -0.8) \]
\[ d^+_G(MN) = d^+_G(M) + d^+_G(N) - 2 \mu^-_F(MN) = 0.5 + 0.7 - 2(0.3) = 0.6 \]
\[ d^-_G(MN) = d^-_G(M) + d^-_G(N) - 2 \mu^-_F(MN) = -0.9 - 1.1 - 2(-0.5) = -1 \]
\[ d_G(MN) = (d^+_G(MN), d^-_G(MN)) = (0.6, -1) \]
\[ d_G^+(NO) = d_G^+(N) + d_G^+(O) - 2\mu^-_F(NO) = 0.7 + 0.9 - 2(0.4) = 0.8 \]
\[ d_G^-(NO) = d_G^-(N) + d_G^-(O) - 2\mu^+_F(NO) = -1.1 - 1.3 - 2(-0.6) = -1.2 \]
\[ d_G^+(OK) = d_G^+(O) + d_G^+(K) - 2\mu^-_F(OK) = 0.9 + 0.6 - 2(0.5) = 0.5 \]
\[ d_G^-(OK) = d_G^-(O) + d_G^-(K) - 2\mu^+_F(OK) = -1.3 - 1.0 - 2(-0.7) = -0.9 \]
\[ d_G^+(OK) = (d_G^+(NO), d_G^-(NO)) = (0.8, -1.2) \]
\[ d_G^+(OK) = (d_G^+(NO), d_G^-(OK)) = (0.5, -0.9) \]
\[ td_G^+(KL) = d_G^+(K) + d_G^+(L) - \mu^+_F(K) = 0.6 + 0.3 - (0.1) = 0.8 \]
\[ td_G^+(KL) = d_G^+(K) + d_G^+(L) - \mu^-_F(K) = -1.0 - 0.7 - (-0.3) = -1.4 \]
\[ td_G^+(KL) = (td_G^+(KL), td_G^+(KL)) = (0.8, -1.4) \]
\[ d_G^+(LM) = d_G^+(L) + d_G^+(M) - \mu^+_F(LM) = 0.3 + 0.5 - (0.2) = 0.6 \]
\[ d_G^+(LM) = d_G^+(L) + d_G^+(M) - \mu^-_F(LM) = -0.7 - 0.9 - (-0.4) = -1.2 \]
\[ td_G^+(LM) = (td_G^+(LM), td_G^+(LM)) = (0.6, -1.2) \]
\[ d_G^+(MN) = d_G^+(M) + d_G^+(N) - \mu^+_F(MN) = 0.5 + 0.7 - (0.3) = 0.9 \]
\[ d_G^+(MN) = d_G^+(M) + d_G^+(N) - \mu^-_F(MN) = -0.9 - 1.1 - (-0.5) = -1.5 \]
\[ td_G^+(MN) = (td_G^+(MN), td_G^+(MN)) = (0.9, -1.5) \]
\[ td_G^+(NO) = d_G^+(N) + d_G^+(O) - \mu^+_F(NO) = 0.7 + 0.9 - (0.4) = 1.2 \]
\[ td_G^+(NO) = d_G^+(N) + d_G^+(O) - \mu^-_F(NO) = -1.1 - 1.3 - (-0.6) = -1.8 \]
\[ td_G^+(NO) = (td_G^+(NO), td_G^+(NO)) = (1.2, -1.8) \]
\[ td_G^+(OK) = d_G^+(O) + d_G^+(K) - \mu^+_F(OK) = 0.9 + 0.6 - (0.5) = 1 \]
\[ td_G^-(OK) = d_G^-(O) + d_G^-(K) - \mu^-_F(OK) = -1.3 - 1.0 - (-0.7) = -1.6 \]
\[ td_G^+(OK) = (td_G^+(OK), td_G^+(OK)) = (1, -1.6) \]

**Definition 3.2.** If every vertex in a bipolar fuzzy graph \( G = (W, F) \) has the same degree \((l_1, l_2)\) then \( G = (W, F) \) is called regular bipolar fuzzy graph or bipolar fuzzy graph of degree \((l_1, l_2)\).

**Definition 3.3.** If every edge in a bipolar fuzzy graph \( G = (W, F) \) has the same degree \((l_1, l_2)\) then \( G = (W, F) \) is called \((l_1, l_2)\)−edge regular bipolar fuzzy graph.

**Definition 3.4.** If every edge in a bipolar fuzzy graph \( G = (W, F) \) has the same total degree \((l_1, l_2)\) then \( G = (W, F) \) is called a totally \((l_1, l_2)\)−edge regular bipolar fuzzy graph.

**Example 3.2.**
Consider the bipolar fuzzy graph $G = (W, F)$ as follows, where $W = \{a, b, c, d\}$ and $F = \{ab, bc, cd, da\}$ (see Fig. 2).

Then $d_G(ab) = d_G(bc) = d_G(cd) = d_G(da) = (0.1, -0.7)$.

**Theorem 3.1.** Let $G = (W, F)$ be an edge regular bipolar fuzzy graph on a cycle $G^* = (V, E)$.

Then $\sum_{w_i \in V} d_G(w_i) = \sum_{w_i \in V} d_G(w_i w_j)$.

**Proof.** Suppose that $G = (W, F)$ is an edge regular bipolar fuzzy graph and $G^*$ be a cycle $w_1 w_2 w_3 \ldots w_n w_1$. Then

$$\sum_{i=1}^{n} d_G(w_i w_{i+1}) = \left( \sum_{i=1}^{n} d_G^+(w_i w_{i+1}), \sum_{i=1}^{n} d_G^-(w_i w_{i+1}) \right)$$

Now, we get

$$\sum_{i=1}^{n} d_G^+(w_i w_{i+1}) = d_G^+(w_1 w_2) + d_G^+(w_2 w_3) + \ldots + d_G^+(w_n w_1),$$

where $w_{n+1} = w_1$

$$= d_G^+(w_1) + d_G^+(w_2) - 2\mu_F^+(w_1 w_2) + d_G^+(w_2) + d_G^+(w_3) - 2\mu_F^+(w_2 w_3) + \ldots + d_G^+(w_n) - 2\mu_F^+(w_n w_1)$$

$$= 2d_G^+(w_1) + 2d_G^+(w_2) + \ldots + 2d_G^+(w_n) - 2\left( \mu_F^+(w_1 w_2) + \mu_F^+(w_2 w_3) + \ldots + \mu_F^+(w_n w_1) \right)$$

$$= 2\sum_{i=1}^{n} d_G^+(w_i) - 2\sum_{i=1}^{n} \mu_F^+(w_i w_{i+1})$$

$$= \sum_{i=1}^{n} d_G^+(w_i) + 2\sum_{i=1}^{n} \mu_F^+(w_i w_{i+1}) - 2\sum_{i=1}^{n} \mu_F^+(w_i w_{i+1})$$

$$= \sum_{i=1}^{n} d_G^+(w_i)$$

Similarly

$$\sum_{i=1}^{n} d_G^-(w_i w_{i+1}) = \sum_{i=1}^{n} d_G^-(w_i).$$

So

$$\sum_{i=1}^{n} d_G^+(w_i w_{i+1}) = \left( \sum_{i=1}^{n} d_G^+(w_i), \sum_{i=1}^{n} d_G^-(w_i) \right) = \left( \sum_{i=1}^{n} d_G(w_i) \right).$$

**Remark 3.1.** Let $G = (W, F)$ be an edge regular bipolar fuzzy graph on a crisp graph $G^*$. Then

$$\sum_{w_i \in F} d_G(w_i w_j) \in \left( \sum_{w_i \in F} d_G^+(w_i w_j) \mu_F^-(w_i w_j), \sum_{w_i \in F} d_G^-(w_i w_j) \mu_F^+(w_i w_j) \right).$$

Where $d_G^+(w_i w_j) = d_G^+(w_i) + d_G^-(w_j) - 2$ for all $w_i w_j \in F$. 
Theorem 3.2. Let $G = (W, F)$ be an edge regular bipolar fuzzy graph on a $c$-regular crisp graph $G^*$. Then

$$
\sum_{w_i, w_j \in F} d_G(w_i, w_j) = \left( (c-1) \sum_{w_i \in F} d_G^+(w_i), (c-1) \sum_{w_i \in F} d_G^-(w_i) \right).
$$

Proof. From Remark 3.1. We get

$$
\sum_{w_i, w_j \in F} d_G(w_i, w_j) = \left( \sum_{w_i, w_j \in F} d_G^+(w_i, w_j), \sum_{w_i, w_j \in F} d_G^-(w_i, w_j) \right).
$$

Since $G^*$ is a regular crisp graph, we have the degree of every vertex in $G^*$ is $c$ i.e., $d^*_G(w_i) = c$, so

$$
\sum_{w_i, w_j \in F} d_G(w_i, w_j) = \left( (c+c-2) \sum_{w_i, w_j \in F} \mu^+_F(w_i, w_j), (c+c-2) \sum_{w_i, w_j \in F} \mu^-_F(w_i, w_j) \right)
$$

Theorem 3.3. Let $G = (W, F)$ be an edge regular bipolar fuzzy graph on a crisp graph $G^*$. Then

$$
\sum_{w_i, w_j \in F} td_G(w_i, w_j) = \left( \sum_{w_i, w_j \in F} d_G^+(w_i, w_j) + \mu^+_F(w_i, w_j), \sum_{w_i, w_j \in F} d_G^-(w_i, w_j) + \mu^-_F(w_i, w_j) \right).
$$

Proof. From the definition of total open neighborhood edge degree of $G$, we get

$$
\sum_{w_i, w_j \in F} td_G(w_i, w_j) = \left( \sum_{w_i, w_j \in F} d_G^+(w_i, w_j) + \mu^+_F(w_i, w_j), \sum_{w_i, w_j \in F} d_G^-(w_i, w_j) + \mu^-_F(w_i, w_j) \right).
$$

From Remark 3.1 We have

$$
\sum_{w_i, w_j \in F} td_G(w_i, w_j) = \left( \sum_{w_i, w_j \in F} d_G^+(w_i, w_j) + \mu^+_F(w_i, w_j), \sum_{w_i, w_j \in F} d_G^-(w_i, w_j) + \mu^-_F(w_i, w_j) \right).
$$

Theorem 3.4. Let $G = (W, F)$ be a bipolar fuzzy graph. Then $\left( \mu^+_F, \mu^-_F \right)$ is a constant function if and only if the next conditions are equivalent.

(i) $G$ is an edge regular bipolar fuzzy graph.

(ii) $G$ is totally edge regular bipolar fuzzy graph.
Proof. Assume that \((\mu^+, \mu^-)\) is a constant function. Then \(\mu^+ (w_i w_j) = k_1\) and \(\mu^- (w_i w_j) = k_2\), \(\forall w_i w_j \in E\), where \(k_1\) and \(k_2\) are constants. Let \(G\) be an \((r_1, r_2)\) edge regular bipolar fuzzy graph. Then for all \(\forall w_i w_j \in E, d_G (w_i w_j) = (r_1, r_2)\).

Now \(td_G (w_i w_j) = (d_G^+ (w_i w_j) + \mu^+ (w_i w_j), d_G^- (w_i w_j) + \mu^- (w_i w_j)) = (r_1 + k_1, r_2 + k_2)\) \(\forall w_i w_j \in E\). Then \(G\) is a totally edge regular. Now, let \(G\) be a \((h_1, h_2)\) - totally edge regular bipolar fuzzy graph. Then \(td_G (w_i w_j) = (h_1, h_2)\), for all \(w_i w_j \in E\).

So, we have \(td_G (w_i w_j) = (d_G^+ (w_i w_j) + \mu^+ (w_i w_j), d_G^- (w_i w_j) + \mu^- (w_i w_j)) = (h_1, h_2)\). Hence, \((d_G^+ (w_i w_j), d_G^- (w_i w_j)) = (h_1 - \mu^+ (w_i w_j), h_2 - \mu^- (w_i w_j)) = (h_1 - k_1, h_2 - k_2)\).

Then, \(G\) is an \((h_1 - k_1, h_2 - k_2)\) edge regular bipolar fuzzy graph. Conversely suppose that conditions (i) and (ii) are equivalent. We have to prove that the function \((\mu^+, \mu^-)\) is a constant function. In a contrary way we suppose that \((\mu^+, \mu^-)\) is not a constant function. Then \(\mu^+ (w_i w_j) \neq \mu^+ (w_i w_j)\) and \(\mu^- (w_i w_j) \neq \mu^- (w_i w_j)\) for at least one pair of edges \(w_i w_j, w_i w_j \in E\).

Let \(G\) be an \((r_1, r_2)\) edge regular bipolar fuzzy graph. Then, \(d_G (w_i w_j) = d_G (w_i w_j) = (r_1, r_2)\). So for all \(w_i w_j \in E\) and for all \(w_i w_j \in E\).

\[
\begin{align*}
\text{Now } td_G (w_i w_j) &= (d_G^+ (w_i w_j) + \mu^+ (w_i w_j), d_G^- (w_i w_j) + \mu^- (w_i w_j)) = (r_1 + \mu^+ (w_i w_j), r_2 + \mu^- (w_i w_j)) \\
&= (r_1 + \mu^+ (w_i w_j), r_2 + \mu^- (w_i w_j)) \\
&= (h_1 - \mu^+ (w_i w_j), h_2 - \mu^- (w_i w_j))
\end{align*}
\]

Since \(\mu^+ (w_i w_j) \neq \mu^+ (w_i w_j)\) and \(\mu^- (w_i w_j) \neq \mu^- (w_i w_j), we have td_G (w_i w_j) \neq td_G (w_i w_j)\).

Hence, \(G\) is not a totally edge regular bipolar fuzzy graph. This is a contradiction to our assumption. Hence \((\mu^+, \mu^-)\) is a constant function. In the same way we can prove that \((\mu^+, \mu^-)\) is a constant function, when \(G\) is a totally edge regular bipolar fuzzy graph.

**Theorem 3.5.** Let \(G^*\) be a \(h\) - regular crisp graph and \(G = (W, F)\) be a bipolar fuzzy graph on \(G^*\). Then, \((\mu^+, \mu^-)\) is a constant function if and only if \(G\) is both regular bipolar fuzzy graph and totally edge regular bipolar fuzzy graph.

Proof. Let \(G = (W, F)\) be a bipolar fuzzy graph on \(G^*\) and let \(G^*\) be a \(h\) - regular crisp graph. Assume that \((\mu^+, \mu^-)\) is a constant function. i.e., \(\mu^+ (w_i w_j) = k_1\) and \(\mu^- (w_i w_j) = k_2\) for all \(w_i w_j \in E\) where \(k_1, k_2\) are constants. From definition of degree of a vertex,

We get \(d_G (w_i) = (d_G^+ (w_i), d_G^- (w_i)) = \left(\sum_{w_{ij} \in E} \mu^+ (w_{ij}), \sum_{w_{ij} \in E} \mu^- (w_{ij})\right)\)

\[
= \left(\sum_{w_{ij} \in E} k_1, \sum_{w_{ij} \in E} k_2\right) = (hk_1, hk_2)
\]

For every \(w_i \in V\). So, \(d_G (w_i) = (hk_1, hk_2)\). therefore \(G\) is regular bipolar fuzzy graph.

Now, \(td_G (w_i w_j) = (td_G^+ (w_i w_j), td_G^- (w_i w_j))\). Where

\[
\begin{align*}
td_G^+ (w_i w_j) &= \sum_{w_{ij} \in E} \mu^+ (w_{ij}) + \sum_{w_{ij} \in E} \mu^- (w_{ij}) \ddot{w}_{ij} \\
&= \sum_{w_{ij} \in E} k_1 + \sum_{w_{ij} \in E} k_1 + k_1 = k_1(h - 1) + k_1(h - 1) + k_1 = k_1(2h - 1).
\end{align*}
\]
Similarly \( t\mu^*_G(w_iw_j) = k_2(2h - 1) \), for all \( w_iw_j \in E \).

Hence \( G \) is also totally edge regular bipolar fuzzy graph.

Conversely, assume that \( G \) is both regular and edge regular bipolar fuzzy graph. Now we have to prove that \( \left( \mu^*_G, \mu^-_G \right) \) is a constant function. Since \( G \) is regular, \( d_G(w_i) = (k_i, k_i) \), for all \( w_i \in V \). Also \( G \) is totally edge regular. Hence \( td_G(w_iw_j) = (h_i, h_j) \), for all \( w_iw_j \in E \). From the definition of totally edge degree we get \( td_G(w_iw_j) = \left( t\mu^*_G(w_iw_j), t\mu^-_G(w_iw_j) \right) \), where \( td_G(w_iw_j) = d^*_G(w_i) + d^*_G(w_j) - \mu^*_G(w_iw_j) \) for all \( w_iw_j \in E \), \( h_i = k_i + k_i - \mu^*_G(w_iw_j) \) so, \( \mu^-_G(w_iw_j) = 2k_i - h_i \). Similarly we have \( \mu^-_G(w_iw_j) = 2k_2 - h_2 \), for all \( w_iw_j \in E \). Hence, \( \left( \mu^*_G, \mu^-_G \right) \) is a constant function.

**Definition 3.5** A bipolar fuzzy graph \( G = (W = \{w_1, w_2, ..., w_n\}, F) \) is said to be strongly regular, if:

(i) \( G \) is \( (h_1, h_2) \)-regular bipolar fuzzy graph.

(ii) The sum of membership values and non-membership values of the common neighborhood vertices of any pair of adjacent vertices and non-adjacent vertices \( w_i, w_j \) of \( G \) has the same weight and is denoted by \( \lambda = (\lambda_1, \lambda_2), \delta = (\delta_1, \delta_2) \) respectively.

Any strongly bipolar fuzzy graph \( G \) is denoted by \( G = (n, h, \lambda, \delta) \).

**Theorem 3.6** If \( G = (W, F) \) is a complete bipolar fuzzy graph with \( \left( \mu^*, \mu^- \right) \) and \( \left( \mu^*_G, \mu^-_G \right) \) as constant functions, then \( G \) is a strongly regular bipolar fuzzy graph.

**Proof.** Let \( G = (W, F) \) be a complete bipolar fuzzy graph where \( W = \{w_1, w_2, ..., w_n\} \).

Since \( \mu^*, \mu^- \) and \( \mu^*_G, \mu^-_G \) are constant functions, hence, \( \mu^*(w_i) = l, \mu^-(w_i) = m \), for all \( w_i \in W \) and \( \mu^*_G(w_iw_j) = k_1, \mu^-_G(w_iw_j) = k_2 \) for all \( w_iw_j \in F \) where \( l, m, k_1, k_2 \) are constants. Now we have to show that \( G \) is a strongly regular bipolar fuzzy graph. Now since is \( G \) complete, we have

\[
d_G(w_i) = \left( d^*_G(w_i), d^-_G(w_i) \right) = \left( \sum_{w_1w_j \in F} \mu^*_G(w_iw_j), \sum_{w_1w_2 \in F} \mu^-_G(w_iw_j) \right) = ((n - 1)k_1, (n - 1)k_2)
\]

Hence, \( G \) is \( \left( (n - 1)k_1, (n - 1)k_2 \right) \) regular bipolar fuzzy graph.

Now here \( G \) is complete bipolar fuzzy graph sum the membership values and non-membership values of common neighborhood vertices of any pair of adjacent vertices \( \lambda = ((n - 2)l, (n - 2)m) \) are the equal and the sum of membership values and non-membership values of common neighborhood vertices of any pair of non-adjacent vertices \( \delta = 0 \) are the same.

**Definition 3.6** A bipolar fuzzy graph \( G = (W, F) \) is said to be a biregular bipolar fuzzy graph if

(i) \( G \) is a \( (h_1, h_2) \)-regular bipolar fuzzy graph

(ii) \( W = W_1W_2 \) be the bipartition of \( W \) and every vertex in \( W_1 \) has the same neighborhood degree \( P = (P_1, P_2) \) and every vertex in \( W_2 \) has the same neighborhood degree \( Q = (Q_1, Q_2) \), where \( P \) and \( Q \) are constants.

**Example 3.3**
Consider a bipolar fuzzy graph \((W,F)\) (see Fig.3.) Table.1. and Table.2. gives the membership values of vertices and edges and also neighborhood degree of vertices.

**Table 1. Vertices with neighborhood degree.**

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Membership Values</th>
<th>Neighborhood Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
<td>(0.4,-0.6)</td>
<td>(1.5,-2.1)</td>
</tr>
<tr>
<td>W2</td>
<td>(0.5,-0.7)</td>
<td>(1.2,-1.8)</td>
</tr>
<tr>
<td>W3</td>
<td>(0.4,-0.6)</td>
<td>(1.5,-2.1)</td>
</tr>
<tr>
<td>W4</td>
<td>(0.5,-0.7)</td>
<td>(1.2,-1.8)</td>
</tr>
<tr>
<td>W5</td>
<td>(0.5,-0.7)</td>
<td>(1.2,-1.8)</td>
</tr>
<tr>
<td>W6</td>
<td>(0.4,-0.6)</td>
<td>(1.5,-2.1)</td>
</tr>
<tr>
<td>W7</td>
<td>(0.5,-0.7)</td>
<td>(1.2,-1.8)</td>
</tr>
<tr>
<td>W8</td>
<td>(0.4,-0.6)</td>
<td>(1.5,-2.1)</td>
</tr>
</tbody>
</table>

**Table 2. Edges with membership values**

<table>
<thead>
<tr>
<th>Edges</th>
<th>Membership Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1W2</td>
<td>(0.2, -0.5)</td>
</tr>
<tr>
<td>W1W4</td>
<td>(0.3, -0.6)</td>
</tr>
<tr>
<td>W1W5</td>
<td>(0.4, -0.5)</td>
</tr>
<tr>
<td>W2W6</td>
<td>(0.4, -0.5)</td>
</tr>
<tr>
<td>W2W3</td>
<td>(0.3, -0.6)</td>
</tr>
<tr>
<td>W3W4</td>
<td>(0.2, -0.5)</td>
</tr>
<tr>
<td>W3W7</td>
<td>(0.4, -0.5)</td>
</tr>
<tr>
<td>W4W8</td>
<td>(0.4, -0.5)</td>
</tr>
<tr>
<td>W5W6</td>
<td>(0.3, -0.6)</td>
</tr>
<tr>
<td>W5W8</td>
<td>(0.2, -0.5)</td>
</tr>
<tr>
<td>W6W7</td>
<td>(0.2, -0.5)</td>
</tr>
<tr>
<td>W7W8</td>
<td>(0.3, -0.6)</td>
</tr>
</tbody>
</table>

Then, \(n = 8, h = (h_1, h_2) = (0.9, -1.6), W_1 = \{w_1, w_5, w_6, w_8\}, W_2 = \{w_2, w_4, w_5, w_7\}, P = (P_1, P_2) = (1.5, -2.1) \text{ and } Q = (Q_1, Q_2) = (1.2, -1.8).\)

**Theorem 3.7.** If \((W,F)\) is a strongly regular bipolar fuzzy graph which is strong then \(\overline{G}\) is a \((h_1, h_2)\)-regular.

**Proof.** Let \((W,F)\) be a strongly regular bipolar fuzzy graph. Then by definition \(G\) is \((h_1, h_2)\)-regular. Since \(G\) is strong, we have...
Now, since $G$ is strong, the degree of a vertex $w_i$ in $G$ is $d_{G}(w_i) = \left( d_{G}^{+}(w_i), d_{G}^{-}(w_i) \right)$ where $d_{G}^{+}(w_i) = \sum_{\forall w_j, w_iw_j \in F} \mu_{F}(w_i, w_j)$, $d_{G}^{-}(w_i) = \sum_{\forall w_j, w_iw_j \notin F} \mu_{F}(w_i, w_j)$. Hence $d_{G}(w_i) = (h_1, h_2)$ for all $w_i \in V$. So $G$ is a $(h_1, h_2)$-regular bipolar fuzzy graph.

**Theorem 3.8.** Let $G=(W,F)$ be a strongly regular bipolar fuzzy graph. Then $G$ is a strongly regular graph if and only if $\overline{G}$ is a strongly regular bipolar fuzzy graph.

**Proof.** Suppose that $G=(W,F)$ is a strongly regular bipolar fuzzy graph. Then we have to prove that $\overline{G}$ is a strongly regular bipolar fuzzy graph. If $G$ is strongly regular bipolar fuzzy graph and which is strong then $\overline{G}$ is a $(h_1, h_2)$ regular bipolar fuzzy graph by theorem 3.7. Next, let $D_1$ and $D_2$ be the sets of all adjacent vertices and non-adjacent vertices of $G$. That is, $S_1 = \{ w_iw_j / w_iw_j \in F \}$ where $w_i$ and $w_j$ have same common neighborhood $\lambda = (\lambda_1, \lambda_2)$ and $S_2 = \{ w_iw_j / w_iw_j \notin F \}$ where $w_i$ and $w_j$ have same common neighborhood $\delta = (\delta_1, \delta_2)$. Then $S_1 = \{ w_iw_j / w_iw_j \in \overline{F} \}$ where $w_i$ and $w_j$ have same common neighborhood $\delta = (\delta_1, \delta_2)$. And $S_2 = \{ w_iw_j / w_iw_j \notin \overline{F} \}$, where $w_i$ and $w_j$ have same common neighborhood $\lambda = (\lambda_1, \lambda_2)$. Which implies $\overline{G}$ is a strongly regular. Similarly we can prove the converse.

**Theorem 3.9.** A strongly regular bipolar fuzzy graph $G$ is a biregular bipolar fuzzy graph if the adjacent vertices have the same common neighborhood $\lambda = (\lambda_1, \lambda_2) \neq 0$ and the non-adjacent vertices have the same common neighborhood $\delta = (\delta_1, \delta_2) \neq 0$.

**Proof:** Let $G=(W,F)$ be a strongly regular bipolar fuzzy graph. Then we have $d_{G}(w_i) = (h_1, h_2)$ for all $w_i \in W$. Assume that the adjacent vertices have the same common neighborhood $\delta = (\delta_1, \delta_2) \neq 0$. Let $S$ be the set of all non-adjacent vertices. That is $S = \{ w_iw_j / i \neq j, w_i, w_j \in W \}$. Now the vertex partition of $G$ is $W_1 = \{ w_i / w_i \in S \}$ and $W_2 = \{ w_j / w_j \in S \}$. Then $W_1$ and $W_2$ have the same neighborhood degree, since $G$ is a strongly regular. Hence $G$ is a bi-regular bipolar fuzzy graph.

**Definition 3.7.** (i) If the underlying graph $G^*$ is an edge regular graph then $G$ is said to be a partially edge regular bipolar fuzzy graph.

(ii) If $G$ is both edge regular and partially edge regular bipolar fuzzy graph, then $G$ is said to be a fully edge regular bipolar fuzzy graph.
Theorem 3.10. Let $G$ be a bipolar fuzzy graph on $G^*$ such that $(\mu_F^+, \mu_F^-)$ is a constant function. If $G$ is full regular, then $G$ is full edge regular bipolar fuzzy graph.

Proof. Let $(\mu_F^+, \mu_F^-)$ be a constant function. Then, $\mu_F^+(w_jw_j) = c_1$ and $\mu_F^-(w_jw_j) = c_2$ for every $w_jw_j \in F$. where $c_1$ and $c_2$ are constants. Suppose that $G$ is full regular bipolar fuzzy graph then $d_{G}^-(w_jw_j) = (h_1, h_2) = k$ and $d_{G}^+(w_jw_j) = r$, for all $w_j \in V$, where $k$ and $r$ are constants.

$d_{G}^-(w_jw_j) = d_{G}^+(w_jw_j) - 2 = 2r - 2 = \text{Constant}$. Hence $G^*$ is an edge regular graph. Now, since $G$ is regular $d_G^-(w_jw_j) = \left(d_G^+(w_jw_j), d_G^-(w_jw_j)\right)$, for all $w_jw_j \in E$. Where, $d_G^+(w_jw_j) = d_G^+(w_jw_j) + d_G^+(w_jw_j) - 2\mu_F^-(w_jw_j) = k_1 + k_1 - 2c_1 = 2k_1 - 2c_1 = \text{constant}$. Similarly, for all $w_jw_j \in E$, $d_G^-(w_jw_j) = 2k_2 - 2c_2 = \text{constant}$.

Hence, $G^*$ is an edge regular bipolar fuzzy graph. Therefore, $G$ is full edge regular bipolar fuzzy graph.

Theorem 3.11. Let $G$ be a $t$-totally edge regular bipolar fuzzy graph and $t$-partially edge regular bipolar fuzzy graph. Then $S(G) = \frac{nt}{1+t_1}$, where $n = |E|$

Proof: The size of bipolar fuzzy graph $G$ is $S(G) = \left(\sum_{w_jw_j \in E} \mu_F^-(w_jw_j), \sum_{w_jw_j \in E} \mu_F^+(w_jw_j)\right)$. Since $G$ is $t$-totally edge regular bipolar fuzzy graph i.e., $td_G^+(w_jw_j) = t$ and $G^*$ is $t_1$-Partially edge regular bipolar fuzzy graph i.e. $d_{G}^+(w_jw_j) = t_1$. Thus

$\sum_{w_jw_j \in E} td_G^+(w_jw_j) = \left(\sum_{w_jw_j \in E} d_G^+(w_jw_j) \mu_F^+(w_jw_j) + \sum_{w_jw_j \in E} d_G^-(w_jw_j) \mu_F^-(w_jw_j)\right) + S(G).qt = t_1S(G) + S(G)$.

Hence, $S(G) = \frac{nt}{1+t_1}$.

4. Conclusion

Any dissimilar fuzzy graph hypothesis needs large data for training to be able to help in decision making which is crucial to utilitarian research in science and technology. The new method developed in this paper based on the pattern of unique cases help us to make a better choice in contrast to the established fuzzy graph solutions.

References

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