

**STATISTICAL DESCRIPTION OF EVOLUTION OF THE
QUANTUM PENDULUM**

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ABSTRACT. The mathematical pendulum is the simplest system having all the basic properties inherent in dynamic stochastic systems. In the present paper we investigate the mathematical pendulum with the aim to reveal the properties of a quantum analogue of dynamic stochasticity or, in other words, to obtain the basic properties of quantum chaos.

It is shown that periodic perturbation of the quantum pendulum (similarly to the classical one) in the neighbourhood of the separatrix can bring about irreversible phenomena. As a result of recurrent passages between degenerate states, the system gets self-chaotized and passes from the pure state to the mixed one. Chaotization involves the states, the branch points whose levels participate in a slow "drift" of the system along the Mathieu characteristics. This "drift" is caused by a slowly changing variable field. Recurrent relations are obtained for populations of levels participating in the irreversible evolution process. It is shown that the entropy of the system first grows and, after reaching the equilibrium state, acquires a constant value.

1. INTRODUCTION. FORMULATION OF THE PROBLEM

Dynamic stochasticity is directly connected with the assumption that classical equations of motion may contain nonlinearities which arise when the (exponential) repulsion of phase trajectories occurs at a sufficiently quick rate. In the case of quantum consideration, the dynamics of a system is described by a wave function that obeys a linear equation, while the notion of a trajectory is not used at all. Hence, at first sight it seems problematic to find out the quantum properties of systems whose classical consideration reveals their dynamic stochasticity. A quantum analogue of classical stochastic motion is usually called quantum chaos.

On the other hand, it is of practical interest to investigate parametrically dependent Hamiltonians $H(Q, P, l)$, where (Q, P) is the set of canonical coordinates, l is the parameter describing how the system is related to the external field. The interest in such systems is explained by their use in the study of quantum points and other problems of mesoscopic physics [1].

In most of the papers that deal with parametrically dependent systems, their authors consider the following situation. For $l=0$, the Hamiltonian $H(Q, P, 0)$ is exactly integrable. As l increases, the Hamiltonian $H(Q, P, l)$ becomes nonintegrable and, for a certain value of l_0 , solutions of the classical equations corresponding to $H(Q, P, l_0)$ become chaotic. In the case of quantum consideration, eigenvalues $E_n(l_0)$ and eigenfunctions $\psi_n(l_0)$ are found in the above-mentioned area of parameter values by using the method of numerical diagonalization. In that case, we show interest in the dependence of the parametrical kernel $P(n/m) = \left| \langle \psi_n(l_0 + \delta l) | \psi_m(l_0) \rangle \right|^2$ on a parameter displacement $\delta l \ll l$. The value $P(n/m)$,

averaged statistically over states n , $\bar{\rho}$ can be interpreted as the local density of states. The introduction of $P(r)$ means that we pass from the quantum-mechanical description to the quantum statistical description [2 - 5] carried out by an intuitive reasoning.

In the problems considered in the above-listed papers, the Hamiltonian $H(Q, P, l)$ displays chaos for both parameter values $l = l_0$ and $l = l_0 + \delta l$.

As different from these papers, in the present paper we investigate the situation, in which the Hamiltonian $H(Q, P, l)$ is integrable and becomes nonintegrable after adding a strictly periodic perturbation $\delta l(t)$. As the basic Hamiltonian we take the Hamiltonian of the mathematical pendulum (universal Hamiltonian).

As is known, the Schrodinger quantum-mechanical equation for the universal Hamiltonian is written in the form of the Mathieu equation. The Mathieu-Schrodinger equation for an atom, which is under the action of optical pumping in the area of large quantum numbers, was for the first time obtained by G. Zaslavsky and G. Berman [6]. These authors also performed analysis of quasiclassical states of the Mathieu-Schrodinger equation [6].

The main objective of the present paper is to investigate the behavior of the quantum mathematical pendulum in the area of dynamic stochasticity parameters. As is known [7], this area, called the stochastic layer, lies in the neighborhood of the separatrix of the classical pendulum.

We show here that with the appearance of quantum chaos the pure state passes to the mixed one. In other words, the reversible quantum process transforms to the reversible process of quantum chaos which can be described by a kinetic equation. The common feature of classical dynamic chaos and quantum chaos is, as will be shown below, the irreversibility of their states.

2. A PARAMETRICALLY DEPENDENT HAMILTONIAN

After writing the stationary Schrodinger equation

$$\hat{H}\psi_n = E_n \psi_n \quad (1)$$

for the universal Hamiltonian of the atom+pumping system

$$\hat{H} = -\frac{\partial^2}{\partial\varphi^2} + V$$

$$V = l \cos 2\varphi ,$$

we come to the equation coinciding with the Mathieu equation [8, 9]

$$\frac{d^2\psi_n}{d\varphi^2} + (E_n - V(l, \varphi))\psi_n = 0 , \quad (2)$$

where $E_n \rightarrow \frac{8E_n}{-2\omega'}$ are the introduced dimensionless values, l is the dimensionless amplitude of pumping, $\omega' = \frac{d\omega(I)}{dI}$ is the derivative of nonlinear oscillation frequency $\omega(I)$ with respect to the action I [9].

The Mathieu-Schrodinger equation is characterized by a specific dependence of the spectrum of eigenvalues $E_n(l)$ and eigenfunctions $\psi_n(\varphi, l)$ on the parameter l (see Fig.1). On the plane (E, l) with the spectral characteristics (so-called Mathieu characteristics [9]) of the problem, this specific feature manifests itself in the alternation of areas of degenerate G_{\pm} and nondegenerate G states (see [10], Figs. 3,4). The boundaries between these areas pass through the branch points of energy terms $E_n(l)$.

Degenerate and nondegenerate states of the quantum mathematical pendulum were established by studying the symmetry properties of the Mathieu-Schrodinger equation. In [10], by using the symmetry properties of the Mathieu-Schrodinger equation and applying the group theory methods, the eigenvalues for each of the areas G_+ ; G_- ; G were found:

$$G_- \rightarrow \psi_{2n+1}^{\pm}(\varphi) = \frac{\sqrt{2}}{2} (ce_{2n+1}(\varphi) \pm ise_{2n+1}(\varphi))$$

$$\psi_{2n}^{\pm}(\varphi) = \frac{\sqrt{2}}{2} (ce_{2n}(\varphi) \pm ise_{2n}(\varphi)); \quad (3G_-)$$

$$G \rightarrow ce_{2n}(\varphi); ce_{2n+1}(\varphi); se_{2n}(\varphi); se_{2n+1}(\varphi) \quad (3G)$$

$$G_+ \rightarrow \xi_{2n}^{\pm}(\varphi) = \frac{1}{\sqrt{2}}(ce_{2n}(\varphi) \pm ise_{2n+1}(\varphi));$$
$$\zeta_{2n+1}^{\pm}(\varphi) = \frac{1}{\sqrt{2}}(ce_{2n+1}(\varphi) \pm ise_{2n+2}(\varphi)); \quad (3G_+)$$

Here $ce_n(\varphi)$, $se_n(\varphi)$ denote the Mathieu functions [9].

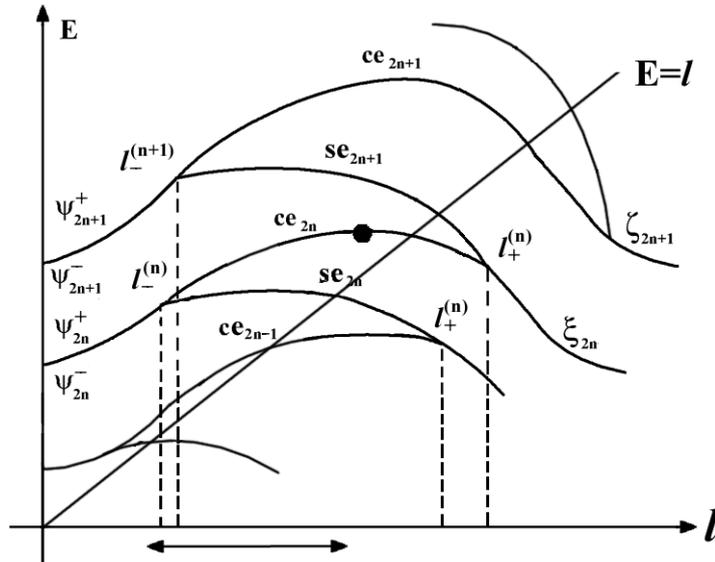


Fig.1. A fragment of the parameter-dependent energy spectrum of the quantum mathematical pendulum (1)

The wave functions $(3G_{\pm})$ and $(3G)$ form the bases of irreducible representations of the respective groups. Each of the four functions $(3G)$ forms a one-dimensional irreducible representation of the Klein group V , while the functions $|$ from $(3G_{-})$ and $|$ from $(3G_{+})$ form the two-dimensional irreducible representations of two invariant subgroups of the group V [10].

Let us assume that the pumping amplitude is modulated by a slowly changing electromagnetic field. The influence of modulation can be taken into account by making a replacement in the Mathieu-Schrodinger equation

$$I(t) \rightarrow I_0 + \Delta I \cos vt, \quad (4)$$

where ΔI is the modulation amplitude expressed in dimensionless units, v is the modulation frequency.

We assume that a gradual change of $I(t)$ may involve some N branch points on the left and on the right side of the separatrix (Fig.1):

$$\Delta I \geq \left| I_+^n - I_-^n \right|, \quad n = 1, 2, \dots, N. \quad (5)$$

After making replacement (4) in the universal Hamiltonian, we obtain

$$\hat{H} = \hat{H}_0 + \hat{H}'(t),$$

$$\hat{H}_0 = -\frac{\partial^2}{\partial \varphi^2} + I_0 \cos 2\varphi, \quad (6)$$

$$\hat{H}'(t) = \Delta I \cos 2\varphi \cos vt. \quad (6')$$

Simple calculations show that the matrix elements of perturbation $\hat{H}'(t)$ with respect to the wave functions (3G) of the nondegenerate area G are equal to zero

$$\langle ce_n | \hat{H}'(t) | se_n \rangle \sim \Delta I \int_0^{2\pi} ce_n(\varphi) \cos 2\varphi se_n(\varphi) d\varphi = 0, \quad (7)$$

where n is any integer number. Therefore perturbation (6') cannot bring about passages between nondegenerate levels.

The interaction $\hat{H}'(t)$, not producing passages between levels, should be inserted in the unperturbed part of the Hamiltonian. The Hamiltonian obtained in this manner can be considered as slowly depending on time.

Thus, in the nondegenerate area G the Hamiltonian can be written in the form

$$\hat{H} = -\frac{\partial^2}{\partial\varphi^2} + l(t)\cos 2\varphi,$$

$$l(t) = l_0 + \Delta l \cos vt. \quad (8)$$

As it has been mentioned above, to different areas on the plane (E, l) we can assign different eigenfunctions $(3G_-)$, $(3G)$, $(3G_+)$. Because of the modulation of the parameter $l(t)$ the system passes from one area to another, getting over the branch points.

3. PASSAGE FROM THE QUANTUM-MECHANICAL DESCRIPTION TO THE KINETIC DESCRIPTION. IRREVERSIBLE PHENOMENA

As different from the nondegenerate states area G , in the areas of degenerate states G_- and G_+ , the nondiagonal matrix elements of perturbation $\hat{H}'(t)$ (6) are not equal to zero. For example, if we take the matrix elements with respect to the wave functions $\psi_{2n+1}^\pm(\varphi)$ (see $(3G_-)$), then for the left degenerate area G_- it can be shown that

$$H'_{+-} = H'_{-+} = \langle \psi_{2n+1}^+ | \hat{H}'(t) | \psi_{2n+1}^- \rangle \approx \Delta l \int_0^{2\pi} \psi_{2n+1}^+ \psi_{2n+1}^{-*} \cos 2\varphi d\varphi \neq 0 \quad (9)$$

Note that the value H'_{+-} has order equal to the pumping modulation depth ΔI .

Analogously to (9), we can write an expression for even $2n$ states as well.

An explicit dependence of $\hat{H}'(t)$ on time given by the factor $\cos vt$ is assumed to be slow as compared with the period of passages between degenerate states that are produced by the nondiagonal matrix elements H'_{+-} . Therefore below the perturbation H'_{+-} will be assumed to be the time-independent perturbation that can bring about passages between degenerate states.

Thus, in a degenerate area the system may be in the time-dependent superpositional state

$$\psi_{2n}(t) = C_n^+(t)\psi_{2n}^+ + C_n^-(t)\psi_{2n}^- . \quad (10)$$

The probability amplitudes $C_n^\pm(t)$ are defined by means of the fundamental quantum-mechanical equation expressing the causality principle [11]. We write such equations for a pair of doubly degenerate states:

$$\begin{cases} -i\Gamma \frac{dC_n^+}{dt} = (E_{0n} + H'_{++})C_n^+ + H'_{+-}C_n^- \\ -i\Gamma \frac{dC_n^-}{dt} = H'_{+-}C_n^+ + (E_{0n} + H'_{--})C_n^- \end{cases} \quad (11)$$

Let us solve system (11). In our case it can be assumed that $H'_{++} = H'_{--}$ and $H'_{+-} = H'_{-+}$. Let us investigate changes that occurred in the state of the system during time ΔT while the system was in the area G_- , assuming that ΔT is a part of the modulation period T , $\Delta T \leq T$.

For arbitrary initial values system (11) has the solution

$$C_n^+(t) = e^{-\frac{1}{2}Et} \left(C_+ \cos\left(\frac{H'}{2}t\right) + iC_- \sin\left(\frac{H'}{2}t\right) \right),$$

$$C_n^-(t) = e^{\frac{i}{2}Et} \left(C_- \cos\left(\frac{H'}{2}t\right) + iC_+ \sin\left(\frac{H'}{2}t\right) \right), \quad (12)$$

where we redenote $\frac{1}{2}H'_{+-} \rightarrow H'$. A slow dependence of the interaction $\hat{H}'(t)$ (6) on time can be taken into account in (12) if we use the replacement $H' = H' \cos \nu t$.

Let motion begin from the state ψ_{2n}^- of the degenerate area. Then as the initial conditions we take

$$C_n^-(0) = 1, \quad C_n^+(0) = 0. \quad (13)$$

Having substituted (13) into (12), for the amplitudes $C_n^\pm(t)$, we obtain

$$C_n^+(t) = i \exp\left(\frac{i}{\tau} Et\right) \sin(\omega t),$$

$$C_n^-(t) = \exp\left(\frac{i}{\tau} Et\right) \cos(\omega t), \quad (14)$$

where $\omega = \frac{2\pi}{\tau} = \frac{H'}{I}$ is the frequency of passages between degenerate states, τ is the passage time.

Note that the parameter ω , which is connected with the modulation depth ΔI , has (like any other parameter) a certain small error $\delta\omega$, which during the time of one passage $t \sim 2\pi/\omega$, leads to an insignificant correction in the phase $2\pi(\delta\omega/\omega)$. But during the time $t \sim \Delta T$, there occurs a great number of oscillations (phase incursion takes place) and, in the case $\Delta T \gg \tau$, a small error $\delta\omega$ brings to the uncertainty of the phase $\sim \Delta T \delta\omega$ which may have order 2π . Then we say that the phase is self-chaotized.

Let us introduce the density matrix averaged over a small dispersion $\delta\omega$:

$$\rho_n^{+-}(t) = \begin{pmatrix} W_n^+(t) & iF_n(t) \\ -iF_n^*(t) & W_n^-(t) \end{pmatrix}, \quad (15)$$

where $W_n^\pm(t) = \overline{|C_n^\pm(t)|^2}$, $\overline{}$ The overline denotes the averaging over a small dispersion $\delta\omega$

$$\overline{A(\omega, t)} = \frac{1}{2\delta\omega} \int_{\omega-\delta\omega}^{\omega+\delta\omega} A(x, t) dx \quad (16)$$

To solve (16) we can write that

$$W_n^+(t) = \overline{\sin^2(\omega t)}, \quad W_n^-(t) = \overline{\cos^2(\omega t)}, \quad F_n(t) = \frac{1}{2} \overline{\sin 2\omega t} \quad (17)$$

After a simple integration of the averaging (16), for the matrix element (17) we obtain

$$W_n^\pm(t) = \frac{1}{2}(1 + f(2\delta\omega t) \cos 2\omega t),$$

$$F_n(t) = F_n^*(t) = \frac{1}{2} f(2\delta\omega t) \sin 2\omega t, \quad (18)$$

$$f(2\delta\omega t) = \frac{\sin 2\delta\omega t}{2\delta\omega t}.$$

At small values of time t , insufficient for self-chaotization ($f(2\delta\omega t) \approx 1$), we obtain

$$W_n^+(t \ll \bar{\tau}) = \sin^2(\omega t), \quad W_n^-(t \ll \bar{\tau}) = \cos^2(\omega t),$$

$$F_n(t \ll \bar{\tau}) = \frac{1}{2} \sin 2\omega t$$

Comparing these values with the initial values (17) of the density matrix elements, we see that the averaging procedure (16), as expected, does not affect them. Thus, for small times we have

$$\rho_n^{+-}(t \ll \bar{\tau}) = \begin{pmatrix} \sin^2 \omega t & \frac{i}{2} \sin 2\omega t \\ -\frac{i}{2} \sin 2\omega t & \cos^2 \omega t \end{pmatrix}. \quad (19)$$

One can easily verify that matrix (19) satisfies the condition $\rho^2(t \ll \bar{\tau}) = \rho(t \ll \bar{\tau})$, which is a necessary and sufficient condition for the density matrix of the pure state.

For times even smaller than $t \ll \tau \ll \bar{\tau}$, when passages between degenerate states practically fail to occur, by taking the limit $\omega t \ll 1$ in (19), we obtain the following relation for the density matrix:

$$\rho_n^{+-}(t=0) = \rho_n^{+-}(t \ll \tau) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (20)$$

This relation corresponds to the initial relation (13) when the system is in the eigenstate ψ_{2n}^- . Let us now investigate the behavior of the system at times $t \geq \bar{\tau}$ when the system gets self-chaotized.

On relatively large time intervals $t \geq \bar{\tau}$, in which the self-chaotization of phases takes place, for the matrix elements we should use general expressions (18). The substitution of these expressions for the matrix elements (18) into the density matrix (15) gives

$$\rho_n^{+-}(t) = \frac{1}{2} \begin{pmatrix} 1 - f(2\delta\omega t) \cos 2\omega t & if(2\delta\omega t) \sin 2\omega t \\ -if(2\delta\omega t) \sin 2\omega t & 1 + f(2\delta\omega t) \cos 2\omega t \end{pmatrix}. \quad (21)$$

Hence, for times $t \geq \bar{\tau}$ during which the phases get completely chaotized, after passing to the limit $\delta\omega t \gg 1$ in (21), we obtain

$$\rho_n^{+-}(t) = \frac{1}{2} \begin{pmatrix} 1 - O(\varepsilon) & iO(\varepsilon) \\ -iO(\varepsilon) & 1 + O(\varepsilon) \end{pmatrix}, \quad (22)$$

where $O(\epsilon)$ is an infinitesimal value of order $\epsilon = \frac{1}{2\delta\omega t}$.

The state described by the density matrix (22) is a mixture of two quantum states ψ_{2n}^+ and ψ_{2n}^- with equal weights. The comparison of the corresponding matrix elements of matrices (22) and (21) shows that they differ in the terms that play the role of quickly changing fluctuations. When the limit is $t \gg \bar{\tau}$, fluctuations decrease as $\sim \frac{1}{2\delta\omega t}$ (see Fig.2 and Fig.3).

Thus the system, which at the time moment $t = 0$ was in the pure state with the wave function ψ_{2n}^- (20), gets self-chaotized with a lapse of time $t \gg \bar{\tau}$ and passes to the mixed state (22). In other words, at the

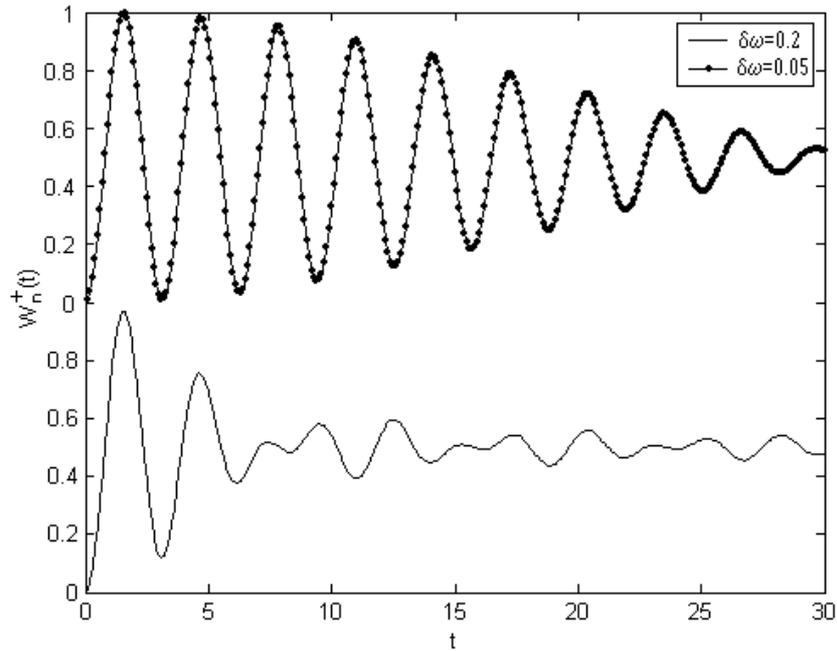


Fig.2. Time-dependence of the diagonal matrix element $W_n^+(t)$ of the density matrix (15), constructed by means of formulas (15),

(18) for the parameter values $\omega = 1/\tau = 1$, $C_n^+(0) = 1$,

$C_n^-(0) = 0$. As clearly seen from the Figure, the higher the dispersion value of the parameter $\delta\omega$, the sooner the stationary

value $W_n^+\left(t > \bar{\tau} \sim \frac{1}{\delta\omega}\right) = \frac{1}{2}$ is achieved

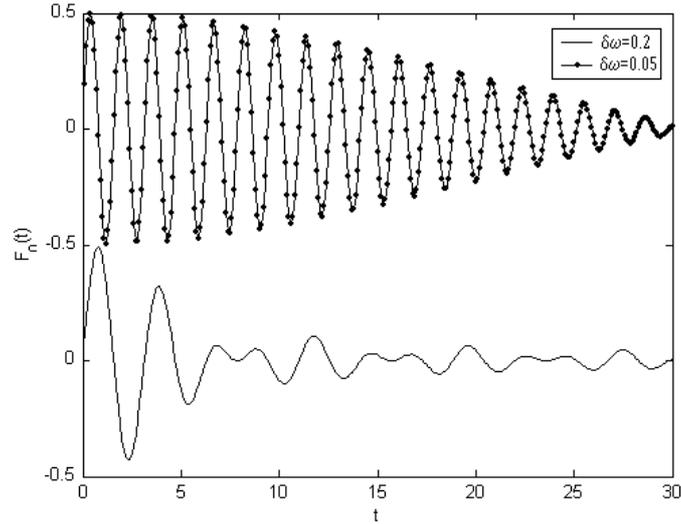


Fig.3. The vanishing of nondiagonal matrix elements of the density matrix (15) with a lapse of time $t > \bar{\tau}$ while the system remained in the degenerate area G_- . The graph is constructed for the parameter values $\omega = 1/\tau = 1$,

with the aid of formulas (15) and (18).

initial moment the system had a certain definite "order" expressed in the form of the density matrix $\rho_n^{+-}(0)$ (20). With a lapse of time the system got self-chaotized and the fluctuation terms appeared in the density matrix (21). For large times $t \gg \bar{\tau}$ a new "order" looking like a macroscopic order is formed, which is defined by matrix (22).

After a halfperiod the system passes to the area of nondegenerate states G (20). In passing through the branch point, there arise nonzero probabilities for passages both to the state ce_{2n} and to the state se_{2n} . Both states ψ_{2n}^+ and ψ_{2n}^- will contribute to the probability that the system will pass to either of the states ce_{2n} and se_{2n} . For the total probability of passage to the states ce_{2n} and se_{2n} we obtain respectively

$$= \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} \psi_{2n}^+(\varphi) ce_{2n}(\varphi) d\varphi \right|^2 + \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} \psi_{2n}^-(\varphi) ce_{2n}(\varphi) d\varphi \right|^2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}, \quad (23)$$

$$= \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} \psi_{2n}^+(\varphi) \text{se}_{2n}(\varphi) d\varphi \right|^2 + \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} \psi_{2n}^-(\varphi) \text{se}_{2n}(\varphi) d\varphi \right|^2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

Thus, in the nondegenerate area the mixed state is formed, which is defined by the density matrix

$$\rho_{2n}^{ik} \left(t \sim \frac{T}{2} \gg \tau \right) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (24)$$

where i and k number two levels that correspond to the states ce_{2n} and se_{2n} .

As follows from (24), at this evolution stage of the system, the populations of two nondegenerate levels get equalized. It should be noted that though the direct passage (7) between the nondegenerate levels is not prohibited, perturbation (6') essentially influences "indirect" passages. Under "indirect" passages we understand a sequence of events consisting of a passage $G \rightarrow G_-$ through the branch point, a set of passages between degenerate states in the area G_- and the reverse passage through the branch point $G_- \rightarrow G$. The "indirect" passages occurring during the modulation halfperiod $T/2$ result in the equalization (saturation) of two nondegenerate levels.

As to the nondegenerate area, the role of perturbation $\hat{H}'(t)$ in it reduces to the displacement of the system from the left branch point to the right one.

It is easy to verify that after states (24) pass to the states of the degenerate area G_+ , we obtain the mixed state which involves four states $\xi_{2n}^{\pm}(\varphi)$ and $\zeta_{2n+1}^{\pm}(\varphi)$ (see Fig.1).

Let us now calculate the probability of four passages from the mixed state ρ_{2n}^{ik} (22) to the states $\xi_{2n}^{\pm}(\varphi)$ and $\zeta_{2n-1}^{\pm}(\varphi)$

$$P(\rho_{2n}^{ik} \rightarrow \xi_{2n}^{\pm}) = \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} (c e_{2n}(\varphi) + s e_{2n}(\varphi)) \xi_{2n}^{\pm}(\varphi) d\varphi \right|^2 = \frac{1}{4},$$

$$P(\rho_{2n}^{ik} \rightarrow \zeta_{2n-1}^{\pm}) = \frac{1}{2} \left| \frac{1}{\pi} \int_0^{2\pi} (c e_{2n}(\varphi) + s e_{2n}(\varphi)) \zeta_{2n-1}^{\pm}(\varphi) d\varphi \right|^2 = \frac{1}{4}. \quad (25)$$

As a result of these passages, in the area G_+ we obtain the mixed state described by the four-dimensional density matrix

$$\rho_{2n,2n+1}^{+-}(t \sim T \gg \tau) = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (26)$$

where the indices of the density matrix (26) show that the respective matrix elements are taken with respect to the wave functions $\xi_{2n}^{\pm}(\varphi)$ and $\zeta_{2n+1}^{\pm}(\varphi)$ of degenerate states of the area G_+ .

It is easy to foresee a further evolution course of the system. At each passage through the branch point, the probability that an energy level will get populated is equally divided between branched states. We can see the following regularity of the evolution of populations for the next time periods.

After odd halfperiods, the population of any n -th nondegenerate level is defined as an arithmetic mean of its population and the population of the nearest upper level, while after even halfperiods -- as an arithmetic mean of its population and the nearest lower level. This population evolution rule can be represented both in the form of Table 1 and in the form of recurrent relations

$$P(n, 2k) = P(n+1, 2k) = \frac{1}{2} (P(n, 2k-1) + P(n+1, 2k-1)), \quad (27)$$

$$P(n+1, 2k+1) = P(n+2, 2k+1) = \frac{1}{2} (P(n+1, 2k) + P(n+2, 2k)),$$

where $P(n, k)$ is the population value of the n -th level after time $k \frac{T}{2}$, where k is an integer number. The creeping of populations among nondegenerate levels is illustrated in Fig.4.

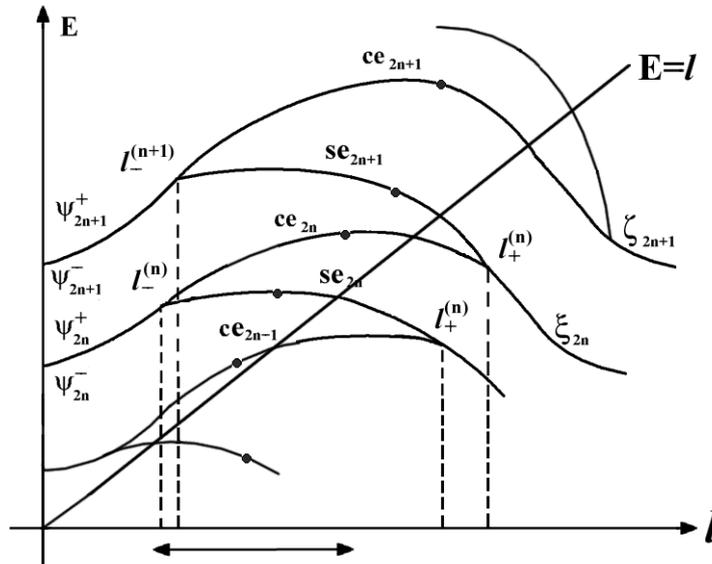


Fig.4. A fragment of the energy spectrum depending on the slowly changing parameter (4) of the quantum mathematical pendulum (6). With a lapse of time $t \gg T$ the stationary state is achieved, for which all levels are populated with an equal probability

Table 1. Evolution of populations of nondegenerate levels

n_0+4	0	0	0	0	0	0	0	0
n_0+3	0	0	0	0	1/16	1/16	3/32	3/32
n_0+2	0	0	0	1/8	1/16	1/8	3/32	1/8
n_0+1	0	0	1/4	1/8	3/16	1/8	5/32	1/8
n_0	1	1/2	1/4	1/4	3/16	3/16	5/32	5/32
n_0-1	0	1/2	1/4	1/4	3/16	3/16	5/32	5/32
n_0-2	0	0	1/4	1/8	3/16	1/8	5/32	1/8
n_0-3	0	0	0	1/8	1/16	1/8	3/32	1/8
n_0-4	0	0	0	0	1/16	1/16	3/32	3/32
n_0-5	0	0	0	0	0	0	0	0
N	$t=0$	$t=T/2$	$t=T$	$t=3T/2$	$t=2T$	$t=5T/2$	$t=3T$	$t=7/2T$

This Table is a logical extrapolation of the analytical results obtained in Subsection 3. It shows how the population concentrated initially on one level n_0 gradually spreads to other levels. It is assumed that the extreme upper level $n_0 + 4$ and the extreme lower level $n_0 - 5$ are forbidden by condition (5) and do not participate in the process.

The results of numerical calculations by means of formulas (27) are given in Fig.5 and Fig.6. Fig.5 shows the distribution of populations of levels $P(n)$ after a long time $t \gg T$ when the population creeping occurs among levels, the number of which is not restricted by (5). Let us assume that at the initial time moment $t = 0$, only one n_0 -th level is populated with probability $P(n_0) = 1$. According to the recurrent relations (27), with a lapse of each period T "indirect" passages will result in the redistribution of populations among the neighboring levels so that, after a lapse of time $t = k\tau \gg T$, populations of the extreme levels will decrease according to the law [10]

$$P(n_0 \pm k) \sim \frac{1}{2^k}.$$

If the number N of levels defined by condition (5) is finite, then, after a lapse of a long time, passages will result in a stationary state in which all N levels are populated with the same probability equal to $1/N$ (see Fig.6). The distribution obtained by us is analogous to the distribution obtained in [12,13] in investigating the problem on a linear oscillator under the action of an electromagnetic field in the conditions of weak chaos.

Let us summarize the results we have obtained above using the notions of statistical physics. After a lapse of time ΔT , that can be called the time of initial chaotization, the investigated closed system (quantum pendulum + variable field) can be considered as a statistical system.

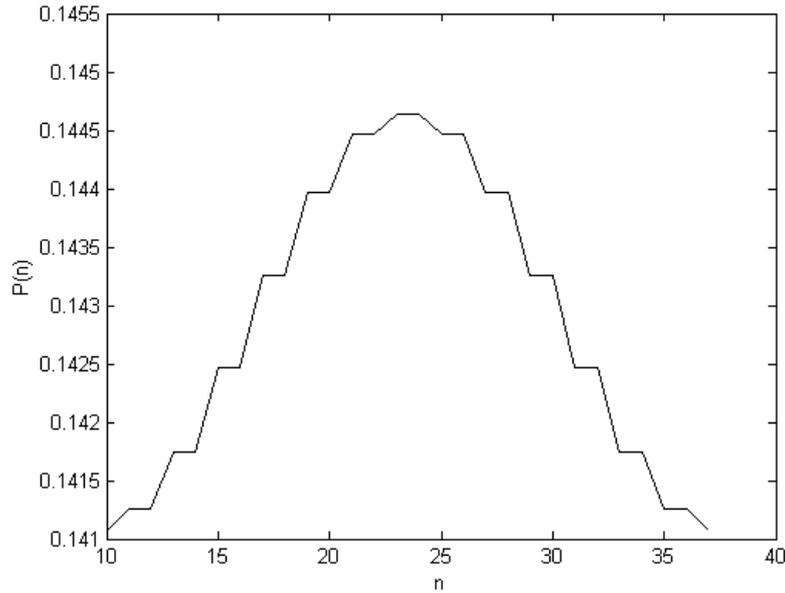


Fig.5. Results of numerical calculations performed by means of recurrent relations (27). Formation of statistical distribution of populations of levels $P(n)$ with a lapse of a large evolution time $t \approx 1000 T$ of the system. The result shown in this figure corresponds to the case for which the level population creeping is not restricted by condition (5)

At that, the closed system consists of two subsystems: the classical variable field (6') that plays the role of a thermostat with an infinitely high temperature and the quantum mathematical pendulum (6). A weak (indirect) interaction of the subsystems produces passages between nondegenerate levels. After a lapse of time $t \gg T$ this interaction ends in a statistical equilibrium between the subsystems. As a result, the quantum pendulum subsystem acquires the thermostat temperature, which in turn leads to the equalization of level populations. The equalization of populations usually called the saturation of passages can be interpreted as the acquisition of an infinite temperature by the quantum pendulum subsystem.

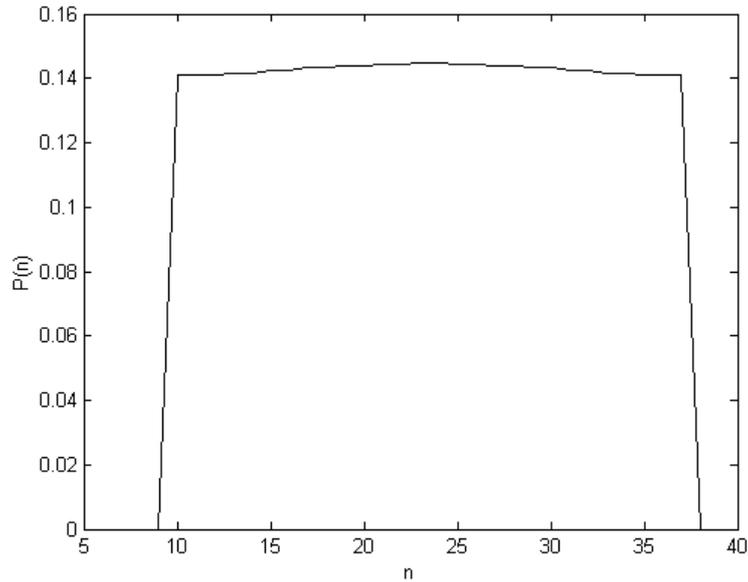


Fig.6. Results of numerical calculations performed by means of recurrent relations (27). With a lapse of a large time interval $t \sim 1000T$ the formation of stationary distribution of

populations among levels takes place. By computer calculations it was found that in the stationary state all N levels satisfying condition (5) were populated with equal probability $1/N$

4. ENTROPY GROWTH OF THE QUANTUM PENDULUM SUBSYSTEM. VARIABLE FIELD ENERGY ABSORPTION

As is known, variation or constancy of entropy can be considered as a criterion of irreversibility and reversibility of processes occurring in a closed system. In the case of irreversible processes, during which the system tends to the equilibrium state, the entropy increases, while in the equilibrium state it remains constant.

Let us use this criterion to clarify the question of reversibility for our problem. As is known, the entropy of an arbitrary quantum system is defined by the operator of the density matrix $\hat{\rho}(t)$ [14]

$$S(t) = - \langle \overline{\hat{\rho}(t)} \ln \overline{\hat{\rho}(t)} \rangle, \quad (28)$$

where the brackets $\langle \dots \rangle$ denote the quantum-mechanical averaging, while the overline denote the averaging over a small dispersion $\delta\omega$ of the passage frequency (9). In the matrix form the right-hand side of formula (24) is written as

$$S(t) = -\text{Tr}(\overline{\hat{\rho}(t)} \ln \overline{\hat{\rho}(t)}) = -\sum_{i,k} \overline{\hat{\rho}^{ik}(t)} \ln \overline{\hat{\rho}^{ik}(t)}. \quad (29)$$

If initially the system is in one of degenerate states, for example, in the pure state ψ_{2n}^- , then $C_n^-(0) = 1$, $C_n^+(0) = 0$

$$\rho_n^{+-}(t=0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore, by (29), the entropy is $S_n(t=0) = 0$.

With a lapse of time $t \gg \tau$, as passages between nondegenerate states are completed and the self-chaotization condition $\Delta T \delta \omega \geq 2\pi$ is fulfilled, the density matrix takes form (22). Then the substitution of the density matrix (22) into the entropy formula (29) gives

$$S_n(t \sim T/2 \gg \tau) = \ln 2 . \quad (30)$$

Thus, on a time interval from $t = 0$ to $0 \leq t \leq \Delta T$, the entropy grows

$$S_n(t \gg \tau) > S_n(t = 0) .$$

This proves that on this time interval the process is irreversible.

Using analytical methods, we have succeeded in establishing only the asymptotic value of entropy. To investigate a complete picture of entropy change on a time interval $0 \leq t \leq \Delta T$, we use expression (15) for $\rho_n^{+-}(t)$. After substituting it into the entropy formula we obtain

$$S_n(t) = -W_n^-(t) \ln|W_n^-(t)| - W_n^+(t) \ln|W_n^+(t)| - \pi F_n(t) . \quad (31)$$

Fig.7 shows the entropy as a function of time constructed with the aid of (18), (31) by numerical methods.

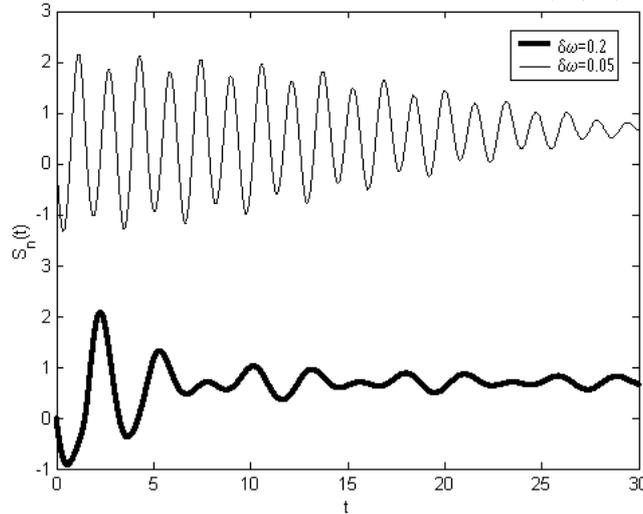


Fig.7. The entropy growth graph constructed with the aid of expression (31), using numerical methods for the parameter

values $\omega = 1/\tau = 1$

To calculate the entropy value with the lapse of one period T , we substitute matrix (26) into the entropy formula (29) and thus obtain

$$S_n(t \sim T) = \ln 4 \cdot$$

The state, in which all accessible levels of the subsystem are populated with the same probability $1/N$ (Fig.6), is the equilibrium state. The corresponding density matrix of dimension N is written as

$$\rho_{2n}^{ik}(t \gg T) = \frac{1}{N} \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (32)$$

After substituting ρ_{2n}^{ik} from (32) into (29), we obtain the maximal entropy value on time intervals $t \gg T$

$$S_n(\infty) = S_n(t \gg T) = \ln N \cdot \quad (33)$$

Thus we see that the entropy constantly grows up to value (33) and after that it stops to grow.

Let us now calculate the energy mean of the quantum pendulum subsystem. It is obvious that for the average energy of the subsystem we can write

$$E(t) = E_0 + \sum_{n=1}^N P_n(t) E_n, \quad (34)$$

where E_0 is the initial energy value defined by the initial ($t=0$) population of the levels, $P_n(t)$ is the probability that the n -th level will be populated at the time moment t , E_n is the energy value in the n -th state defined from the area of nondegenerate states.

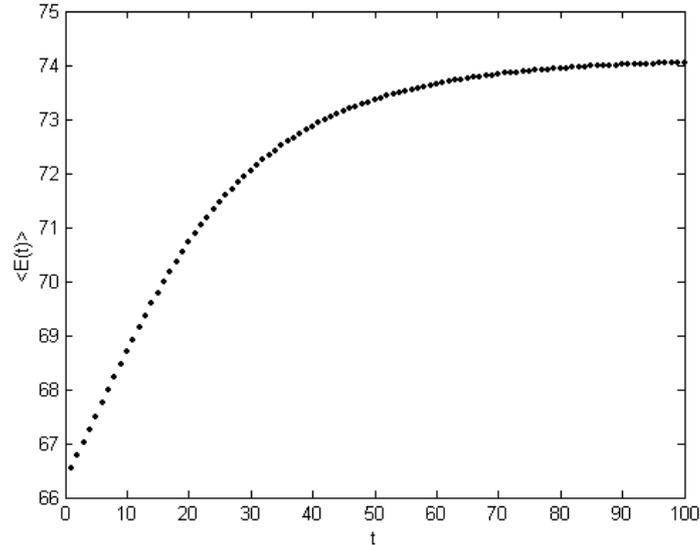


Fig.8. Time-dependence of a mean energy value of the quantum mathematical pendulum subsystem, constructed by numerical methods using formula (34). As clearly seen from the figure, the absorption of optical pumping energy takes place prior to reaching the state of statistical equilibrium

In Fig.8 we see that the subsystem energy first grows and then becomes constant. This result can be explained if we take into account the time dependent trend of population changes which is defined by the recurrent relations (27) or Table 1. At the beginning the subsystem absorbs the field energy (ϵ') and, in doing so, performs "indirect" passages between energy levels mostly in the upward direction. Upon reaching the equilibrium state, in which the subsystem is characterized by the equalization of level populations, it stops to absorb energy.

5. CONCLUSIONS. ANALOGY BETWEEN THE CLASSICAL AND THE QUANTUM CONSIDERATION

The classical mathematical pendulum may have two oscillation modes (rotational and oscillatory), which on the phase plane are separated by the separatrix (see Fig.9a).

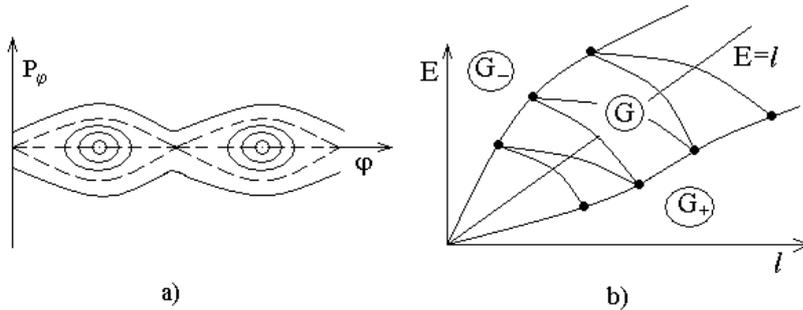


Fig.9. Analogy between the classical

and quantum considerations.

Unperturbed motion:

a) Classical case. Phase plane. Separatrix;

b) Quantum case. Specific dependence of the energy spectrum on the parameter (Mathieu characteristics).

Degenerate G_\pm and non-degenerate G areas of the spectrum

On the plane (E, l) the quantum mathematical pendulum has two areas of degenerate states -- G_- and G_+ . Quantum states from the area G_- possesses translational symmetry in the pendulum phase space. These states are analogous to the classical rotational mode. Quantum states from the degenerate area G_+ possess

symmetry with respect to the equilibrium state of the pendulum and therefore are analogous to the classical oscillatory state [10]. On the plane (E, l) , the area of nondegenerate states G , which lies between the areas G_- and G_+ , contains the line $E = l$ corresponding to the classical separatrix (see Fig.9b). If the classical pendulum is subjected to harmonically changing force that perturbs a trajectory near to the separatrix, then the perturbed trajectory acquires such a degree of complexity that it can be assumed to be a random one. Therefore we say that a stochastic motion layer (so-called stochastic layer) is formed in the neighborhood of the separatrix [7] (see Fig.10a).

In the case of quantum consideration, the periodic perturbation (6) brings about passages between degenerate states. As a result of repeated passages, before passing to the area G the system gets self-chaotized, passes from the pure state to the mixed one and further evolves irreversibly. While it repeatedly passes through the branch points, the redistribution of populations by the energy spectrum takes place. Only the levels whose branch points satisfy condition (5), participate in the redistribution of populations (see Fig.10b).

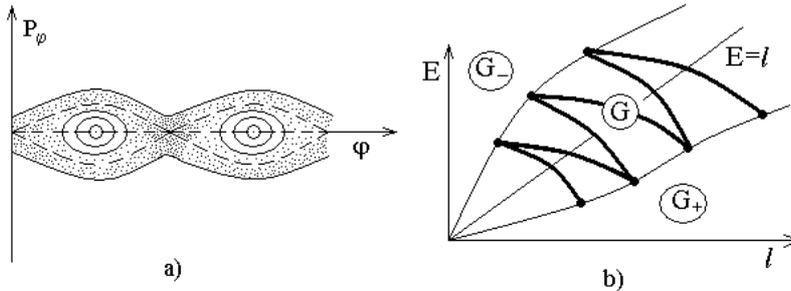


Fig.10. Analogy between the classical and quantum considerations.
Perturbed motion:

- a) Classical case. Stochastic trajectories in the neighborhood of the separatrix form the stochastic layer (cross-hatched area);
- b) Quantum case. The mixed state was formed as a result of population of nondegenerate levels situated on both sides of the classical separatrix.

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