On One Iterative Method for Solving the Difference Scheme for Quasilinear Elliptic Equation

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Abstract

The present article considers the iterative method for the numerical solution of the Dirichlet problem for quasilinear elliptic equation. For constructing the iterative method, the difference analog of the Grin function for the Laplace operator is used.

The uniform convergence of the iterative method is proved, as well as the convergence of the solution of the difference scheme to the exact solution of the initial differential problem.

Keywords: iterative method, difference scheme, elliptic equation, convergence.

Many established processes of different physical nature lead to partial differential equations of elliptic type. It is enough to specify stationary problems of thermal conductivity and diffusion, theory of elasticity, theory of filtration, etc.

Exact (analytical) solutions of boundary problems for elliptic equations are possible to get only in special cases. The universal method of solving such problems is a method of finite differences. The present paper is devoted to the investigation of one difference scheme for solving the quasilinear elliptic equation.

There is a substantial literature devoted to the difference methods of the solution of boundary problems for elliptic equations. The authors indicate a selected bibliography [1] – [5] with cited there references.

Let us consider the Dirichlet problem for quasilinear elliptic differential equation in p-dimensional parallelepiped $R_p = \{x = (x_1, x_2, \cdots, x_p) \mid 0 \le x_\alpha \le l_\alpha, \quad \alpha = 1,2,\cdots,p\}$ with boundary Γ:

$$Lu \equiv \sum_{\alpha=1}^{p} L_\alpha u = -\varphi(x,u), \quad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}, \tag{1}$$

$$u|_\Gamma = 0. \tag{2}$$

Suppose the problem (1)-(2) has unique, sufficiently smooth solution (see e.g. [6]). Later, we will impose some additional restrictions on the function $\varphi(x,u)$.

Let us introduce the difference grid, uniform to each direction x_α, with boundary $\gamma = \{x \in \Gamma\}$:

$$\bar{\omega}_h = \{x = (i_1 h_1, \cdots, i_p h_p) \in R_p, \quad i_\alpha = 0,\ 1,\ 2, \dots, N_\alpha;\ h_\alpha N_\alpha = l_\alpha,\ \alpha = \overline{1,\ p}\}.$$

Further, we will use the notations from [7].

For the approximation of the problem (1), (2) we consider the following difference scheme:

$$\Lambda y + \varphi(x,u) = 0,\ \ x \in \omega_h\ ,\ \ y|_\gamma = 0, \tag{3}$$

$$\Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha,\ \ \Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha}.$$

Denote by $G(x,\xi)$ the Grin function for the operator Λ with zero values at the boundary of the region, that is function which satisfies the following conditions:

$$\Lambda G(x,\xi) = -\frac{\delta(x,\xi)}{H},\ \ x \in \omega_h,\qquad \xi \in \omega_h,\ \ H = \prod_{\alpha=1}^{p} h_\alpha,$$

$$\delta(x,\xi) = \begin{cases} 1, & when\ x = \xi \\ 0, & when\ x \neq \xi \end{cases}$$

$$G(x,\xi) = 0,\ if\ \ x \in \omega_h,\ \ \xi \in \gamma.$$

For $G(x,\xi)$ the following formula is true (see e.g. [8],[9]):

$$G(x,\xi) = \sum_{\alpha=1}^{N} \frac{y_i(x)y_i(\xi)}{\lambda_i^{(h)}}, \tag{4}$$

where $y_i, \lambda_i^{(h)},\ i = 1,2,\cdots,N$ – eigenfunctions and eigenvalues of the following problem:

$$\Lambda y + \lambda^{(h)} y = 0,\ \ \ x \in \omega_h,\ y = 0,\ if\ x \in \gamma,$$

and N is a number of interior nodes $N = \prod_{\alpha=1}^{p}(N_\alpha - 1)$.

It’s possible to write down explicitly eigenfunctions and eigenvalues for the Laplace difference operator in the parallelepiped (see e.g. [10], [11]):

$$y_k(x) = \mu_{1,k_1}(x_1)\,\mu_{2,k_2}(x_2)\cdots\mu_{p,k_p}(x_p), \tag{5}$$

where

$$\mu_{\alpha,k_\alpha}(x_\alpha) = \frac{\sqrt{2}}{\sqrt{l_\alpha}}\sin\frac{k_\alpha \pi x_\alpha}{l_\alpha},\ \ \ k_\alpha = 1,2,\cdots,N_\alpha - 1.$$

$$\lambda_K^{(h)} = \lambda_{1k_1}^{(h)} + \lambda_{2k_2}^{(h)} + \cdots + \lambda_{pk_p}^{(h)},\ \ \lambda_{\alpha k_\alpha}^{(h)} = \frac{4}{h_\alpha^2}\sin^2\frac{k_\alpha \pi h_\alpha}{2l_\alpha}, \tag{6}$$

where $K = (k_1,k_2,\cdots,k_p),\ \ k_\alpha = 1,2,\cdots,N_\alpha - 1,\ \ \alpha = 1,2,\cdots,p.$

Here $\mu_{\alpha,k_\alpha}(x_\alpha),\ \lambda_{\alpha k_\alpha}^{(h)}$ are the eigenfunctions and eigenvalues of the following problem

$$\Lambda_\alpha u_\alpha(x_\alpha) + \lambda_K^{(h)} u_\alpha(x_\alpha) = 0,\qquad 0 < x_\alpha < l_\alpha,\qquad u_\alpha(0) = u_\alpha(l_\alpha) = 0.$$

Note, that the eigenfunctions (5) for the operator are orthonormal.

Thus, considering (5) and (6), we will get

$$y_k(x) = \left(2^{p/2}/\sqrt{\bar{l}}\right)\prod_{\alpha=1}^{p}\sin\frac{k_\alpha \pi x_\alpha}{l_\alpha},\qquad \bar{l} = \prod_{\alpha=1}^{p} l_\alpha\,,$$

$$\lambda_K^{(h)} = \sum_{\alpha=1}^{p} \frac{4}{h_\alpha^2} \sin^2 \frac{k_\alpha \pi h_\alpha}{2 l_\alpha}, \quad K = (k_1, k_2, \cdots, k_p).$$

The last equality allows us to write down the Grin function evidently for the Laplace operator with zero boundary conditions.

Suppose $z(x)$ and $v(x)$ are some grid functions, defined on ω_h. Let us define a scalar product of these functions using the following formula:

$$(z, v) = \sum_{x \in \omega_h} z(x) v(x) H$$

For these functions we will introduce the following norm:

$$\|z\|_C = \max_{x \in \bar{\omega}_h} |z(x)|.$$

Let us prove the following Lemma.

Lemma. For all $x \in \omega_h$ when $p = 2,3$, the following estimation is valid:

$$(G^2(x,\xi), 1) \le \frac{l_0^4}{8}, \qquad where \quad l_0 = \max_\alpha l_\alpha. \tag{7}$$

Proof. We can get the estimation (7) if we use the known estimations of eigenfunctions and eigenvalues of the difference operator for the equality (4) (see [10]). Indeed, due to the orthonormality of eigenfunctions we will get

$$(G^2(x,\xi), 1) = \sum_{x \in \omega_h} G^2(x,\xi) H = \sum_{K=1}^{N} \left(\frac{y_K}{\lambda_K^{(h)}} \right)^2.$$

But when $p = 2,3$, the following estimation takes place

$$\sum_{K=1}^{N} \left(\lambda_K^{(h)} \right)^{-2} \le \frac{l_0^4}{16} \left(p^{-2} + \frac{\pi}{2(4-p)} \right) < \frac{l_0^4}{8},$$

which allows to get the desired estimation.

After construction of the difference Grin function, instead of the linear system (3) we can consider the equivalent system

$$y(x) = \Big(G(x,\xi), \varphi\big(\xi, y(\xi)\big) \Big), \; x \in \bar{\omega}_h. \tag{8}$$

Substituting the value of $y(\xi)$ in the right-side of the equality (8), we will get

$$y(x) = \bigg(G(x,\xi), \varphi\Big(\xi, \big(G(\xi,\eta), \varphi(\eta, y(\eta)) \big)\Big) \bigg), \; x \in \bar{\omega}_h. \tag{9}$$

For the investigation of nonlinear system (9) and construction of its approximate solution the following iterative method is used:

$$y^{(r+1)}(x) = \bigg(G(x,\xi), \varphi\Big(\xi, \big(G(\xi,\eta), \varphi(\eta, y^{(r)}(\eta)) \big)\Big) \bigg), \tag{10}$$

$$x \in \bar{\omega}_h, \; r = 0, 1, 2, \cdots.$$

where the superscript denotes the roll number of iteration.

Let us investigate the convergence of the iterative method (10). Suppose the function $\varphi(x,u)$ in the area of definition $\{x \in R;\ -\infty < u < +\infty\}$ satisfies the Lipschitz condition

$$|\varphi(x,u_1) - \varphi(x,u_2)| \leq L|u_1 - u_2|. \quad (11)$$

Let us consider the set of functions Φ_h, which consists of the functions $u_h \equiv u_h(x),\ x \in \bar{\omega}_h$, defined on the grid $\bar{\omega}_h$. We will define a distance between two elements u_k and v_k of this set using the following equality

$$\rho(u_k, v_k) = \max_{x \in \bar{\omega}_h} |u_k(x) - v_k(x)|.$$

This will lead to the normal space Φ_h, for each element of which the formula (9) defines the map $z_k = Au_k$, which maps the space Φ_h to itself.

If two elements $u_h \in \Phi_h$ and $\bar{u}_h \in \Phi_h$ are given, we will have

$$z_h(x)\text{-}\bar{z}_h(x)=\Big(G(x,\xi),\Big[\varphi\Big(\xi,\big(\eta,\varphi(\eta,u_h(\eta))\big)\Big) - \varphi\Big(\xi,\big(G(\xi,\eta),\varphi(\eta,\bar{u}_h(\eta))\big)\Big)\Big]\Big).$$

From this equality, considering the estimation (5) of the difference Grin function and inequality (11), after some transformation we will get

$$\rho(z_h,\bar{z}_h) \leq l_0^4\, L^2 \rho(u_h,\bar{u}_h) = \alpha^2 \rho(u_h,\bar{u}_h), \quad (12)$$

where $\alpha = L\, l_0^2$.

If $0 < \alpha < 1$, then from the inequality (10) follows that the map (10) is a shrinking map and based on Banach theorem [12] one can conclude that the nonlinear system of equations (9) has a unique solution $y(x)$. This solution is a limit of the sequence $\{y^{(r)}(x)\}, r = 0,1,2,\cdots$, generated by the iterative process (10). The build of sequential approximations can be started from an arbitrary element of the space Φ_h. Besides this the following estimation is true:

$$\rho\big(z^{(r)},z\big) \leq \frac{\alpha^{2r}}{1-\alpha^2}\rho\big(z^{(0)},z^{(1)}\big). \quad (13)$$

Thus, we proved the following theorem

Theorem 1. If the condition (11) is true, the iterative process (10) converges when $L\, l_0^2 < 1$ and $p = 2,3$, with a speed of infinitely decreased geometric progression.

Let us now investigate convergence of the solution of the difference scheme (3) to the solution of the differential problem (1)-(2) and estimate an error.

We will consider the function $z = y - u$, where y is a solution of the problem (3), and u is a solution of the problem (1)-(2). Then we will get the following problem for the function z:

$$\Lambda z + [\varphi(x,u) - \varphi(x,y)] + |h|^2 p(x) = 0$$

where $|p(x)| \leq E$. E is a defined constant - it depends on maximal values of the fourth order derivatives of the function $u(x)$, and

$$|h| = \prod_{\alpha=1}^{p} h_\alpha^2.$$

Repeating abovementioned discussion for function z we will get the following equality

$$z(x) = \Big(G(x,\xi),\Big[\varphi\Big(\xi,\big(G(\xi,\eta),\varphi(\eta,u(\eta)) - |h|^2 p(\eta)\big)\Big) - \varphi\Big(\xi,\big(G(\xi,\eta),\varphi(\eta,y(\eta))\big)\Big)\Big]\Big)$$

After some easy transformations, considering the estimation of difference Grin function, from the last equality follows, that

$$\|z\|_C \leq L^2\, l_0^4\, \|z\|_C + |h|^2\, l_0^4\, EL + |h|^2\, l_0^4\, E^2,$$

or

$$\|z\|_C \leq |h|^2 \frac{l_0^4\, EL + l_0^4\, E^2}{1 - L^2\, l_0^4}, \qquad (14)$$

as due to the sufficient condition of convergence of iterative process (10), the inequality $L^2\, l_0^4 < 1$ is valid.

Thus, if the sufficient condition of the convergence of iterative process (10) is true then the estimation (14) of the error is valid and the solution of the difference problem converges uniformly to the correspondent solution of the boundary problem (1)-(2).

It means that we proved the following theorem.

Theorem 2. If $L^2\, l_0^4 < 1$, then when $p = 2,3$, the solution of the system (9) converges uniformly to the solution of the boundary problem (1)-(2) with a speed $O(h^2)$.

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