On Construction and Justification of Systems of von Karman type for Binary Mixture of Piezo-Elastic Plates

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One of the most principle objects in development of nonlinear mechanics and many branches of modern mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by von Karman in 1910. This system with corresponding boundary conditions represents the most essential part of the main manuals in elasticity theory and building mechanics (between them those of Kirchhof, Filon, Love, Fliugge, Timoshenko, Donnell, Landau and Lifchitz, Novozhilov, ...). In spite of this in 1978 Truesdell expressed an idea about unfounded of "Physical Soundness" of von Karman system. This circumstance generated the problem of justification of von Karman system. In the following period this problem was discussed by many authors, but with most attention and in details it was studied by Ciarlet ([1], ch. V). The main result obtained here is given as follows: "The von Karman equations may be given a full justification by means of the leading term of a formal asymptotic expansions of the exact 3D equations of nonlinear elasticity associated with a specific class of boudary conditions" [1,p.368]. This result obviously is not sufficient for justification of "Physical Soundness" of von Karman system, as the basic terms of these expansions are coefficients of power series but not the terms having "Physical Soundness".

On the basis of work [2-3] below there is given the direct method of constructing such anisotropic inhomogeneous 2D models of von Karman-Mindlin-Reissner(KMR) type, by means of which corresponding terms take quite determined "Physical Soundness". These terms are: the averaged components of the displacement vector, bending and twisting moments, shearing forces, normal rotations, surface efforts. Further it is shown, that these new models depend from the arbitrary parameters, by choosing of which in the isotropic homogeneous case from KMR systems the von Karman system as one of possible models having continuum cardinality is obtained. This method gives full system of differential equations and we underline that constructing by classical method one of the equations of von Karman system is not independent and represents a compatibility condition of Saint-Venant-Beltrami type (for nonlinear case) between deformation components (see also the recent work of Podio-Guidugli [4]). This remark is essential in discussing the dynamical problems. In this case along the quantities describing the vertical directions and surface wave processes in the class of constructed models it is necessary to take into account the quantity $\Delta\partial_{tt}\Phi$ *(*Φ *denotes Airy function), corresponding to wave processes in the horizontal direction. It also should be mentioned, that from KMR type system in the linear case when elastic plates are isotropic or generalized transversal isotropic a uniform representation for all boundary-value problems (considered for example in [5-6]) in terms of planar expansion and rotation is obtained. Corresponding systems of inhomogeneous equations have Cauchy-Riemann type operator as a principal part.*

The work is devoted to the matter of constructing the KMR type two-dimensional mathematical model with respect to spatial variables for binary mixture in case of elastic plate. In the first part on the basis of works [7-8] there will be introduced nonlinear dynamic three-dimensional (with respect to spatial variables) mathematical model in an elastic case. For simplicity and clearness in the work there is considered the case of isotropic mixture, but analogous models can be easily constructed for anisotropic elastic plate with variable thickness, using the methods developed in [3]. We should emphasize that we essentially rely on the works [2,3,9], using discussions and formulas from the monograph [2] without quotations.

I. The principle system of 3D equations with respect to spatial variables

We denote the domain in three-dimensional Euclidean space $\mathbb{R}^3$ by $\Omega -$. In the Cartesian coordinates we denote the point by $x=(x_1,x_2,x_3)$ or (x,y,z). Time changes in the following interval $t\in(0,T)$, $Q_T=\Omega\times(0,T)$.

Equations of dynamic equilibrium of binary mixture have the following form

$$\partial_j\left(\sigma_{ij}+\sigma_{kj}\otimes u_{i,k}\right)=\rho\partial_t^2 u_i+f_i,\ (x,t)\in Q_T. \tag{1.1}$$

Here $\sigma_{ij}=(\sigma'_{ij},\sigma''_{ij})^T$, $u_i=(u'_i,u''_i)^T$, $f_i=(f'_i,f''_i)^T$ – are column-matrices constructed by components of stress tensors, displacement and volume forces vectors, respectively. ρ is a density matrix.

$$\rho=\begin{pmatrix}\rho_1 & \rho_3\\ \rho_3 & \rho_2\end{pmatrix},\quad \rho_\alpha>|\rho_3|\geq 0,$$

ρ_1,ρ_2- are partial densities of components of mixture, $\otimes$ symbol denotes the following operation

$$(a_1,a_2)^T\otimes(b_1,b_2)^T=(a_1b_1,\quad a_2b_2)^T.$$

The initial and boundary conditions have the following form, respectively

$$u(x,0)=u_0(x),\quad \partial_t u(x,0)=\dot u_0(x),\quad x\in\Omega, \tag{1.2}$$

$$l[\partial_1,\partial_2,\partial_3][u]=g,\ (x,t)\in\partial\Omega\times(0,T), \tag{1.3}$$

where $u=(u',u'')^T$ – is a matrix of displacement vector, l – is a linear 6×6 matrix-operator, components of which include operation of derivation of at most first order.

The relations between components of strain tensor and displacement vector have the following form

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}\otimes u_{k,j}\right),\quad (x,t)\in Q_T. \tag{1.4}$$

ε_{ij} is a column-matrix $\varepsilon_{ij}=(\varepsilon'_{ij},\varepsilon''_{ij})^T$. Thus, in the theory of mixture for two isotropic elastic material in each point of body there is considered two tensors of stress and strain and two vectors of displacement.

Hooke's generalized law is written as follows

$$\sigma_{ij}=(k+\Lambda\varepsilon_{kk})\delta_{ij}+2M\varepsilon_{ij}+2\lambda_5\hbar_{ij},\ (x,t)\in\overline{Q}_T, \tag{1.5}$$

where δ_{ij} – is Kronecker symbol.

$$k=(-\alpha_2,\alpha_2)^T=(\lambda_4-\lambda_3,\lambda_3-\lambda_4)^T,\ \Lambda=\{\lambda_{ij}\}_{2\times2},\ \hbar_{ij}=(h_{ij},h_{ji})^T,$$

$$\lambda_{11}=\lambda_1-\frac{\alpha_2\rho_2}{\rho_+},\ \lambda_{22}=\lambda_2+\frac{\alpha_2\rho_1}{\rho_+},\ \lambda_{12}=\lambda_{21}=\lambda_3-\frac{\alpha_2\rho_1}{\rho_+}=\lambda_4+\frac{\alpha_2\rho_2}{\rho_+},\ \rho_+=\rho_1+\rho_2,$$

$$M=\begin{pmatrix}\mu_1 & \mu_3\\ \mu_3 & \mu_2\end{pmatrix},$$

$\lambda_1,\cdots,\lambda_5,\ \mu_1,\ \mu_2,\ \mu_3$ – numbers are elastic components which characterize the mechanic properties of mixture, h_{ij} – are components of so called partial rotation of components of mixture.

$$h_{ij}=\frac{1}{2}\left(u'_{j,i}-u'_{i,j}+u''_{i,j}-u''_{j,i}\right),$$

and for matrix $\hbar_{ij}$ we have the following expression

$$\hbar_{ij}=\frac{1}{2}S\left(u_{j,i}-u_{i,j}\right),\quad S=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

Taking into account the latter formula and (1.4), the relation (1.5) can be rewritten in the following form

$$\sigma_{ij}=(k+\Lambda u_{k,k})\delta_{ij}+(M-\lambda_5 S)u_{i,j}+(M+\lambda_5 S)u_{j,i}+\frac{1}{2}\Lambda A_{mm}\delta_{ij}+MA_{ij}, \tag{1.5′}$$

where exist the denotation

$$A_{ij} = u_{k,i} \otimes u_{k,j} \, .$$

In all the formulas introduced above, Latin deaf indices mean summarization from one to three.

II. Von Karman-Mindlin-Reissner type two-dimensional equations with respect to spatial variables

Let us consider the case, when Ω represents the plate of constant thickness $2h$. We denote this domain as follows: $\Omega_h = D(x_1, x_2) \times]-h, h[$, where $D(x_1, x_2)-$ is a flat bound domain, $S^{\pm} = D \times \{\pm h\} -$ are surfaces of plate, $S = \partial D \times [-h, h] -$ is a lateral surface. In connection with domain's Ω_h structure, the condition on surfaces $S^{\pm}$ is signed out. Thus:

$$\sigma_{i3} + \sigma_{j3} \otimes u_{i,j} = g_i^{\pm}, \; (x,t) \in S^{\pm} \times (0,T) \, . \tag{2.a}$$

Our objective is to obtain systems corresponding to bending and extension (compression) processes.

II.1. Let us begin with constructing the system corresponding to equations of bending. We introduce the following means of Reissner (we note too, that for purpose of strictness and brevity of writing, in this work numbers of formulas and methods of proving totally coincide with those in [2]):

$$M_{\alpha} = \int_{-h}^{h} z(\sigma_{\alpha\alpha} + \sigma_{k\alpha} \otimes u_{\alpha,k}) dz, \tag{2.1}$$

$$M_{\alpha\beta} = \int_{-h}^{h} z(\sigma_{\alpha\beta} + \sigma_{k\beta} \otimes u_{\alpha,k}) dz, \tag{2.2}$$

$$Q_{ij} = \int_{-h}^{h} (\sigma_{ij} + \sigma_{kj} \otimes u_{i,k}) dz, \tag{2.3}$$

$$u_{\alpha}^{*} = \frac{3}{2h^3} \int_{-h}^{h} z u_{\alpha} dz \, , \tag{2.4}$$

$$u_3^{*} = \frac{3}{4h^3} \int_{-h}^{h} (h^2 - z^2) u_3 dz, \tag{2.5}$$

$$\psi_{ij} = \frac{1}{2} \int_{-h}^{h} (h^2 - z^2) \sigma_{ij} dz \, , \tag{2.6}$$

where all the quantities represent column matrices of type $a = (a', a'')^T$.

From (1.5′) in virtue of (2.1), (2.5), (2.6) we obtain

$$u_{\alpha}^{*} = -(M - \lambda_5 S)^{-1}(M + \lambda_5 S) u_{3,\alpha}^{*} + \frac{3}{2h^3}(M - \lambda_5 S)^{-1} \psi_{\alpha 3} - (M - \lambda_5 S)^{-1} M B_{\alpha 3}, \tag{2.7}$$

where

$$B_{\alpha 3} = \frac{3}{4h^3} \int_{-h}^{h} (h^2 - z^2) A_{\alpha 3} dz$$

From formula (1.5) we have the following relations

$$\sigma_{\alpha\alpha} = k + (\Lambda + 2M)\varepsilon_{\alpha\alpha} + \Lambda(\varepsilon_{3-\alpha\; 3-\alpha} + \varepsilon_{33}),$$
$$\varepsilon_{33} = (\Lambda + 2M)^{-1}(\sigma_{33} - \Lambda \varepsilon_{\beta\beta} - k).$$

From the latter two formulas we obtain

$$\sigma_{\alpha\alpha} = (\Lambda^* + 2M)\varepsilon_{\alpha\alpha} + \Lambda^* \varepsilon_{3-\alpha\; 3-\alpha} + \Lambda(\Lambda + 2M)^{-1}\sigma_{33} + (I - \Lambda(\Lambda + 2M)^{-1})k, \tag{2.b}$$

where

$$\Lambda^* = \Lambda - \Lambda(\Lambda + 2M)^{-1}\Lambda, \ \mathrm{I}=\{1,1\}.$$

From this according to formula (2.1) and taking into account (2.7) we obtain

$$M_\alpha = -\frac{2h^3}{3}\left[(\Lambda^* + 2M)\partial_\alpha^2 + \Lambda^* \partial_{3-\alpha}^2\right](M - \lambda_5 S)^{-1}(M + \lambda_5 S)u_3^* +$$
$$+ (\Lambda^* + 2M)(M - \lambda_5 S)^{-1}\psi_{\alpha 3,\alpha} + \Lambda^*(M - \lambda_5 S)^{-1}\psi_{3-\alpha,3-\alpha} +$$
$$+ \Lambda(\Lambda + 2M)^{-1}\int_{-h}^{h} z\sigma_{33} dz + M_\alpha^{NL}, \quad (2.8)$$

where

$$M_\alpha^{NL} = \int_{-h}^{h} z\sigma_{k\alpha} \otimes u_{\alpha,k} dz + \frac{2h^3}{3}\left\{(\Lambda^* + 2M)\left[A_\alpha^* - (M - \lambda_5 S)^{-1} MB_{\alpha,\alpha}\right] + \Lambda^*\left[A_{3-\alpha}^* -\right.\right.$$
$$\left.\left. - (M - \lambda_5 S)^{-1} MB_{3-\alpha,3-\alpha}\right]\right\},$$
$$A_{\alpha\beta}^* = \frac{3}{4h^3}\int_{-h}^{h} zA_{\alpha\beta} dz, \quad A_{\alpha\alpha}^* = A_\alpha^*.$$

In virtue of formulas (2.2) and (1.5′) we have

$$M_{\alpha\beta} = \int_{-h}^{h} z\left[(M - \lambda_5 S)u_{\alpha,\beta} + (M + \lambda_5 S)u_{\beta,\alpha} + MA_{\alpha\beta}\right]dz = \frac{2h^3}{3}\left[(M - \lambda_5 S)u_{\alpha,\beta}^*\right.$$
$$\left. + (M + \lambda_5 S)u_{\beta,\alpha}^* + 2MA_{\alpha\beta}^*\right].$$

If we insert (2.7) expression of u_α^* – in the latter formula, we obtain

$$M_{\alpha\beta} = -\frac{4h^3}{3}M(M - \lambda_5 S)^{-1}(M + \lambda_5 S)u_{3,\alpha\beta}^* + I\psi_{\alpha 3.\beta}$$
$$+ (M + \lambda_5 S)(M - \lambda_5 S)^{-1}\psi_{\beta 3,\alpha} + M_{\alpha\beta}^{NL}, \quad (2.9)$$

where

$$M_{\alpha\beta}^{NL} = \int_{-h}^{h} z\sigma_{k\beta} \otimes u_{\alpha,k} dz + \frac{2h^3}{3}\left[2MA_{\alpha\beta}^* - MB_{\alpha,\beta} - (M + \lambda_5 S)(M - \lambda_5 S)^{-1} MB_{\beta,\alpha}\right].$$

From the first two equations of equilibrium system (1.1), in virtue of formulas (2.2) and (2.3), we obtain

$$Q_{\alpha 3} = M_{\alpha\beta.\beta} + h\left(g_\alpha^+ + g_\alpha^-\right) - \frac{2h^3}{3}\rho\partial_t^2 u_\alpha^* - \int_{-h}^{h} zf_\alpha dz.$$

If in this equality we insert expressions (2.8) and (2.9) of M_α and $M_{\alpha\beta}$, respectively, we obtain

$$Q_{\alpha 3} = -\frac{2h^3}{3}(\Lambda^* + 2M)(M - \lambda_5 S)^{-1}(M + \lambda_5 S)\Delta u_{3,\alpha}^* + I\Delta\psi_{\alpha 3} +$$
$$+ (\Lambda^* + M + \lambda_5 S)(M - \lambda_5 S)^{-1}\psi_{\beta 3,\alpha\beta} + \Lambda(\Lambda + 2M)^{-1}\int_{-h}^{h} z\sigma_{33,\alpha} dz +$$
$$+ h(g_\alpha + g_\alpha^-) - \frac{2h^3}{3}\rho\partial_t^2 u_\alpha^* - \int_{-h}^{h} zf_\alpha dz + Q_{\alpha 3}^{NL}, \quad (2.10)$$

where for nonlinear part we have the following expression

$$Q_{\alpha 3}^{NL} = \int_{-h}^{h} z\partial_\beta(\sigma_{k\beta} \otimes u_{\alpha,k})dz + \frac{2h^3}{3}\left[(\Lambda^* + 2M)A_{\alpha,\alpha}^* + \Lambda^* A_{3-\alpha,\alpha}^* +\right.$$
$$\left. + 2MA_{\alpha 3-\alpha,3-\alpha}^* - M\Delta B_\alpha - (\Lambda^* + M + \lambda_5 S)(M - \lambda_5 S)^{-1} MB_{\beta,\alpha\beta}\right].$$

In all the above mentioned formulas by Greek deaf indices summarization is done from one to two. In addition, if in some term exist the same indices in different sides of equality, then summarization does not take place.

Let us integrate the third equation of equilibrium system (1.1). If we take into account formulas (2.3) and boundary conditions (2.a) on the surfaces of plate, we will have

$$Q_{3\alpha,\alpha} = 2h\rho\partial_t^2 \bar{u}_3 + \int_{-h}^{h} f_3 dz - g_3^+ + g_3^- \,, \tag{2.c}$$

where exist the following denotation

$$\bar{u}_j = \frac{1}{2h}\int_{-h}^{h} u_j dz$$

For calculation of the entities $\int_{-h}^{h} z\sigma_{33}dz,\ \psi_{\alpha 3}, \psi_{3\alpha}$ involved in (2.7)-(2.10) we use Simpson, trapezoid or Gauss formulas

$$\int_{-h}^{h} z\sigma_{33}dz = \frac{h^2}{3}(I+2\Gamma)(g_3^+ - g_3^-) + r_1[z\sigma_{33};\Gamma], \tag{2.11}$$

where

$$r_1[z\sigma_{33};\Gamma] = ((I-\Gamma)\rho_{Sm} + \Gamma\rho_{tr})[z(\sigma_{33} + \sigma_{j3}\otimes u_{3,j})] - \int_{-h}^{h} z\sigma_{j3}\otimes u_{3,j}dz,$$

Here ρ_{Sm}, ρ_{tr} – are the remainder term of corresponding quadratic formula, $\rho_{Sm} = (\rho'_{Sm}, \rho''_{Sm})^T$, $\rho_{tr} = (\rho'_{tr}, \rho''_{tr})^T$, $\Gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}$, γ', γ'' – are arbitrary parameters.

Analogously we obtain

$$\psi_{ij} = \frac{h^2}{3}(I+2\Gamma)Q_{ij} + r_2\left[z\int_0^z \sigma_{ij}dz;\Gamma\right], \tag{2.12}$$

where

$$r_2\left[z\int_0^z \sigma_{ij}dz;\Gamma\right] = ((I-\Gamma)\rho_{Sm} + \Gamma\rho_{tr})\left[z\int_0^z \sigma_{ij}dz\right] - \frac{h^2}{3}(I+2\Gamma)\int_{-h}^{h}\sigma_{kj}\otimes u_{i,k}dz$$

Now let us determine the relation between the entities $\psi_{\alpha 3}$ and $\psi_{3\alpha}$. Analogously to formula (2.7) we obtain

$$u_\alpha^* = -(M+\lambda_5 S)^{-1}(M-\lambda_5 S)u_{3,\alpha}^* + \frac{3}{2h^3}(M+\lambda_5 S)^{-1}\psi_{3\alpha} - (M+\lambda_5 S)^{-1}MB_{\alpha 3}.$$

From the latter and by above mentioned formulas we obtain the following relation

$$\psi_{3\alpha} = \frac{2h^3}{3}\left\{-[M'(M+\lambda_5 S) - M + \lambda_5 S]u_{3,\alpha}^* + (I-M')MB_{\alpha 3}\right\} + M'\psi_{\alpha 3}, \tag{2.13}$$

where

$$M' := (M+\lambda_5 S)(M-\lambda_5 S)^{-1}$$

From the formulas (2.13), (2.11) and (2.c) we will have the following equality

$$\psi_{\beta 3,\beta} = \frac{2h^3}{3}M'^{-1}\Big\{[M'(M+\lambda_5 S) - M + \lambda_5 S]\Delta u_3^* + (I+2\Gamma)\rho\partial_t^2\bar{u}_3 +$$

$$\frac{1}{2h}(I+2\Gamma)\left[\int_{-h}^{h} f_3 dz - g_3^+ + g_3^-\right] + \frac{3}{2h^3}\partial_\beta r_2[\tau_{3\beta};\Gamma] + (M'-I)MB_{\beta 3,\beta}\Big\}. \tag{2.14}$$

From this latter formula taking into account again formula (2.12) we obtain (we imply, that $\Gamma \neq -\frac{1}{2} I$)

$$Q_{\beta 3,\beta} = 2h(I+2\Gamma)^{-1} M'^{-1}\{[M'(M+\lambda_5 S) - M + \lambda_5 S]\Delta u_3^* +$$
$$+(I+2\Gamma)\rho\partial_t^2 \bar{u}_3 + \frac{1}{2h}(I+2\Gamma)\times\left(\int_{-h}^{h} f_3 dz - g_3^+ + g_3^-\right) +$$
$$+\frac{3}{2h^3}\partial_\beta\left(r_2[\tau_{3\beta};\Gamma] - M' r_2[\tau_{\beta 3};\Gamma]\right) + (M'-I)MB_{\beta 3,\beta}\}, \qquad (2.15)$$

where we have the following denotations

$$\tau_{ij} := z\int_0^z \sigma_{ij} dz \, .$$

Now again multiply the both sides of the third equation of equilibrium system (1.1) on $\frac{1}{2}(h^2 - z^2)$ – and integrate it from $-h$ – to h – . Taking into account formula (2.11), we will have

$$\psi_{3\beta,\beta} = \frac{2h^3}{3}\rho\partial_t^2 u_3^* + \frac{1}{2}\int_{-h}^{h}(h^2 - z^2) f_3 dz - \frac{h^2}{3}(I+2\Gamma)(g_3^+ - g_3^-) - r_1[z\sigma_{33};\Gamma]$$
$$-\int_{-h}^{h} z\sigma_{k3}\otimes u_{3,k} dz - \int_{-h}^{h} zdz\int_0^z(\sigma_{k3}\otimes u_{3,k})_{,\beta} dz.$$

From the latter taking into account formula (2.13) we obtain

$$\psi_{\beta 3,\beta} = \frac{2h^3}{3} M'^{-1}\{[M'(M+\lambda_5 S) - M + \lambda_5 S]\Delta u_3^* + \rho\partial_t^2 u_3^* - \frac{1}{2h}(I+2\Gamma)(g_3^+ - g_3^-)$$
$$+\frac{3}{4h^2}\int_{-h}^{h}(h^2 - z^2) f_3 dz - \frac{3}{2h^2}\left(\int_{-h}^{h} z\sigma_{k3}\otimes u_{3,k} dz + \int_{-h}^{h} zdz\int_0^z(\sigma_{k3}\otimes u_{3,k})_{,\beta} dz + r_1[z\sigma_{33};\Gamma]\right)$$
$$+(M'-I)MB_{\beta 3,\beta}\}\}. \qquad (2.14')$$

If we insert formulas (2.12) and (2.13) into (2.8), for the bending moment we will have the following expression

$$M_\alpha = -\frac{2h^3}{3}\left[(\Lambda^* + 2M)\partial_\alpha^2 + \Lambda^*\partial_{3-\alpha}^2\right]M'' u_3^* + (\Lambda^* + 2M)(M - \lambda_5 S)^{-1}$$
$$\times\left[\frac{h^2}{3}(I+2\Gamma)Q_{\alpha 3,\alpha} + \partial_\alpha r_2[\tau_{\alpha 3};\Gamma]\right] + \Lambda^*(M-\lambda_5 S)^{-1}$$
$$\times\left[\frac{h^2}{3}(I+2\Gamma)Q_{3-\alpha\ 3,3-\alpha} + \partial_{3-\alpha} r_2[\tau_{3-\alpha 3};\Gamma]\right] + \frac{h^2}{3}\Lambda(\Lambda+2M)^{-1}$$
$$\times\left[(I+2\Gamma)(g_3^+ - g_3^-) + \frac{3}{h^2} r_1[z\sigma_{33};\Gamma]\right] + M_\alpha^{NL}, \qquad (2.8a)$$

where

$$M'' = (M - \lambda_5 S)^{-1}(M + \lambda_5 S).$$

If we insert the same formulas into (2.9), we obtain the expression for twisting moment

$$M_{\alpha\beta} = -\frac{4h^3}{3} MM'' u_{3,\alpha\beta}^* + \frac{h^2}{3}\left[(I+2\Gamma)Q_{\alpha 3,\beta} + M'(I+2\Gamma)Q_{\beta 3,\alpha}\right] +$$
$$+\partial_\beta r_2[\tau_{\alpha 3};\Gamma] + M\partial_\alpha r_2[\tau_{\beta 3};\Gamma] + M_{\alpha\beta}^{NL}. \qquad (2.9a)$$

Now let us obtain equations for the quantities $Q_{\alpha 3}=(Q'_{\alpha 3},Q''_{\alpha 3})^T$ and $u_3^*=(u_3^{*\prime},u_3^{*\prime\prime})^T$, which represent vectors consisted of sharing forces and average bending, respectively.

Let us derive the both sides of formula (2.14) by x_α – and insert it into right-hand side of (2.10), in the same formula substitute $\psi_{\alpha 3}$ by expression (2.12). With respect to $Q_{\alpha 3}$ – we obtain the following equation

$$Q_{\alpha 3}-\frac{h^2}{3}(I+2\Gamma)\Delta Q_{\alpha 3}=-\frac{2h^3}{3}\dot{M}\Delta u^*_{3,\alpha}+\frac{h^2}{3}\left[\Lambda(\Lambda+2M)^{-1}-\overline{M}\right](I+2\Gamma)(g_3^+-g_3^-)_{,\alpha}$$
$$+h(g_\alpha^++g_\alpha^-)+\frac{2h^3}{3}\left[\overline{M}(I+2\Gamma)\rho\partial_t^2\overline{u}_{3,\alpha}-\rho\partial_t^2u_\alpha^*\right]+\frac{h^2}{3}\overline{M}(I+2\Gamma)\int_{-h}^{h}f_{3,\alpha}dz-$$
$$-\int_{-h}^{h}zf_\alpha dz+\overline{M}\partial^2_{\alpha\beta}r_2\left[\tau_{3\beta};\Gamma\right]+\Lambda(\Lambda+2M)^{-1}\partial_\alpha r_1\left[z\sigma_{33};\Gamma\right]+\Delta r_2\left[\tau_{\alpha 3};\Gamma\right]$$
$$+\frac{2h^3}{3}\overline{M}(M'-I)MB_{\beta 3,\alpha\beta}+Q^{NL}_{\alpha 3}, \qquad (2.16)$$

where we have the following denotations

$$\dot{M}:=(\Lambda^*+2M)M''-(\Lambda^*+M+\lambda_5S)(M-\lambda_5S)^{-1}(M+\lambda_5S-M'^{-1}(M-\lambda_5S)),$$
$$\overline{M}:=(\Lambda^*+M+\lambda_5S)(M-\lambda_5S)^{-1}M'^{-1}.$$

We can obtain the equation (2.16) also by using (2.14′). In this case we will have

$$Q_{\alpha 3}-\frac{h^2}{3}(I+2\Gamma)\Delta Q_{\alpha 3}=-\frac{2h^3}{3}\dot{M}\Delta u^*_{3,\alpha}+\frac{h^2}{3}\left[\Lambda(\Lambda+2M)^{-1}-\overline{M}\right](I+2\Gamma)\partial_\alpha(g_3^+-g_3^-)+$$
$$+h(g_\alpha^++g_\alpha^-)+\frac{2h^3}{3}\left[M'^{-1}\rho\partial_t^2u^*_{3,\alpha}+\rho\partial_t^2u_\alpha^*\right]+\frac{1}{2}\overline{M}\int_{-h}^{h}(h^2-z^2)f_{3,\alpha}dz$$
$$-\overline{M}\left(\int_{-h}^{h}z(\sigma_{k3}\otimes u_{3,k})_{,\alpha}dz+\int_{-h}^{h}zdz\int_0^z(\sigma_{k3}\otimes u_{3,k})_{,\beta}dz\right)+\left[\Lambda(\Lambda+2M)^{-1}-\overline{M}\right]\partial_\alpha r_1\left[z\sigma_{33};\Gamma\right]$$
$$-\frac{2h^3}{3}\overline{M}(I-M')MB_{\beta 3,\alpha\beta}-\int_{-h}^{h}zf_\alpha dz+Q^{NL}_{\alpha 3}\;. \qquad (2.16')$$

From the formula (2.10) we obtain

$$Q_{\beta 3,\beta}=-\frac{2h^3}{3}(\Lambda^*+2M)M''\Delta^2u_3^*+(I+\overline{M}M')\Delta\psi_{\beta 3,\beta}+\Lambda(\Lambda+2M)^{-1}\int_{-h}^{h}z\Delta\sigma_{33}dz+$$
$$+h(g_\beta^++g_\beta^-)_{,\beta}-\frac{2h^3}{3}\rho\partial_t^2u^*_{\beta,\beta}-\int_{-h}^{h}zf_{\beta,\beta}dz+Q^{NL}_{\beta 3,\beta}.$$

Let us insert in the latter formula (2.14) and equalize the obtained equation to the right-hand side of equality (2.15). We will obtain the following equation for the entity u_3^*

$$\frac{2h^3}{3}D'\Delta^2u_3^*+2h(I+2\Gamma)^{-1}D''\Delta u_3^*=L(\Delta,M',\overline{M},\Gamma)\{(I+2\Gamma)\rho\partial_t^2\overline{u}_3$$
$$+\frac{1}{2h}(I+2\Gamma)\left(\int_{-h}^{h}f_3dz-g_3^++g_3^-\right)+\frac{3}{2h^3}\partial_\beta r_2\left[\tau_{3\beta};\Gamma\right]+(M'-I)MB_{\beta 3,\beta}\}$$
$$-\frac{2h^3}{3}\partial_t^2u^*_{\beta,\beta}+h(g_\beta^++g_\beta^-)_{,\beta}+\frac{h^2}{3}\Lambda(\Lambda+2M)^{-1}\left[(I+2\Gamma)\Delta(g_3^+-g_3^-)\right.$$
$$\left.+\frac{3}{h^3}\Delta r_1\left[z\sigma_{33};\Gamma\right]\right]-\int_{-h}^{h}zf_{\beta,\beta}dz+\frac{3}{h^2}(I+2\Gamma)^{-1}r_2\left[\tau_{\beta 3};\Gamma\right]+Q^{NL}_{\beta 3,\beta}. \qquad (2.17)$$

where we have the following denotations

$$D' = (\Lambda^* + 2M)M'' - (M'^{-1} + \overline{M})(M'(M + \lambda_5 S) - M + \lambda_5 S),$$
$$D'' = (M + \lambda_5 S - M'^{-1}(M - \lambda_5 S)),$$
$$L(\Delta, M', \overline{M}, \Gamma) = \frac{2h^3}{3}(M'^{-1} + \overline{M})\Delta - 2h(I + 2\Gamma)^{-1} M'^{-1}.$$

when $\lambda_5 = 0$, then $M' = M'' = I \Rightarrow D'' = 0$ and $D' = \Lambda^* + 2M$, $\overline{M} = \Lambda^* + M$.

For this case (2.17) will transform into biharmonic equation

$$\frac{2h^3}{3}(\Lambda^* + 2M)\Delta^2 u_3^* = F. \qquad (2.18)$$

II.2. Let us construct the equations corresponding to extension-compression processes. Let us multiply the first two equations of (1.1) on $1/2h-$ and integrate it from $-h-$ to $h-$, we will obtain

$$\frac{1}{2h}\int_{-h}^{h} \partial_\beta(\sigma_{\alpha\beta} + \sigma_{k\beta} \otimes u_{\alpha,k}) = \rho\partial_t^2 \overline{u}_\alpha + \frac{1}{2h}\int_{-h}^{h} f_\alpha dz - \frac{1}{2h}(g_\alpha^+ - g_\alpha^-),$$

from which on the basis of relations (1.5) and (2.a) we write

$$(\Lambda^* + 2M)\overline{\varepsilon}_{\alpha\alpha,\alpha} + \Lambda^* \ \overline{\varepsilon}_{3-\alpha\ 3-\alpha,\alpha} + 2M\overline{\varepsilon}_{\alpha 3-\alpha,3-\alpha} + 2\lambda_5 \overline{\hbar}_{\alpha 3-\alpha,3-\alpha} = \rho\partial_t^2 \overline{u}_\alpha +$$
$$+ \frac{1}{2h}\int_{-h}^{h} f_\alpha dz - \frac{1}{2h}(g_\alpha^+ - g_\alpha^-) - \frac{1}{2h}\Lambda(\Lambda + 2M)^{-1}\int_{-h}^{h}\sigma_{33,\alpha}dz + \frac{1}{2h}\int_{-h}^{h}\partial_\beta(\sigma_{k\beta} \otimes u_{\alpha,k})dz, \qquad (2.19)$$

where

$$\overline{\varepsilon}_{ij} = \frac{1}{2h}\int_{-h}^{h}\varepsilon_{ij}dz, \quad \overline{\hbar}_{ij} = \frac{1}{2h}\int_{-h}^{h}\hbar_{ij}dz.$$

Let us rewrite the formula (2.19) in the following form

$$(\Lambda^* + 2M)\partial_\alpha \overline{\varepsilon}_{\beta\beta} + 2M\partial_{3-\alpha}(\overline{\varepsilon}_{\alpha 3-\alpha} - \partial_{3-\alpha}^{-1}\partial_\alpha \overline{\varepsilon}_{3-\alpha\ 3-\alpha}) + 2\lambda_5 \partial_{3-\alpha}\overline{\hbar}_{\alpha 3-\alpha} = \overline{F}_\alpha$$

where $\overline{F}_\alpha$ – denotes the right-hand side of expression (2.19).

If in the latter formula we calculate the expression in brackets and insert value of $\overline{\hbar}_{\alpha 3-\alpha}$ we will obtain

$$(\Lambda^* + 2M)\partial_\alpha \overline{\varepsilon}_{\beta\beta} + (M - \lambda_5 S)\partial_{3-\alpha}(u_{\alpha,3-\alpha} - u_{3-\alpha,\alpha}) = \rho\partial_t^2 \overline{u}_\alpha + \frac{1}{2h}\int_{-h}^{h} f_\alpha dz - \frac{1}{2h}(g_\alpha^+ - g_\alpha^-) -$$
$$- \frac{1}{2h}\Lambda(\Lambda + 2M)\int_{-h}^{h}\sigma_{33,\alpha}dz + \frac{1}{2h}\int_{-h}^{h}\partial_\beta(\sigma_{k\beta} \otimes u_{\alpha,k})dz + M(\partial_\alpha(\overline{u}_{k,3-\alpha} \otimes \overline{u}_{k,3-\alpha}) -$$
$$- \partial_{3-\alpha}(\overline{u}_{k,\alpha} \otimes \overline{u}_{k,3-\alpha})), \qquad \alpha = 1,\ 2,$$

If we derive the both sides of the obtained equation by the coordinate x_α and summarize it, we obtain

$$(\Lambda^* + 2M)\Delta\overline{\varepsilon}_{\beta\beta} = \rho\partial_t^2 \overline{u}_{\alpha,\alpha} + \frac{1}{2h}\int_{-h}^{h} f_{\alpha,\alpha}dz - \frac{1}{2h}(g_\alpha^+ - g_\alpha^-)_{,\alpha} - \frac{1}{2h}\Lambda(\Lambda + 2M)\int_{-h}^{h}\Delta\sigma_{33}dz +$$
$$+ \frac{1}{2h}\int_{-h}^{h}\partial_{\alpha\beta}^2(\sigma_{k\beta} \otimes u_{\alpha,k})dz - M[u_3^*, u_3^*] - R_1[\varepsilon], \qquad (2.20)$$

where $[u, v]$ is so called Monje-Amper operator,

$$[u,v] = \partial_{11}u \otimes \partial_{22}v - 2\partial_{12}u \otimes \partial_{12}v + \partial_{11}v \otimes \partial_{22}u$$
$$R_1[\varepsilon] = M\left([\bar{u}_3, \bar{u}_3] - [u_3^*, u_3^*] + [\bar{u}_\alpha, \bar{u}_\alpha]\right).$$

To calculate the integral $\int_{-h}^{h} \sigma_{33} dz$ involved in the obtained relations approximately we use Euler-Maclaurin formula

$$\int_{-h}^{h} (\sigma_{33} + \sigma_{k3} \otimes u_{3,k}) dz = h[g_3^+ + g_3^-] - \frac{h^2}{3}[g_{3,3}^+ - g_{3,3}^-] + o(h^\nu), \quad \nu \le 5 \tag{2.21}$$

where we have the denotation

$$g_3(z) = \sigma_{33} + \sigma_{k3} \otimes u_{3,k},$$

and ν depends on the order of smoothness of function $g_3(z)$.

If we consider the case of statics and there exists a limit

$$\lim_{z \to \pm h} (f_3 - \partial_\alpha(\sigma_{3\alpha} + \sigma_{k\alpha} \otimes u_{3,k}), \tag{2.22}$$

then

$$g_{3,3}^\pm = f_3^\pm - g_{\alpha,\alpha}^\pm$$

Therefore the considered integral will be calculated by the same precision, as above (2.11).

Let us consider the nonstationary case. In virtue of the third equation of equilibrium system and formula (2.22) we write

$$g_{3,3}^\pm = f_3^\pm - g_{\alpha,\alpha}^\pm + \rho\partial_t^2(u_3^+ - u_3^-) = f_3^\pm - g_{\alpha,\alpha}^\pm + \rho\partial_t^2 \int_{-h}^{h} u_{3,3} dz. \tag{2.23}$$

where $u_3^\pm = u_3(x_1, x_2, \pm h, t)$.

From Hooke law we have

$$u_{3,3} = (\Lambda + 2M)^{-1}(\sigma_{33} - \Lambda\varepsilon_{\beta\beta} - k) - \frac{1}{2}u_{k,3} \otimes u_{k,3}. \tag{2.24}$$

Integrating the latter equality we obtain

$$\int_{-h}^{h} u_{3,3} dz = (\Lambda + 2M)^{-1} \int_{-h}^{h} (\sigma_{33} + \sigma_{k3} \otimes u_{3,k} - \sigma_{k3} \otimes u_{3,k} - \Lambda\varepsilon_{\beta\beta} - k) + \int_{-h}^{h} u_{k,3} \otimes u_{k,3} dz.$$

If we denote $\sigma_{33} + \sigma_{k3} \otimes u_{3,k} \equiv g_3(z)$ and insert (2.23) in formula (2.21), we obtain

$$\int_{-h}^{h} g_3(z) dz = h[g_3^+ + g_3^-] - \frac{h^2}{3}[f_3^+ - f_3^-(g_{\alpha,\alpha}^+ - g_{\alpha,\alpha}^-)] - \frac{h^2}{3}\int_{-h}^{h} u_{3,3} dz.$$

Using (2.24) in this formula gives:

$$\int_{-h}^{h} g_3(z) dz = h[g_3^+ + g_3^-] - \frac{h^2}{3}[f_3^+ - f_3^- - (g_{\alpha,\alpha}^+ - g_{\alpha,\alpha}^-)] - \frac{h^2}{3}\rho\partial_t^2\left[(\Lambda + 2M)^{-1}\int_{-h}^{h} g_3(z) dz\right] +$$
$$+ \frac{h^2}{3}\rho\partial_t^2\left[(\Lambda + 2M)^{-1}\int_{-h}^{h} (\Lambda\varepsilon_{\alpha\alpha} + \sigma_{k3} \otimes u_{3,k}) dz + \frac{1}{2}\int_{-h}^{h} u_{k3} \otimes u_{k,3} dz\right].$$

Let us rewrite the obtained equality in the following form

$$\int_{-h}^{h} g_3 dz = A - \frac{h^2}{3}\rho(\Lambda + 2M)^{-1}\partial_t^2 \int_{-h}^{h} g_3 dz, \tag{2.25}$$

where

$$A = h[g_3^+ + g_3^-] - \frac{h^2}{3}[f_3^+ - f_{3k}^- - (g_{\beta,\beta}^+ - g_{\beta,\beta}^-)] - \frac{h^2}{3}\rho\partial_t^2(\Lambda + 2M)^{-1}\Lambda\bar{\varepsilon}_{\beta,\beta} + r(A),$$

$$r(A)=\frac{h^2}{3}\rho\partial_t^2\left[(\Lambda+2M)^{-1}\int_{-h}^{h}\sigma_{k3}\otimes u_{3,k}dz+\frac{1}{2}\int_{-h}^{h}u_{k,3}\otimes u_{k,3}dz\right]+o(h^{\nu}).$$

From (2.25) we obtain

$$\int_{-h}^{h}g_3dz=\left(I+h^2D_{tt}\right)^{-1}A,$$

where

$$D_{tt}=\frac{1}{3}\rho(\Lambda+2M)^{-1}\partial_t^2.$$

Let us return again to the bending equations and note the following: M_α, $Q_{\alpha 3}$ and In the nonlinear part of the expressions of entities $Q_{3\alpha}$ is isolated integral $\int_{-h}^{h}\sigma_{\beta\delta}\otimes u_{3,\beta\delta}dz$, which equals to

$$\int_{-h}^{h}\sigma_{\beta\delta}\otimes u_{3,\beta\delta}dz=2h\left[u_3^*,F^*\right]+u_{3,11}^*\otimes P_{11}-2u_{3,12}^*\otimes P_{12}+u_{3,22}^*\otimes P_{22}+\int_{-h}^{h}\sigma_{\beta\delta}\otimes\left(u_{3,\beta\delta}-u_{3,\beta\delta}^*\right)dz, \qquad (2.d)$$

where

$$\sigma_{\alpha\beta}=(-1)^{\alpha+\beta}\left[\partial_{3-\alpha}\partial_{3-\beta}F(x_1,x_2,x_3,t)+\frac{1}{2h}P_{\alpha\beta}(x_1,x_2,t)\right],$$

and we must take the function $P_{\alpha\beta}$ in the following form

$$P_{\alpha\alpha,\alpha}-P_{\alpha\beta,3-\alpha}=2h\,\rho\,\partial_t^2\,\bar{u}_\alpha+\int_{-h}^{h}\left[f_\alpha-\left(\sigma_{\beta\delta}\otimes u_{3,k}\right)_{,\beta}\right]dz-g_\alpha^++g_\alpha^-,$$

and

$$F^*=\frac{1}{2h}\int_{-h}^{h}Fdz$$

From the right-hand side of equation (2.17) due to expression (2.d) the mentioned equation will have the following form

$$\frac{2h^3}{3}D'\Delta^2u_3^*+2h(I+2\Gamma)^{-1}D''\Delta u_3=\left\{(I+2\Gamma)^{-1}M^{/-1}-\frac{h^2}{3}\left[\overline{M}+M^{/-1}-\Lambda(\Lambda+2M)^{-1}\right]\Delta\right\}(I+2\Gamma)\left(g_3^+-g_3^-\right)$$

$$+2h\left\{I-\frac{h^2}{3}\left[I+\left(\Lambda^*+M-\lambda_5S\right)(M+\lambda_5S)^{-1}(I+\Gamma)\right]\Delta\right\}\left[u_3^*,F^*\right]+\tilde{f}, \qquad (2.17')$$

where for brevity $\tilde{f}$ denotes all other terms in right-hand side of (2.17).

The system of differential equations corresponding to bending can be written in the classical form, when as a unknown functions there are chosen average deflection of Reissner and rotation of normals. For this we use the equilibrium equations:

$$\begin{cases} M_{1,1}+M_{12,2}-Q_{13}=\dfrac{2h^3}{3}\rho\partial_t^2u_1^*+\displaystyle\int_{-h}^{h}zf_1dz-h(g_1^++g_1^-) \\ M_{21,1}+M_{2,2}-Q_{23}=\dfrac{2h^3}{3}\rho\partial_t^2u_2^*+\displaystyle\int_{-h}^{h}zf_2dz-h(g_2^++g_2^-) \\ Q_{31,1}+Q_{32,2}=2h\rho\partial_t^2\bar{u}_3+\displaystyle\int_{-h}^{h}f_3dz-(g_3^+-g_3^-). \end{cases} \qquad (2.26)$$

Taking into account (2.13) from formulas (2.7) and (2.7′) we obtain

$$Q_{\alpha 3}-\int_{-h}^{h}\sigma_{k3}\otimes u_{\alpha,k}dz=2h(I+2\Gamma)^{-1}\left[(M-\lambda_5 S)u_\alpha^*+(M+\lambda_5 S)u_{3,\alpha}^*+MB_{\alpha 3}\right] \qquad (2.27)$$

$$Q_{3\alpha}-\int_{-h}^{h}\sigma_{k\alpha}\otimes u_{3,k}dz=2h(I+2\Gamma)^{-1}\left[(M+\lambda_5 S)u_\alpha^*+(M-\lambda_5 S)u_{3,\alpha}^*+MB_{\alpha 3}\right] \qquad (2.28)$$

Inserting (2.27) in formulas (2.8) and (2.9) correspondingly we obtain

$$M_\alpha=\frac{2h^3}{3}\Bigg\{(\Lambda^*+2M)u_{\alpha,\alpha}^*+\Lambda^* u_{3-\alpha,3-\alpha}^*+\frac{1}{2h}\Lambda(\Lambda+2M)^{-1}(I+2\Gamma)(g_3^+-g_3^-)+$$
$$+(\Lambda^*+2M)(M-\lambda_5 S)^{-1}\left[\frac{1}{2}(I+2\Gamma)\int_{-h}^{h}\left(\sigma_{k3}\otimes u_{\alpha,k}\right)_{,\alpha}+MB_{\alpha 3,3}\right]+$$
$$+\Lambda^*(M-\lambda_5 S)^{-1}\left[\frac{1}{2}(I+2\Gamma)\int_{-h}^{h}\left(\sigma_{k3}\otimes u_{3-\alpha,k}\right)_{,3-\alpha}+MB_{3-\alpha 3,3-\alpha}\right]\Bigg\}+M_\alpha^{NL}. \qquad (2.29)$$

$$M_{\alpha\beta}=\frac{2h^3}{3}\Bigg\{(M+\lambda_5 S)u_{\beta,\alpha}^*+(M-\lambda_5 S)u_{\alpha,\beta}^*+\frac{1}{2}(I+2\Gamma)\left(\sigma_{k3}\otimes u_{\alpha,k}\right)_{,\beta}dz+$$
$$+\frac{1}{2}M'(I+2\Gamma)\int_{-h}^{h}(\sigma_{k3}\otimes u_{\beta,k})_{,\alpha}dz+MB_{\alpha 3,\beta}+M'MB_{\beta 3,\alpha}\Bigg\}+M_{\alpha\beta}^{NL}. \qquad (2.30)$$

Inserting the formulas (2.27)-(2.30) in system (2.26) we obtain

$$\begin{cases}(M-\lambda_5 S)\Delta u_\alpha^*+(\Lambda^*+M+\lambda_5 S)\partial_\alpha u_{\beta,\beta}^*-\dfrac{3}{h^2}(I+2\Gamma)^{-1}\left[(M-\lambda_5 S)u_\alpha^*+(M+\lambda_5 S)u_{3,\alpha}^*\right]=\\ =\rho\partial_t^2 u_\alpha^*+\dfrac{3}{2h^3}\displaystyle\int_{-h}^{h}zf_\alpha dz-\dfrac{3}{2h^2}(g_\alpha^++g_\alpha^-)-\dfrac{1}{2h}\Lambda(\Lambda+2M)^{-1}(I+2\Gamma)\partial_\alpha(g_3^+-g_3^-)-\\ -\Phi_{\alpha,\alpha}-\Phi_{\alpha\, 3-\alpha,3-\alpha}+\dfrac{3}{h^2}(I+2\Gamma)^{-1}B_{\alpha 3}+\dfrac{3}{2h^3}\displaystyle\int_{-h}^{h}\sigma_{k3}\otimes u_{\alpha,k}dz,\qquad \alpha=1,2\\ (M-\lambda_5 S)\Delta u_3^*+(M+\lambda_5 S)u_{\beta,\beta}^*=(I+2\Gamma)\Big\{\rho\partial_t^2\overline{u}_3+\dfrac{1}{2h}\displaystyle\int_{2h}^{h}f_3 dz-\dfrac{1}{2h}(g_3^+-g_3^-)-\\ -(I+2\Gamma)^{-1}MB_{\beta 3,\beta}+\dfrac{1}{2h}\displaystyle\int_{-h}^{h}(\sigma_{k\beta}\otimes u_{3,k})_{,\beta}dz\Big\},\end{cases} \qquad (2.31)$$

where Φ_α and $\Phi_{\alpha\beta}$ denotes nonlinear terms involved in quantities M_α and $M_{\alpha\beta}$.

II.3.Let us consider he problem if construction of KMR type systems for binary mixture of piezo-elastic plates. For this aim we used results of [2,section 9].

Let a vector of electrical induction-D, a tension of electrical field-E and an electrical potential φ have two-component representation as in I section .

The following result is true:

If equations of states of type (2.36), (2.37) of monograph [10] or generalized Hooke's law [2,p.117] is supposed to have a form of (1.5) then the scheme of section II is applicable for constructing and justifying (in Physical Soundness sense) linear systems of KMR type for binary mixtures of piezoelectrics and electrically conductive transversally isotropic elastic plates.

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