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Note on Chobanyan-Pecherski condition for the series in Banach spaces

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Abstract.

Chobanyan-Pecherski condition is a sufficient condition for the affinity of the sum range of a series in a normed space. This condition is automatically fulfilled for the null-sequences in finite dimensional normed spaces. In this paper we describe some classes of null-sequences in an infinite dimensional normed space $\boldsymbol{l_p, 1 \le p < \infty}$, *satisfying the mentioned condition.*

Key words: *series, sum range, affine subspace, permutation, collection of signs, Banach space.*

1. Introduction.

Let X be a topological vector space and $\sum_{k=1}^{\infty} x_k$ be a series with terms in X. The set of all permutations (bijections) $\sigma: M \to M$ of a set M, we denote by $\boldsymbol{Sym(M)}$. For $\sigma \in Sym(\mathbb{N})$,

$$\sum_{k=1}^{\infty} x_{\sigma(k)}$$

denotes the ***rearranged series***.

Consider such $x \in X$ that $\sum_{k=1}^{\infty} x_{\sigma(k)}$converges to x for some permutation $\sigma \in Sym(\mathbb{N})$. The set of all such elements is called the ***sum range*** (or the ***set of sums***) of the series $\sum_{k=1}^{\infty} x_k$ and is denoted by $\boldsymbol{SR(\sum_{k=1}^{\infty} x_k)}$. If $\sum_{k=1}^{\infty} x_{\sigma(k)}$ diverges for all $\sigma \in Sym(\mathbb{N})$ then $SR(\sum_{k=1}^{\infty} x_k)$ is an empty set.

The famous ***Riemann's rearrangement theorem*** says that the sum range of a conditionally convergent series of real numbers is the whole real axes. According to **P. Levy [1]** and **E. Steinitz [2],** in a finite dimensional normed space $SR(\sum_{k=1}^{\infty} x_k)$ is an affine subspace, i.e. a shifted subspace (by the convention an empty set is affine). The same holds in the metrizable locally convex nuclear spaces (**[3, 4, 5, 6, 7]**).

This theorem fails in the infinite dimensional normed spaces. Additional condition on the series in L_p-spaces for the affinity of $SR(\sum_{k=1}^{\infty} x_k)$ was first given by **M. Kadets [8]**. The result of Kadets was refined by **Nikishin [9]**. Later **Chobanyan [10]** found the condition for an abstract normed space, which implied all known results for the infinite dimensional normed spaces:

$$\sum_{k=1}^{\infty} x_k r_k \text{ converges almost sure,} \tag{1.1}$$

where $r_k, k = 1,2, \dots, \infty$ are the Rademacher functions. We denote by $\mathcal{R}(\boldsymbol{X})$ the set of all sequences $(x_k)_{k=1}^{\infty}$in a topological vector space X satisfying the condition (1.1).

Final (in this context) condition for a normed space was obtained by **Chobanyan [11]** and independently by **Pecherski [12].**

We say that a sequence $(x_k)_{k=1}^{\infty}$ in a topological vector space X satisfies the ***Chobanyan-Pecherski Condition****, or* ***CP-condition*** in brief, if for any $\sigma \in Sym(\mathbb{N})$, there exists a collection of signs $\theta_k = \pm 1, k = 1,2, \dots, \infty$ such that the series $\sum_{k=1}^{\infty} \theta_k x_{\sigma(k)}$ converges.

Note that the CP-condition is a sufficient condition for a series in locally bounded metrizable spaces **[13]** and in Frechet spaces **[14]** as well.

Duality between the permutations and signs is established due to the Chobanyan's celebrated Transference Theorem **[11]** (or Pecherski's lemma in **[12]**):

Theorem 1.1. (Chobanyan's Transference Theorem ([11])). Let $x_1, x_2, \cdots, x_n \in X$ be any finite collection of elements of a normed space X and let $\sum_{i=1}^{n} x_i = 0$. Then there exists a permutation $\sigma \in Sym([1, \dots, n])$ such that

$$\max_{1 \le j \le n} \left\| \sum_{i=1}^{j} x_{\sigma(i)} \right\| = \min_{\theta} \max_{1 \le j \le n} \left\| \sum_{i=1}^{j} \theta_i x_{\sigma(i)} \right\|, \tag{1.2}$$

where the minimum is taken over the collections of signs $\theta_k = \pm 1, k = 1,2, \dots, n$. In fact, permutation σ in (1.2) is the one, which minimizes the left-hand side expression.

In some references *CP*- condition is called as the $(\boldsymbol{\sigma}, \boldsymbol{\theta})$***-condition***. Below, for the brevity, we denote by $\boldsymbol{CP(X)}$ the set of all sequences $(x_k)_{k=1}^{\infty}$in X satisfying the *CP-condition*. Obviously we have $CP(X) \subset c_0(X)$, where $c_0(X)$ denotes the set of null-sequences of X. For the F-space (in particular for the normed space) X the following holds **[15]**:

$$\mathcal{R}(X) \subset CP(X).$$

Converse inclusion is not valid even for the real numbers, where $CP(\mathbb{R}) = c_0(\mathbb{R})$ and (1.1) is equivalent to $\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \alpha_k \in \mathbb{R}$.

A sequence $(x_k)_{k=1}^{\infty}$ in a topological vector space X is called **sign-convergent**, if the series $\sum_{k=1}^{\infty} \theta_k x_k$ converges for some collection of signs $\theta_k = \pm 1, k = 1,2, \dots, \infty$. The set of all sign-convergent sequences we denote by $\boldsymbol{S(X)}$**.** Obviously

$$CP(X) \subseteq S(X) \subseteq c_0(X).$$

It's easy to see that if $S(X) = c_0(X)$ then $CP(X) = c_0(X)$.

According to **Dvoretzky, Hanani [16]** and **Barany, Grinberg [17]**

$$CP(\mathbb{R}^d) = S(\mathbb{R}^d) = c_0(\mathbb{R}^d), \qquad 1 \le d < \infty.$$

This celebrated theorem is often called as the "**DH-theorem**" (see e.g. **[18]**). Thus, the *CP*-condition is automatically fulfilled for any null-sequence in a finite dimensional normed space. Moreover, due to **[5]** this holds true for the countable product of real lines - $\mathbb{R}^{\mathbb{N}}$

$$CP(\mathbb{R}^{\mathbb{N}}) = c_0(\mathbb{R}^{\mathbb{N}}).$$

In **[7]** and **[14]** the validity of this statement for the general metrizable locally convex nuclear spaces is conjectured.

For an infinite dimensional normed space, the algebraic structure of $CP(X)$ and $S(X)$ is not well investigated. One can provide an example showing that $S(X)$ is not linear. Worthy to note **Beck's** results **[19, 20]** describing some type of sequences in $CP(l_\infty)$.

Characterization of the Rademacher condition (1.1) in Banach spaces can be given in terms of a type and a cotype of the underlying space. We can say even more when X is a Banach sublattice of L_0 with finite cotype (e.g., X is $L_p, 1 \le p < \infty$ or a certain Orlicz space). In this case (see **[21]**) condition (1.1) is equivalent to the condition

$$\left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2} \in X. \tag{1.3}$$

In contrast, we do not have "good" characterization of the CP-condition yet. The main aim of the paper is to give some specific examples of sequences $(x_k)_{k=1}^{\infty}$ belonging to $CP(l_p)$ provided that the Rademacher condition (1.1) does not hold, i.e. $(x_k)_{k=1}^{\infty} \in CP(l_p) \setminus \mathcal{R}(l_p)$.

It should be noted that the CP-condition is not a necessary condition for the affinity of the sum range in the infinite dimensional normed spaces. Appropriate example is constructed in a Hilbert space **[14]. Barany [22]** suggests the method of construction of series with the affine sum range in the spaces $l_p, 1 \le p \le \infty$, which seems to be independent of the Rademacher condition (1.1) and the CP-condition. For more details about the development of the topic the reader is advised to see **[23, 24, 15].**

2. CP-condition in the spaces $l_p, 1 \le p < \infty$.

Let $X = l_p, 1 \le p < \infty$, the Banach space of p-absolutely summing sequences of real numbers with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} |\alpha_j|^p\right)^{\frac{1}{p}}, x = (\alpha_j)_{j=1}^{\infty}.$$

Let us construct a series in l_p which satisfy the CP-condition but does not satisfy (1.1). Thus, we construct a sequence

$$x_n = (\alpha_{nj})_{j=1}^{\infty} \in l_p, \ \ n \in \mathbb{N} \tag{2.1}$$

with the following properties:

$$(x_n)_{n=1}^{\infty} \notin R(l_p),$$

which, according to (1.3), in our case is equivalent to

$$\sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} \alpha_{nj}^2\right)^{\frac{p}{2}} = \infty, \tag{2.2}$$

the negation of condition (1.1), and

$$(x_n)_{n=1}^{\infty} \in CP(l_p). \tag{2.3}$$

Consider the set of all sequences of real numbers such that the corresponding series absolutely converges:

$$c_{abs} = \left\{(a_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n| < \infty\right\}.$$

For any given sequence $A \equiv (a_n)_{n=1}^{\infty}$, denote

$$D_A \equiv \{(b_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}: |b_n| \le |a_n|\}.$$

Obviously $D_A \subset c_{abs}$ for any $A \in c_{abs}$. Moreover, the following lemma holds:

Lemma 2.1. Let $A \in c_{abs}$ and $B_j = (b_{nj})_{n=1}^{\infty}, j \in \mathbb{N}$, be any sequence of sequences $B_j \in D_A$. Then for any permutation $\sigma \in Sym(\mathbb{N})$ and a collection of signs $(\varepsilon_n)_{n=1}^{\infty}$

$$\lim_{k\to\infty} \sup_j \sum_{n=k}^{\infty} |b_{\sigma(n)j}| = 0, \tag{2.4}$$

$$\lim_{k\to\infty} \sup_j \left|\sum_{n=k}^{\infty} \varepsilon_n b_{\sigma(n)j}\right| = 0. \tag{2.5}$$

The proof follows from the domination inequality

$$\left|\sum_{n=k}^{\infty}\varepsilon_n b_{\sigma(n)j}\right| \le \sum_{n=k}^{\infty}\left|b_{\sigma(n)j}\right| \le \sum_{n=k}^{\infty}\left|a_{\sigma(n)}\right| \; k,j \in \mathbb{N}.$$

Proposition 2.2. Let $1 \le p < \infty$, $A \in c_{abs}$ and let $d \in \mathbb{N}$. Fix some sequences $B_j \in D_A, j \in \mathbb{N}$. Let further $c_{ni}, n \in \mathbb{N}, i = 1,2,..,d$, and $\beta_j, j \in \mathbb{N}$, be real numbers so that

$$\lim_{n\to\infty} c_{ni} = 0, \qquad \sum_{n=1}^{\infty} c_{ni}^2 = \infty, \qquad i = 1,2,\dots,d,$$

$$\sum_{j=1}^{\infty}\left|\beta_j\right|^p < \infty.$$

Consider the sequences $(\alpha_{nj})_{j=1}^{\infty}, n \in \mathbb{N}$, where

$$\alpha_{nj} = c_{nj} \; when \; 1 \le j \le d \; and \; \alpha_{nj} = b_{n,j-d} \; when \; j > d.$$

Then the sequence $(x_n)_{n=1}^{\infty}, x_n = (\alpha_{nj}\beta_j)_{j=1}^{\infty}$, satisfies (2.1), (2.2) and (2.3).

Proof. Clearly (2.1) and (2.2) hold. To prove (2.3), note first that due to the DH theorem, for some collection of signs $(\theta_n)_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty}\theta_n x_n'$ converges in $\mathbb{R}^d$, where x_n' are the appropriate projections. Besides, (2.5) implies that

$$\lim_{k\to\infty}\left|\sum_{n=k}^{\infty}\varepsilon_n\alpha_{nj}\right| = 0 \;\; uniformly\; for\; (\varepsilon_n)_{n=1}^{\infty} \in \{-1,1\}^{\mathbb{N}} \; and \;\; j > d.$$

Thus, there exists a strictly increasing sequence of indices $(n_l)_{l=1}^{\infty}$ so that

$$\left|\sum_{n=n_l}^{m}\theta_n\alpha_{nj}\right| \le 2^{-l} \; as \; m \ge n_l \; and \; j \in \mathbb{N}.$$

This implies that for any $m, n_l \le m < n_{l+1}$

$$\left\|\sum_{n=n_l}^{m}\theta_n x_n\right\|^p = \sum_{j=1}^{\infty}\left|\sum_{n=n_l}^{m}\theta_n\alpha_{nj}\beta_j\right|^p \le \sum_{j=1}^{\infty}\left|\beta_j\right|^p\left|\sum_{n=n_l}^{m}\theta_n\alpha_{nj}\right|^p \le \sum_{j=1}^{\infty}\left|\beta_j\right|^p 2^{-lp} \le C2^{-lp}.$$

Hence

$$\sum_{l=1}^{\infty}\max_{n_l\le m<n_{l+1}}\left\|\sum_{n=n_l}^{m}\theta_n x_n\right\| < \infty.$$

Consequently, the series $\sum_{n=1}^{\infty}\theta_n x_n$ converges in l_p and as due to (2.5), conditions are invariant with respect to the permutations, (2.3) is proved.

Remark 2.3. In the above construction one can take e.g.

$$|\alpha_{n1}| = \frac{1}{\sqrt{n}}, \qquad |\alpha_{nj}| = \frac{1}{n^{1+\omega_j}}, j = 2,3, \dots,$$

where ω_j are real numbers with $inf\; \omega_j > 0$.

Proposition 2.2 and Chobanyan-Pecherski theorem readily imply

Corollary 2.4. Let $(x_n)_{n=1}^{\infty}$ be a sequence of the Proposition 2.2. Then $SR(\sum_{n=1}^{\infty} x_n)$ is an affine, closed subspace of l_p .

REFERENCES

[1] Lèvy P. Sur les series semi-convergentes. Nouv. Ann. Math., 1905, (4) 5, 506-511.
[2] Steinitz E. Bedingt convergente Reihen konvexe Systeme. J. Reine Angew. Math., 1913, 143,128-175.
[3] Wald A. Bedingt konvergente Reihen von Vektoren im R_{ω}. Ergebnisse Math. Kolloqu. (1933), 5, 13-14.
[4] Trojanski S. On conditionally convergent series in certain F-spaces, Teor. Funktsii, Funktsional. Anal. i Prilozhen. (1967), 5, 102-107 (Russian).
[5] Katznelson Y. , McGehee O. C. Conditionally convergent series in R^{∞}. Michigan Math. J., (1974), 21, no. 1, 97-106.
[6] Banaszczyk W. The Steinitz theorem on rearrangement of series for nuclear spaces. J.Reine Angew. Math., 1990, 403, 187-200.
[7] Banaszczyk W. Additive Subgroups of Topological Vector Spaces. Springer-Verlag, 1991.
[8] Kadets M.I. On conditionally convergent series in L_p-spaces. Uspekhi Matem.Nauk 1954, 9, 107-109.
[9] Nikishin E. M. On convergent rearrangements of functional series. Mat. Zametki (1967), 1, 129–136. (Russian).
[10] Chobanyan S.A. Structure of the set of sums of a conditionally convergent series in a normed space. Mat. Sb. (N.S.), 1985, v.128 (170),no 1(9), 50–65. (English transl. Math. USSR-Sb. 1987, 56, 49-62).
[11] Chobanyan S.A., Giorgobiani G. A problem on rearrangements of summands in normed spaces and Rademacher sums. Probab. Theory on Vector Spaces IV. Proc..Conf.Lancut, 1987, Lecture Notes Math. 1989, 1391, 33-46.
[12] Pecherski D.M. Rearrangements of series in Banach spaces and arrangements of signs. Matem Sb. 1988, 135, 24-35.
[13] Giorgobiani G. Structure of the set of sums of a conditionally convergent series in a p-normed space. Bull. Acad. Sci. Georgian SSR, 1988, 130, 480-484.
[14] Chasco M.-J., Chobanyan S. On rearrangements of series in locally convex spaces. Michigan Math. J. (1997), 44, no. 3, 607-617.
[15] Giorgobiani G. Rearrangements of series. Journal of Mathematical Sciences, 2019, Vol. 239, No. 4, p. 438 -543. DOI 10.1007/s10958-019-04315-9
[16] Dvoretzky A., Hanani C. Sur les changements des signes des termes d'une serie a termes complexes, C.R. Acad. Sci. Paris. 1947, 255, 516-518.
[17] Barany I., Grinberg V. S., On some combinatorial questions in finite-dimensional spaces, Linear Algebra Appl. 41 (1981), p. 1-9.
[18] Garcia J.N. Sobre ciertas variedades de grupos nucleares (on certain types of nuclear groups). Doctoral thesis, 2001.
[19] Beck J. Balancing families of integer sequences. Combinatorica 1981, 1(3), 209 – 216.
[20] Beck J. A Discrepancy Problem: Balancing Infinite Dimensional Vectors. Chapter in the book: Number Theory – Diophantine Problems, Uniform Distribution and Applications (Festschrift in Honour of Robert F. Tichy's 60th Birthday), Editors C. Elsholtz and P. Grabner, 2017, p. 61-82.
[21] Gorgadze Z. G., Tarieladze V. I., Chobanyan S. A. Gaussian covariances in Banach sublattices of the space L_0. (Russian) Dokl. Akad. Nauk SSSR. (1978), 241, no. 3, 528-531.

[22] Barany I. Permutations of series in infinite-dimensional spaces, Math. Notes 46. (1989) 895-900.
[23] Halperin I., Ando T. Bibliography: Series of Vectors and Riemann sums. Hokkaido University, Sapporo, 1989.
[24] Kadets M.I., Kadets V.M. Series in Banach Spaces. Birkhauser. 1997.

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