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QUANTIZATION OF SINE-GORDON EQUATION SOLITON AND φ4 KINK EQUATION USING EIKONAL APPROACH

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ABSTRACT. **We offer nonperturbed scheme of S-G equation soliton and briser, φ4 equation of kink-antikink couple for obtaining of the spectrum of bound state in eikonal approach using Glauber method. Correspondingly, analytical expression of profile functions have been obtained. This gives the possibility to express scattering amplitude on briser by meson of the theory and on kink-antikink couple of their constituent solution, antisoliton and kink-antikink scattering amplitudes without using potential.**

It is known that collective excitons of particle systems having nonlinear potential (due to strong self-influence of the scalar field) are considered as physical solitons [1].

According to the hypothesis the soliton like particles are quantum corpuscles of such field. [2].

At first from the activity function by means of variation principle for the classical nonlinear fields, Euler-Lagrange equation is obtained, where soliton like solutions are nonperturbed. Thus, we gradually imagine the quantization of the classical field by means of nonperturbation method, particularly eikonal approach.

In this approach, when the scattering is observed only on the small angles, the field inside interaction area is equal to the field, which is elastic scattered, and this one far from scattering is the diffraction field of Fraungofer. In conditions of ermitivity of the scattering potential the inside field gives the binding states i.e. negative energies $E < 0$, if imaginary part of wave vector is $\mathrm{Im}K = i\chi$, $\chi > 0$ [3]. If the "black" $\mathrm{Re}K = 0$ is scattered, an interaction is nonelastic and the inside field is the absorbed one. If $\mathrm{Re}\,K \neq 0$, then the absorbed part corresponds to the field of the binding state of system [4].

To the solutions of sine-Gordon (S-G) equations solitons and antisolitons – Skyrme [5] gave a topological charge +1 and –1 respectively, which agrees with the phase shift, while studying their intercollision [App.A].

A couple solution of soliton and antisoliton i.e. briser is neutral according to the topological charge; it itself is antiparticle and, that is why the meson corresponds to it.

Soliton is surrounded with briser clouds and most part of time it spends in the state of exchange of vacuum with mesons of theory. The meson is the briser of small radius. The free soliton (antisoliton) is absorbed by the vacuum briser and soliton of briser is irradiated. According to the classical theory this process is stochastic and corresponds to the interaction of classical solution field, to its own meson one.

According to nonstochastic approach, the description is possible in the conditions of small deviation from balance position and during small scale of physical system.

In the future approach, using the multitime method, one excludes the secular members in the formal asymptotic setting. This dictate the expression of nonlinear potential U(Φ) in the evaluative equation of Euler-Lagrange, which doesn't reveal any stochastic behaviour. The trajectories in the phase space are smooth, the field $\eta = \langle\Phi\rangle$, where $\langle\cdot\rangle$ means the vacuum average value, Φ is the descriptive field of physical system, which generally may be stochastic. The trajectories on the rotation points of the separatrice are not smooth – the physical system here is still stochastic.

Then one is deriving linearity near η, but it breaks the symmetry of task to the transformation of rotational and translational permanent groups. This leads us to elementary excitation – Goldstone boson type mode with zero energy. New $\Phi - \langle\Phi\rangle \equiv x$ field will change them into the Higgs bosons. The field equation gets the expression of Klein-Gordon linear equation:

$$\Box x + U''(\eta)x = O(x^2)\ .$$

The solutions of these equations are mesons.

If the constant *g* of an interaction is small and the Hamiltonian of interacting fields was renormalized, it would be possible to use the perturbation theory. That time the mesons were described by free fields and their quantum transitions on the classical solitons of mesons or on kink between the conditions of scattering link were due to their interaction.

Faddeev and Takhtajian [2] solved S-G classical equation by means of determination of action-angle variables. In these variables classical Hamiltonian was brought to the sum of free-coupled particles, taking into account a coupled soliton-antisoliton (briser) solution. Hamiltonian depended on variables of activity and not on angles. That is why, the following steps of quantization are trivial. Unfortunately, it's problematic to find these variables. This task may be solved using the method of reverse scattering.

Dashen et al. [6] made quantization of S-G and φ4 systems. The lower energetic level is S-G or φ4.According to this its mass consists of classical part which is followed by quantum correction. In the correction of high order divergence was neglected by counter members; they made quantization of briser in the same way. The setting of the field in the approach of the given soliton solution breaks the symmetry of the tasks in relation to continuous group. Crist and Lee [7] solved the problem of emerged zero modes (which appeared because of breaking continuous rotational and continuous translational symmetries) by introducing collective coordinates as implicit form.

The solutions of the field of quantum theory are correct using the theory of perturbation [5]. The precision of quantum correction with the members of the fourth order from the point of view of the relativity of masses is due to particularity of S-G. In the case of kink it is not so. Though the correction of soliton mass is not precise, the precision of their relativity is unexpected. The existence of the theory of mesons points that an interaction is strong. It is known that if we use the normalized theory of interaction for not small meanings of g, it's impossible to use the theory of perturbation successfully.

In the conditions of strong interaction for the solution of the task of scattering different methods are used. Of them Glauber method of eikonal approach is one of the widespread approaches [8]. It uses the function of profile $\Gamma(\rho)$ (ρ is a target parameter), which allows us to count the scattering amplitude more precisely, than in the Born approach, or using Shvinger's variation functional.

QUANTIZING OF SOLITON TYPE FIELDS

According to the given approach the central subject is a finding of conditions of the scattering of meson on the soliton.

Let us discuss a model of self-acting complex scalar field $\Phi(\mathbf{r},\mathbf{t})$. The acting functional is:

$$S[\Phi(\vec{r},t)]=\int\left[\frac{1}{2}\left|\Phi_t\right|^2-\frac{1}{2}\left|\Phi_x\right|^2-\frac{1}{g^2}U(g|\Phi|)\right]dtd^3r, \tag{1}$$

where g is a bound constant. The Euler-Lagrange equation has a form [5]:

$$\frac{\delta S}{\delta\Phi}=-_{\Gamma}\Phi(\vec{r},t)-U_{|\phi|}\frac{\Phi(\vec{r},t)}{g|\Phi(r,t)|}=0 \tag{2}$$

It means that the nonlinearity of the potential $U[\Phi(\mathbf{r},\mathbf{t})]$ is selected so that the equation has soliton like solution, but soliton like particle is described by a pseudoscalar field $\Phi(\mathbf{r},\mathbf{t})$.

It's assumed in the eikonal approach that transferred impulse is in the perpendicular plane of the particle spreading z, along the radial coordinate ρ. Let's write a field in the cylindric coordinates (z,ρ) of this approach:

$$\Phi(\vec{r},t)=\frac{\langle\Phi(z,t)\rangle}{g}+\left\{\Gamma_R(\rho)+i\Gamma_I(\rho)e^{i\omega t}\right\}e^{ikz}, \tag{3}$$

where $\{\ \}\,e^{ikz}$ is scattered field, $\langle\cdot\rangle$ is classical field and $\Gamma_R(\rho)+i\Gamma_I(\rho)=1-e^{ix(\rho)}$ is the profile function. Inserting the expression (3) into the equation (2), we shall get three following equations:

$$(-\partial,_{zz}+\partial,_{tt})\langle\Phi\rangle=\partial,_{\Phi}u\langle\Phi\rangle, \tag{4}$$

$$(-\partial,_{\rho\rho}+V)\Gamma_R(\rho)=0, \tag{5}$$

$$(-\partial,_{\rho\rho}+V)\Gamma_I(\rho)=\omega^2\Gamma_I(\rho), \tag{6}$$

where $\mathrm{x}(\rho)=-\frac{\mathrm{iE}}{\mathrm{k}}\int\limits_{-\infty}^{\infty}\mathrm{d}z'\mathrm{V}(\vec{\rho},\mathrm{z}')$ [App.B]. The last equations describe bound conditions in the scattering in the point z = 0. From the equation (4), we shall find V, becouse we have known x(ρ) and $\Gamma_R(\rho)$.

From (6) the law of dispersion is obtained:

$$\omega^2=m^2+k^2 \tag{7}$$

SOLITON

If $\mathrm{U}\langle\Phi(\vec{\mathrm{r}},\mathrm{t})\rangle=\frac{\mathrm{m}^2}{\mathrm{g}^2}\left(1-\cos\langle\Phi(\vec{\mathrm{r}},\mathrm{t})\rangle\right)$, then we shall get S-G equation from (4) one:

$$\left(\partial_{\bar{t}\bar{t}}-\partial_{\bar{z}\bar{z}}\right)\langle\Phi\rangle+\sin\langle\Phi\rangle=0\,. \tag{8}$$

Here the variables are changed:

$$mt \to \bar{t}\,, \quad mz \to \bar{z}\,, \quad g\Phi \to \langle \Phi \rangle\,. \tag{9}$$

The soliton solution of this equation is

$$\langle \Phi(\bar{z},t) \rangle = 4\mathrm{arctg}e^{(\bar{z}+v\bar{t})}\,. \tag{10}$$

This means, that the first member or classical field of three equations has the form:

$$\frac{\langle \Phi(\bar{z},t) \rangle}{g} = \frac{4}{g}\mathrm{arctg}e^{\gamma m(z+vt)}\,, \tag{11}$$

where $\gamma \equiv (1-v^2)^{-1/2}$.

Let us appeal equations (5) and (6). Assume that

$$x(\rho) = 2\mathrm{arctg}e^{\rho}\,. \tag{12}$$

Let us appeal the new variables

$$\rho = mz = \bar{z}\,. \tag{13}$$

The profile function will have a form:

$$\Gamma(\rho) = 1-\cos(2\mathrm{arctg}e^{\rho}) - i\sin(2\mathrm{arctg}e^{\rho}) = 1+\mathrm{th}\rho - i\,\mathrm{sech}\rho\,. \tag{14}$$

Equation (5) gives that for a zero mode $\omega_0 = 0$ $\Gamma_R(\rho) = 1+\mathrm{th}\rho$ we shall get

$$\left(-\partial,_{\rho\rho} + V(\rho)\right)(1+\mathrm{th}\rho) = 0\,. \tag{15}$$

The frequency of normal oscillation is determined inserting $V(\rho) = 1-2\,\mathrm{sech}^2\rho$ and $\Gamma_I(\rho) = -\mathrm{sech}\rho$ in the equation (6).

$$(-\partial,_{\rho\rho} + 1 - 2\,\mathrm{sech}^2\rho)\mathrm{sech}\rho = \frac{\omega_1^2}{m^2}\mathrm{sech}\rho \tag{16}$$

i.e. we can rewrite $\omega_1 = 0$ and equation (16) for nonreflective potential $-2\,\mathrm{sec\,h}^2\rho$

$$(\partial_{\rho\rho} + 2\,\mathrm{sech}^2\rho)\,\mathrm{sec\,h}\rho = \frac{\omega_2^2}{m^2}\,\mathrm{sech}\rho\,. \tag{16'}$$

φ4 kink

For two hole potential

$$U\langle\Phi(z,t)\rangle = \frac{m^2}{g^2}\left[\frac{1}{4}\left(\Phi^2(z,t)-1\right)^2\right]$$

The classical field $\langle\Phi\rangle$ will satisfy the equation:

$$\left(\partial_{\bar{t}\bar{t}} - \partial_{\bar{z}\bar{z}}\right)\langle\Phi\rangle = \frac{m^2}{g^2}\left(\langle\Phi\rangle^3 - \langle\Phi\rangle\right) \tag{17}$$

The nonstationar solution of this equation is kink:

$$\langle\Phi(z,t)\rangle = \mathrm{th}\left(\frac{\bar{z}}{\sqrt{2}} + v\bar{t}\right) \tag{18}$$

i.e. the classical field will have a form:

$$\frac{\langle\Phi(z,t)\rangle}{g} = \frac{1}{g}\,\mathrm{th}\left(\frac{\gamma m}{\sqrt{2}}(z+vt)\right). \tag{19}$$

Then appealing to the new variabels $\rho = mz/\sqrt{2} = \bar{z}/\sqrt{2}$. If $x(\rho)$ soliton solution has the form (10)

$$x(\rho) = 4\mathrm{arctg}e^{\rho}\,, \tag{20}$$

we shall have

$$\Gamma(\rho) = \sec^2 \rho - i sh\rho \sec h^2 \rho \,. \tag{21}$$

The equation (5), which describes a zero mode will get a form:

$$\left(-\frac{1}{2}\partial,_{\rho\rho} + V(\rho)\right)\sec h^2 \rho = 0 \,. \tag{22}$$

From this we get $V(\rho)$:

$$V(\rho) = 3th^2\rho - 1 \tag{22'}$$

For the normal mode ω_1, from equation (6) we shall get Shrodinger type equation of scattering form:

$$\left(-\frac{1}{2}\partial,_{\rho\rho} + (3th^2\rho - 1)\right) sh\,\rho \sec h^2 \rho = \frac{\omega_1}{m^2} sh\,\rho \sec h^2 \rho \,, \tag{23}$$

then we shall see that discrete mode

$$\omega_1^2 = \frac{3}{2} m^2 \,, \tag{24}$$

which agrees with results of [6].

As we discuss the scattering on the small angles $(\cos\vartheta \approx 1)$ and assume that the scattering has azimuthal symmetry, that's why of profile functions we shall get the amplitudes of scattering

$$f(\vartheta) = ik \int_0^{+\infty} d\rho \rho \Gamma(\rho) J_0(k\rho\vartheta) \,, \tag{25}$$

where $J_0(k\rho\vartheta)$ is Bessel function of zero order, and their quadrates will give a differential section of the scattering.

QUANTIZATION OF BRISER

The equation of S-G

$$\overline{\Phi} = \sin\overline{\Phi}\,, \tag{26}$$

in the descriptive coordinates $\sigma = x + vt,\ \rho = x - vt$ is written

$$\frac{\partial^2\overline{\Phi}}{\partial\sigma\partial\rho} = \pm\sin\overline{\Phi}\,. \tag{27}$$

Between the solutions of this equation is soliton $\overline{\Phi}_s$, antisoliton $\overline{\Phi}_A = -\overline{\Phi}_s$ and a couple of soliton-antisoliton $\overline{\Phi}_{SA} \equiv \overline{\Phi}_V$ i.e. briser.

To form a briser soliton let is use the theorem of Baclund transformation:

$$\mathrm{tg}\frac{\overline{\Phi}_V - \overline{\Phi}_o}{4} = \frac{1}{U}\mathrm{tg}\frac{\overline{\Phi}_A - \overline{\Phi}_S}{4}\,. \tag{28}$$

For briser a parameter U is imaginary U = iv

$$\overline{\Phi}_v(x,t) = 4\mathrm{arctg}\left\{\frac{1}{v}\frac{\sin\left[vt/\left(1+v^2\right)^{1/2}\right]}{\mathrm{ch}\left[x/\left(1+v^2\right)^{1/2}\right]}\right\}. \tag{29}$$

It is clear that, in briser soliton and antisoliton may be apart each other only in the limit distance.

In the $v = 0$ account system $\Phi_v(x,t)$ is a still wave, but in the system of pair mass center its components: soliton and antisoliton oscillate in relation to each other with the period

$$T = \frac{2\pi v}{\left(1+v^2\right)^{1/2}}, \tag{30}$$

Accounting this (29) may be written as [5,6]:

$$\overline{\Phi}_v(x,t) = 4\operatorname{arctg}\left\{\left(T^2-1\right)^{1/2}\frac{\sin\left[mt/\overline{T}\right]}{\operatorname{ch}\left[mx\left(\overline{T}^2-1\right)^{1/2}/\overline{T}\right]}\right\}, \tag{31}$$

where $\overline{T} \equiv \frac{mT}{2\pi}$.

An assumable Skyrme's charge of briser is zero, but brizer consists of two different named topological unified charges. At the meeting of these particles the charge disappears i.e. annihilates. The other particle emerges. This is Frenkel's exciton i.e. an exciton of small radius. Quantization of doublets using WKB method was made by Dashen et al.[3].

Taking into consideration a meson of theory – briser we have found above the bound conditions on S-G soliton and kink of its elastic scattering.

The consideration of eikonal approach according to Glauber having a form of multiorder scattering allows us to consider bound conditions of briser on the complex objects – briser, φ4 doublet, the soliton and antisoliton, kink and antikink.

Briser will not have topological charge of Skyrme. It is itself antiparticle. That is why the meson corresponds to it.

Thus we think, that briser is a meson of theory. It is special, because it has a mass

$$M_B = \frac{16}{\gamma_o m} \sin \frac{n\gamma_o}{16}, \text{ where } n = 1,2...,N. \tag{32}$$

We shall make quantization using Glauber's method. Really we are accounting briser structure.

From (5) let us appeal the following equation:

$$(-\partial,_{\rho\rho} + V)G_R(\rho) = 0, \tag{33}$$

From (6) we shall get

$$(-\partial,_{\rho\rho} + V)G_I(\rho) = \omega^2 G_I(\rho), \tag{34}$$

where according to Glauber, the profile function of a pair is

$$G^S(p) = \Gamma_S\left(\rho + \frac{1}{2}\vec{r}\right) + \Gamma_{AS}\left(\rho - \frac{1}{2}\vec{r}\right) - \Gamma_S\left(\rho + \frac{1}{2}\vec{r}\right)\Gamma_{AS}\left(\rho - \frac{1}{2}\vec{r}\right), \tag{35}$$

In the system of mass center ρ is a distance in briser between soliton and antisoliton. If we insert Γ_S and Γ_{AS} from (14) and (14') meanings we have

$$G^S(\rho) = \frac{2\mathrm{sh}\frac{1}{2}r}{\mathrm{ch}^2\rho + \mathrm{sh}^2\frac{1}{2}r}\left(\mathrm{sh}\frac{1}{2}r - i\mathrm{ch}\rho\right), \tag{36}$$

For a pair of φ4 kink and antikink we have

$$G^K(\rho) = \frac{4}{\left(\mathrm{ch}^2\rho + \mathrm{sh}^2\frac{1}{2}r\right)^2}\left(2\mathrm{sh}^2\rho\,\mathrm{ch}^2\frac{1}{2}r - i3\mathrm{ch}^3\rho\,\mathrm{sh}\frac{1}{2}r\right). \tag{37}$$

The offered scheme of quantization is true in the eikonal approach, but contrary the need of setting of potential (accounting the member of second order) is not necessary.

APPENDIX A

The quantitative experiments of Skyrme topological charge showed, that soliton and antisoliton attract each other, but solitons (antisolitons) do not. We have analogous results in the case of kinks and antikinks

If we take a mark "+" then 2π soliton satisfies limit conditions $<\varphi>\rightarrow 0$ if $z \rightarrow -\infty$ and $<\varphi>\rightarrow 2\pi$, when $z \rightarrow +\infty$, i.e. 2π soliton undergoes to interpolation from 0 to 2π.

$$<\varphi(\infty, t)> - <\varphi(-\infty, t)> = 2\pi - 0 \cdot$$

4π soliton totally undergoes to interpolation from 2π to 4π and so on.

If we have antisoliton ("-" mark), then $<\varphi>\rightarrow 2\pi$, when $z \rightarrow -\infty$ and $<\varphi>\rightarrow 0$ if $z \rightarrow +\infty$, i.e. it undergoes to interpolation from 2π to 0. According to this Skyrme gave to soliton and antisoliton topological charge, respectively.

In the case of $\varphi 4$ kink interpolation occurs from $-\pi$ to $+\pi$. That is why, topological charge of Skirme for kink will be +1 and for antikink –1. Skirme supposed about equivalence of S-G and MT models, which are proved by Coleman.

In the MT model fermion charge is

$$Q= \int_{-\infty}^{+\infty} J_0 dx = \int_{-\infty}^{+\infty} \overline{\Psi}\gamma_0 \Psi dx \, .$$

Fermion has Q = 1 and antifermion - Q = -1. In the bound (linked) position Q = 0. From the theorem of Coleman it yields that

$$-\frac{\sqrt{\lambda}}{2\pi m}E^{\mu\nu}\partial_{\nu}\Phi = \overline{\Psi}\gamma^{\mu}\Psi \equiv J^{\mu},$$

where charge can be written by means of S-G fields.

$$Q = \int_{-\infty}^{+\infty} -\frac{\sqrt{\lambda}}{2\pi m}E^{01}\partial_x \Phi dx = \frac{\sqrt{\lambda}}{2\pi m}\int_{-\infty}^{+\infty}\partial_x \Phi dx .$$

This is topological charge of S-G model.

APPENDIX B

Let us note that the method does not need the knowledge of potential [9], and it is used successfully to study elastic scattering of pions on protons, deutons and some complex nucleus.

We know that the meson of theory is briser. We have to solve the equation of Klein-Gordon, which describes it. In the stationary case it has a form

$$\nabla^2\psi + \left[(\omega - V)^2 - M^2\right]\psi = 0 \cdot$$

This expression can be written simply in the form of Shrodinger equation if we use the values $K^2 \equiv \left(\omega^2 - M^2\right)$, $U \equiv V(2\omega - V)$, where M is a mass of briser. If we are searching for the solution in the form flat wave $\varphi(r)e^{ikz}$, then in the eikonal approach $\partial_z\phi << ik\partial_z\phi$ we get the equation of diffusion type more exactly of diffraction type $ik\partial_z\phi = \omega V\phi$.

From this expression we have

$$\psi = e^{ikz+\chi(\rho)}.$$

The Shrodinger equation of scattering form

$$\left[\nabla^2 + k^2 - U(r)\right]\Psi = 0$$

could be initially transformed into Lippmann-Schwinger, which in the asymptotic region $\vec{r} \to \infty$ will get a form

$$\psi = e^{ikz} - \int d^3r' \frac{1}{4\pi} \frac{e^{ik\vec{r}}}{r} e^{-ik\frac{rr'}{r}} U(\vec{r})\Psi(\vec{r}) \cdot$$

So, equation (3) is a modified form of Lippmann-Schwinger equation in the eikonal approach.

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