Necessary Conditions of Externality of Initial Moment for One Class Variation Problem with Delay Argument

L. Alkhazishvili, T. Tadumadze
Control Theory Chair, Iv. Javakhishvili Tbilisi State University

Abstract.
Necessary conditions of extremality are obtained in the form of Euler's equation, the condition of Wierstrass-Erdmann and transversality condition. The condition in the initial moment unike the early known condition, contains a new member.

Let $J=[a,b]$ be a finite interval and $O\subset R^n$ be an open set; the function $f(t,x_1,x_2,x_3,)$ is defined on $J\times O\times O\times R^n$ and satisfies the following conditions: for almost all $t\in J$, the function f is continuously differentiable with respect to (x_1,x_2,x_3) for each fixed $(x_1,x_2,x_3)\in O^2\times R^n$, the functions $f, f_{x_i}, i=1,2,3$ are measurable on J; for arbitrary compacts $K\subset O, V\subset R^n$ there exists the summable function $m_{K,V}(t), t\in J$, such that

$$\left|f(t,x_1,x_2,x_3)\right|+\sum_{i=1}^{3}\left|f_{x_i}(\cdot)\right|\le m_{K,V}(t),\ \forall(t,x_1,x_2,x_3)\in J\times K^2\times V.$$

Further, let Φ be a set of absolutely continuous functions $x(t)\in O, t\in J$, satisfying the condition $\left|\dot x(t)\right|\le const$. $\dot\tau(t)>0, t\in J$ is absolutely continuous function satisfying the conditions $\tau(t)\le t$, $\dot\tau(t)>0$; $\varphi(t)\in O,\ \ t\in[\tau(a),b]$ is piecewise continuous function with a finite number of discontinuity points, satisfying the conditon $cl\{\varphi(t):t\in[\tau(a),b]\}\subset O;\ \ a_0, a_1\subset O$ are fixed points.

Let us consider the variational problem

$$I(z)=\int_{t_0}^{t_1} f(t,x(t),x_{t_0}(\tau(t)),\dot x(t))dt\to min,\ z=(t_0,t_1,x(\cdot))\in A=J^2\times O,$$
$$x(t_0)=a_0,\ x(t_1)=a_1, \tag{1}$$

where,

$$x_{t_0}(t)=\begin{cases}\varphi(t), t\in[\tau(a),t_0),\\ x(t), t\in[t_0,b].\end{cases}$$

<u>DEFINITION1.</u> The element $z\in A$ is said to be admissible, if the condition holds. The set of admissible elements will be denoted by A_0.

<u>DEFINITION2.</u> The element $\tilde z=(\tilde t_0,\tilde t_1,\tilde x(\cdot))\in A_0$ is said to be locally extremal, if there exists a number $\delta>0$ such that for an arbitrary element $z\in A_0$ satisfying

$$\left|\tilde t_0-t_0\right|+\left|\tilde t_1-t_1\right|+\max_{t\in J}\left|\tilde x(t)-x(t)\right|\le\delta$$

the inequality $I(\tilde z)\le I(z)$ holds. Variational problem consists in finding locally extremal element.

<u>THEOREM1</u>. *Let* $\tilde z\in A_0$ *be a locally extremal element,*

$$\tilde t_0\in(a,b),\ \tilde t_1\in(a,b],\ y_0=y(\tilde t)\in[\tilde t_0,\tilde t_1\}$$

And there exists the finite limits: $\dot{\tilde x}(\tilde t_0^-),\ \dot{\tilde x}(\tilde t_1^-),\ \dot y(\tilde t_0^-),\ f_{x_3}[\tilde t_1^-],\ f_{x_3}[\tilde t_0^+]$;

$$\lim_{\omega\to\omega_0^-}\tilde f(\omega)=f_0^-,\ \ \omega\in R_{\tilde t_0}^-\times O^2\ \ i=1,2;\ \ \omega_0^-=\left(\tilde t_0,a_0,\varphi\left(\tau\left(\tilde t_0^-\right)\right)\right),\ \ R_{\tilde t_0}^-=\left]-\infty,\tilde t_0\right];$$

$$\lim_{\omega\to\omega_3^-}\tilde f(\omega)=f_2^-,\ \omega\in R_{\tilde t_1-}^-\times O^2,\ \omega_3^-=\left(\tilde t_1,\tilde x(t_1),\tilde x(\tau(t_1))\right).$$

Then the following conditions are fulfilled:

1) for almost all $t \in [\tilde t_0, \tilde t_1]$

$$\tilde f_{x_3}[t] = \tilde f_{x_3}[\tilde t_0^{+}] + \int_{\tilde t_0}^{t} \left(\tilde f_{x_1}[s] + \chi(\gamma(s)) \tilde f_{x_2}[\gamma(s)] \dot\gamma(s) \right) ds,$$

where $\chi(t)$ *is characteristic function of interval* $[\tilde t_0, \tilde t_1]$, $y(t)$ *is the function inverse to* $\tau(t)$, $\tilde f[t] = \tilde f(t, \tilde x(t), \tilde x_{\tilde t_0}(\tau(t)), \dot{\tilde x}(t))$

2) if at poin $t \in (\tilde t_0, \tilde t_1)$ *the function* $\tilde f_{x_3}[t]$ *has the one-side limits, then*

$$\tilde f_{x_3}[t^-] = \tilde f_{x_3}[t^+];$$

3)

$$\begin{cases} \tilde f_{x_3}[\tilde t_0^{+}]\dot{\tilde x}(t_0^-) \le f_0^- + f_1^- \dot\gamma(\tilde t_0^-), \\ \tilde f_x[\tilde t_1^-]\dot{\tilde x}(\tilde t_1^-) \ge f_2^- . \end{cases}$$

<u>THEOREM2</u>. *Let* $\tilde z \in A_0$ *be a locally extremal element,* $\tilde t_0 \in [a,b)$, $\tilde t_1 \in (a,b)$, $\gamma_0 \in [\tilde t_0, \tilde t_1)$; *and there exist the finite limits* $\dot{\tilde x}(\tilde t_0^+)$, $\dot{\tilde x}(\tilde t_1^+)$, $\dot\gamma(\tilde t_0^+)$, $\tilde f_{x_3}[\tilde t_1^-]$, $\tilde f_{x_2}[\tilde t_0^+]$

$$\lim_{\omega \to \omega_0} \tilde f(\omega) = f_0^+,\ \omega \in R_{\tilde t_0}^+ \times O^2,\ i = 1,2;\ \omega_0^+ = (\tilde t_0, a_0, \varphi(\tau(\tilde t_0)));$$

$$\lim_{(\omega_1,\omega_2) \to (\omega_1^0, \omega_2^+)} \left[\tilde f(\omega_1) - \tilde f(\omega_2) \right] = f_1^+,\ \omega_i \in R_{\gamma_0}^+ \times O^2,\ i = 1,2;\ \omega_2^+ = (\gamma_0, \tilde x(\gamma_0), \tilde\varphi(\tilde t_0^+));$$

$$\lim_{\omega \to \omega_3^+} \tilde f(\omega) = f_2^+,\ \omega \in R_{\tilde t_1}^+ \times O^2,\ \omega_3^+ = (\tilde t_1, \tilde x(\tilde t_1), \tilde x(\tau(t_1^+)));$$

Then the conditions 1), 2) are fulfilled and, moreover,

4)

$$\begin{cases} \tilde f_{x_3}[\tilde t_0^+]\dot{\tilde x}(\tilde t_0^+) \ge f_0^+ + f_1^+ \dot\gamma(\tilde t_0^+), \\ \tilde f_{x_3}[\tilde t_1^-]\dot{\tilde x}(\tilde t_1^+) \le f_2^+ . \end{cases}$$

<u>THEOREM3</u>. *Let* $\tilde z \in A_0$ *be a locally extremal element,* $\tilde t_0 \in (a,b)$, $\tilde t_1 \in (a,b)$, $\gamma_0 \in (\tilde t_0, \tilde t_1)$ *and the assumptions of theorems 1,2 are fulfilled. Let, besides:*

$$f_0^+ + f_1^+ \dot\gamma(\tilde t_0^+) = f_0^- + f_1^- \dot\gamma(\tilde t_0^-) = f_1,\quad f_2^+ = f_2^- = f_2,$$

$$\dot{\tilde x}(\tilde t_0^-) = \dot{\tilde x}(\tilde t_0^+) = \dot x_0,\qquad \dot{\tilde x}(\tilde t_1^-) = \dot{\tilde x}(\tilde t_1^+) = \dot x_1.$$

Then the conditions 1) ,2) are fulfilled, and

5)

$$\begin{cases} \tilde f_{x_3}[\tilde t_0^+]\dot x_0 = f_1, \\ \tilde f_{x_3}[\tilde t_1^-]\dot x_1 = f_2 . \end{cases}$$

<u>REMARK.</u> Assume that the function $\dot\gamma(t)$ is continuous at point $\tilde t_0$, the function $\varphi(t)$ is continuous on $[\tau(a), b]$; the function $f(t, x_1.x_2, x_3)$ is continuous at points $(\tilde t_0, a_0, \varphi(\tau(\tilde t_0)))$, $(\gamma_0, \tilde x(\gamma_0, a_0))$, $(\gamma_0, \tilde x(\gamma_0), \varphi(\tilde t_0))$, $(\tilde t_1, \tilde x(\tilde t_1), \tilde x(\tau(\tilde t_1)))$; $\dot{\tilde x}(t)$ is continuous at points $\tilde t_0, \tilde t_1$. Then it is clear that in Theorem 3

$$\dot x_0 = \dot{\tilde x}(\tilde t_0),\quad \dot x_1 = \dot{\tilde x}(\tilde t_1)$$

$$f_1 = \tilde f[\tilde t_0] + \left(f(\gamma_0, \tilde x(\gamma_0), a_0, \dot{\tilde x}(\gamma_0)) - f(\gamma_0, \tilde x(\gamma_0)\varphi(\tilde t_0), \dot{\tilde x}(\gamma_0)) \right) \dot\gamma(\tilde t_0),$$

$$f_2 = \tilde f[\tilde t_1],\quad \tilde f_{x_3}[\tilde t_0^+] = \tilde f_{x_3}[\tilde t_0],\quad \tilde f_{x_3}[\tilde t_1^-] = \tilde f_{x_3}[\tilde t_1] .$$

These theorems have been proved in standard way [1], and are based on necessary conditions of optimality [2].

REFERENCES

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