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ANGULAR MOMENTUM CAUSED BY ANY SHAPE VORTEX LINE

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ABSTRACT. **The contribution of any shape vortex line to the angular momentum of the liquid confined to the axially symmetric vessel is considered. The general formulae, which do not require the knowledge of the velocity distribution of the liquid, are received (only the shape and disposition of vortex and wall are enough).**

1.INTRODUCTION

The contribution $\vec{r} \times \vec{v}\rho dV$ of each element ρdV is to be integrated over the volume V of a liquid to determine its angular momentum $\vec{L}$ (ρ is the density, and v is the velocity). The existence of a vortex line causes the inversely proportional dependence of the velocity on the distance from this line. Because of it the contribution of the remote part of a liquid to angular momentum is more sufficient than the contribution of the vortex core though in its vicinity the velocity tends to infinity (the product rv is finite as well as in far areas, where the velocity tends to zero). That is why the angular momentum cannot be determined by rough but simple and effective estimations with a weak dependence on the form of wall and of the distance to it (such estimations are possible and widely used for energy of the vortex in hydrodynamics of superfluid He II [1], see also [2].

During the derivation of the formula of critical velocity of the first vortex generation Vinen [3] obtained the following expression of the angular momentum of the vortex, which is disposed along the axis of a cylinder:

$$L = \frac{1}{2}\rho\Gamma R^2 H, \tag{1}$$

where H is the height of the cylinder, R is its radius, and Γ is a circulation, which was quantized in Vinen's paper, as well as in [4-7] mentioned below, in the units of $2\pi\hbar/m$ (m is the mass of helium atom). But the structure of the vortex core (the size of this formation is of order of 3 $\overset{0}{A}$) is neglected as if the incompressible liquid was considered. The same approximation is exploited in this paper (if one has in mind the superfluid component of He II denotations ρ and Γ must be substituted by ρ_s and $2\pi\hbar/m$).

The interaction in the incompressible liquid spreads in a moment. Because of this three following expressions of angular momentum are received also in the hydrodynamics of superfluid liquid. Namely, if a vortex is disposed parallel to the axis of a cylinder on the distance r_v from this axis being parallel to it then [4,5]:

$$L = \frac{1}{2}\rho\Gamma\left(R^2 - r_v^{\,2}\right)H. \tag{2}$$

If a vortex is disposed along the axis of rotation of a sphere (its diameter), then [6]:

$$L = \frac{2}{3}\rho\Gamma R^3, \tag{3}$$

where R is the radius of the sphere. If a vortex is disposed along the axis of rotation of two concentric spheres with radii R_1, R_2, then [7]:

$$L = \frac{2}{3}\rho\Gamma\left(R_2^3 - R_1^3\right). \tag{4}$$

Eqs. (1,3,4) were received by the direct integration of $r\rho v 2\pi r dr$ with $v = \Gamma / 2\pi r$. In the case of a displaced vortex its image must be taken into account, and the integration is complicated.

We would not be surprised if in classical hydrodynamics the formulae exist, which determine the angular momentum of a vortex in some other geometry, but we had not seen them in the textbooks. May be the point is that in the classical physics the ideal liquid is treated as the excessive idealization of the properties of real liquid. In the real liquid the core of a vortex diffuses. Unlike this situation the vortex filaments in He II are stable and attract more attention.

In several works are calculated the angular moments of many vortex arrays formed in rotating He II (both in cylindrical and spherical geometry). But in this paper we are interesed only in the angular momentum of a single vortex. Just this problem is connected with some aspects of vortex dynamics including the ones connected with pulsar quakes.

2. THE ANGULAR MOMENTUM OF ANY SHAPE VORTEX LINE CONFINED TO THE AXIALLY SYMMETRIC VESSEL

The main restrictions in the following derivation of Eq. (6) is that the vessel has an axially symmetry (its wall is the surface of revolution), and the vortex is supposed to be on one plane with the z-axis of liquid rotation (Fig.1). The equation of the wall is $r = r_w(z)$, the equation of the vortex line is $r = r_v(z)$ (the cylindrical coordinates $r,\ \alpha,\ z$ are used), and both these dependencies are supposed to be single-valued. The possible generalizations see in Sec.5.

In such conditions the integral mentioned above in the beginning of Sec.1 may be written down as:

$$L_z = L = \rho \int_{z_{min}}^{z_{max}} dz \int_0^{r_w(z)} dr \int_0^{2\pi} v_\alpha r d\alpha\,, \tag{5}$$

$L_r = L_\alpha = 0$ under the supposed conditions.

The last integral in Eq. (5) is the circulation around the z-axis. It is zero if there is no vortex in the circle with the radius r on the height z, and it is equal to $\Gamma = \text{const}$ if there is a vortex in such circle. Therefore

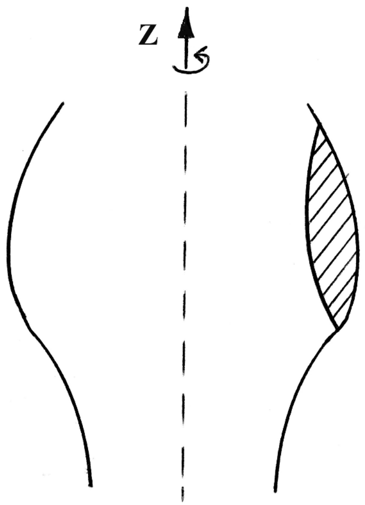


Fig.1. The section of the axially symmetric vessel with a vortex on it. The surface outlined by the vortex and the wall is shaded.

$$L = \frac{1}{2}\rho\Gamma \int_{z_{min}}^{z_{max}} \left(r_w^2 - r_v^2\right) dz\,, \tag{6}$$

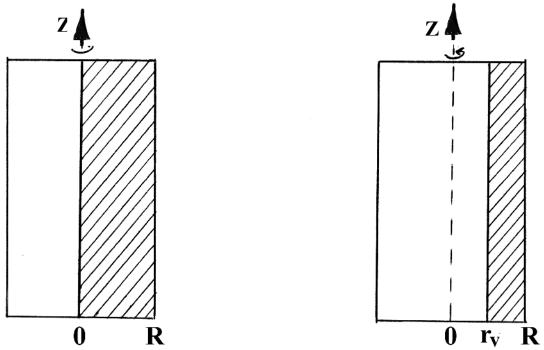
where z_{min} and z_{max} are the limits of the vortex disposition. One can exploit this equation without necessity to determine the velocity distribution in the vessel.

The known formulae (Eqs. (1-4)) may be received from Eq. (6). Using this equation it is easy to get the result for the case where the vortex is disposed on the distance r_v from the axis of rotation of the sphere, being parallel to this axis:

$$L = \frac{2}{3}\rho\Gamma\left(R^2 - r_v^2\right)^{3/2}. \tag{7}$$

3. IMPULSE IN THE SENSE OF KELVIN AND THE ANGULAR MOMENTUM

The impulse in the Kelvin's sense differs on principle from the momentum [8-10]. It is so because the transfer of the momentum which is necessary to set unmoving liquid in the state of given motion (just this is the impulse in the sense of Kelvin) is accompanied by the action of walls which can sufficiently change the momentum of a liquid. In the case of angular momentum the similar situation is less necessary because the action of walls to change the direction of the rotating liquid flow is centripetal and does not change the angular momentum. E.g. let us consider the cases shown in Fig.2. A liquid is rotating around the vortex disposed along the axis of cylinder or parallel to it. To create such a motion one must do a push on the shaded surface in the direction perpendicular to this figure plane. The following centripetal action of walls is oriented radial, and the angular momentum oriented along z-axis remains unchanged.



and parallel to it. The surface which must be pushed to create the considered motion of liquid is shaded.

In general case shown in Fig.1 the Kelvin's impulse is also expressed by the shaded surface [4]:

$$p_K = \rho\Gamma S, \tag{8}$$

where S is the magnitude of the surface limited by the vortex and walls. It is reasonable to think that the angular momentum can be calculated as the product of the Kelvin's impulse on the radius vector R_c of the center of this surface:

$$L = R_c p_K. \tag{9}$$

All preceding formulae for L confirm our proposition. Namely, if the vortex is disposed along the axis of a cylinder then $R_c = R/2$, and $p_K = \rho\Gamma RH$. Their product gives Eq. (1). If a vortex is disposed parallel to the axis of rotation of the cylinder then $R_c = r_v + (R - r_v)/2 = (R + r_v)/2, p_K = \rho\Gamma(R - r_v)H$, and their product gives Eq. (2). If the vortex is disposed along the axis of sphere then $R_c = 4R/3\pi, p_K = \rho\Gamma\pi R^2/2$, and their product gives Eq. (3). If the vortex is disposed along the axis of two concentric spheres then $R_c = 4\left(R_2^3 - R_1^3\right)/3\pi\left(R_2^2 - R_1^2\right)$, $p_K = \rho\Gamma\pi\left(R_2^2 - R_1^2\right)/2$, and their product gives Eq. (4).

Let us prove the equivalence of Eqs. (9) and (6). The center of the imaginary surface which one pushes to create a given motion is on the following distance from the z-axis:

$$R_c = \frac{\int r dS}{\int dS} = \frac{1}{S}\int_{r_{min}}^{r_{max}} zr dr$$

(r_{min}, r_{max} are the minimal and the maximal distances of the surface S from the axis of rotation).

According to Eqs. (8,9) we obtain:

$$L = \rho\Gamma \int_{r_{min}}^{r_{max}} zrdr \, . \tag{10}$$

On the other hand

$$\int_{r_{min}}^{r_{max}} zrdr = -\frac{1}{2}\oint zdr^2 \, .$$

The last integral must be taken along the boundary of the surface S in the positive direction of revolution. Substituting Eqs. (8,10) in Eq. (9) we obtain:

$$L = -\frac{1}{2}\rho\Gamma\oint zdr^2 \, . \tag{11}$$

Eq. (6) may be also transformed in the integral about the same path that gives another expression of L:

$$\int_{z_{min}}^{z_{max}} (r_w{}^2 - r_v{}^2)dz = \oint r^2 dz \, , \tag{12}$$

$$L = \frac{1}{2}\rho\Gamma\oint r^2 dz \, . \tag{13}$$

The equivalence of Eqs. (11,13) means the equivalence of Eqs. (6,9,10). We have obtained these formulae for conditions determined in Sec.2. However some generalizations, described in the following section, provide the possibility to use our general equations (Eqs. (6,9-11,13)) for actually any shape vortex line.

4. GENERALIZATIONS

The simplest generalization, which can be made as compared with Fig.1, is the case where a vortex is placed between the wall and another (also axially symmetric) body. Our general formulae (Eqs. (6,9-11,13)) may be directly exploited in this case. E.g. so may be received Eq. (4) and the following expression for the case where a vortex is disposed between two coaxial cylinders parallel to the axis of their rotation. In this doubly connected area the existence of a circulation Γ_1 is possible on the surface of inner cylinder which gives the additional contribution to the angular momentum. Together with the contribution of a vortex it implies:

$$L = \rho\Gamma_1\left(R_2^2 - R_1^2\right) + \rho\Gamma\left(R_2^2 - r_v^2\right). \tag{14}$$

If a vortex is disposed between two concentric spheres parallel to the axis of their rotation, and the edges of a vortex are placed on the surfaces of both spheres ($r_v < R_1$) then:

$$L = \frac{2}{3}\rho\Gamma\left[\left(R_2^2 - r_v^2\right)^{3/2} - \left(R_1^2 - r_v^2\right)^{3/2}\right], \tag{15}$$

and if the edges of a vortex are placed on the surface only of the outer sphere $r_v > R_1$ then (cf. Eq. (7)):

$$L = \frac{2}{3}\rho\Gamma\left(R_2^2 - r_v^2\right)^{3/2}. \tag{16}$$

We would like to note that in doubly connected area between two cylinders a circulation Γ_1 on the surface of inner cylinder is possible independently from the fact if a vortex exists or not. In the simply connected area between two spheres a circulation on the surface of the inner sphere is possible only if it is caused, according the Stokes theorem, by a vortex that pierces the sphere.

If the functions $r(z)$ which describe walls or a vortex are not single-valued then one must divide the curves r_v, r_w into several single-valued parts and Eq. (6) would be replaced by the sum of several integrals with the own z_{min}, z_{max} and r_{min}, r_{max}. The values of these integrals depend on the absence or existence of a circulation and its sign in the interval considered. In similar way one must deal with Eq. (10) if the dependence $z(r)$ is not single valued. Eqs. (9,11,13) may be used without any changes.

If z-axis divides the shaded area into two parts, then their p_K have opposite signs. But corresponding R_c also have opposite signs. Therefore the sum of two products $p_K R_c$ will appear in Eq. (9). Eqs. (6,11,13) may be used without any changes.

Now let us consider the closed vortex, which has no contact with walls. If we increase the radius of circumference around the axis of rotation, then the part of the vortex, which is on the distance $r_{min}(z)$ from the axis, enters in this circumference earlier than the other ones. This part contributes in Eq. (6) nonzero Γ until the part of vortex with $r_{max}(z)$ and with circulation $-\Gamma$ enters. The contribution of circulation in Eq. (5) for $r_{max} < r < r_w$ is zero. Eq. (6) is valid but r_v, r_w must be substituted by r_{min}, r_{max}. Eqs. (9,11,13) may be used without any changes. It is right also in the case where the axis divides the closed vortex in two parts. But, in this case, in Eq. (9) the product $p_K R_c$ must be substituted by the sum of such products.

5. SUMMARY

The momentary shape and disposition of a vortex and walls completely determine the angular momentum of a liquid at the same moment (let us remind that the interaction in the incompressible liquid spreads in a moment). The motion of a vortex and its stability is not considered in this paper. Eqs. (6,9-11,13) only imply the angular momentum corresponding to definite configuration of a vortex and walls.

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ნებისმიერი ფორმის გრიგალით განპირობებული მოძრაობის რაოდენობის მომენტი

დასკვნა

გრიგალით განპირობებული მოძრაობის რაოდენობის მომენტისათვის ცნობილია მხოლოდ რამდენიმე ფორმულა (ამ სტატიის (1-4)). ეს ფორმულები (გარდა (2)-ისა), გამოიყენება იმ შემთხვე-

ვებისათვის, როდესაც გრიგალი მბრუნავი ჭურჭლის ღერძზეა განლაგებული. ამის მიზეზი, ალბათ, არის ის, რომ, თუ გრიგალი ღერძზე არაა მოთავსებული, ძნელდება სიჩქარეთა განაწილების განსაზღვრა და ინტეგრება. ამ სტატიაში მიღებულია მოძრაობის რაოდენობის მომენტის გამოსათვლელი ფორმულები (6) და (10), რომლებიც არ მოითხოვს სიჩქარეთა განაწილების ცოდნას. საჭიროა ვიცოდეთ მხოლოდ გრიგალისა და კედლის ფორმა (გრიგალის ფორმა ნებისმიერია, კედელისა კი - ბრუნვის ნებისმიერი ზედაპირი). მიღებულია, აგრეთვე, ამავე პირობებში სამართლიანი ფორმულა (9), რომელშიაც გამოყენებულია კელვინის აზრით იმპულსის ცნება. ცნობილია, რომ ეს იმპულსი შეიძლება ძალიან არსებითად განსხვავდებოდეს მოძრაობის რაოდენობისაგან, მაგრამ აქ ნაჩვენებია, რომ აღწერილ პირობებში იგი იძლევა მოძრაობის რაოდენობის მომენტის სწორ მნიშვნელობას. კიდევ ორი ზოგადი ფორმულა (11,13) წარმოგვიდგენს საძიებელ სიდიდეს წირითი ინტეგრალების სახით, რომლებიც გამოითვლება შეკრული გრიგალის მიერ ან გრიგალისა და კედლის მიერ შემოწერილ კონტურებზე. ხუთივე ზოგადი ფორმულა მოიცავს დღემდე ცნობილ ფორმულებს (1-4) და იძლევა სხვა კერძო შემთხვევების განხილვის საშუალებას (ფორმულები (7,14-16)).