Decomposition Sequential-Parallel Scheme of High Degree Precision for Non-Homogenous Evolution Equation

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Abstract

The Cauchy abstract problem for non-homogenous evolution equations is considered in Banach space in case of limited operator. Is constructed sequential-parallel decomposition scheme with third degree precision. For the approximation of solution explicit prior estimations are obtained.

As is well known, decomposition method is sufficiently general for obtaining the economical schemes for the solution of the multidimensional problems of mathematical physics. They can be divided in two groups: the schemes of sequential account (N. N. Ianenko [1], A. A. Samarskii [2], E. G. Diakonov [3], Marchuk G. I. [4], D.G. Gordeziani [5], Temam R. [6], Gegechkori Z. G. and Demidov G. V. [7]) and the schemes of parallel account (D. G. Gordeziani and H. V. Meladze [8], [9], D. G. Gordeziani and A. A. Samarskii [10]).

In the above-stated works the considered schemes are of the first or second precision order. As far as we know, the high degree precision decomposition formulas in case of two addands $\left(A = A_1 + A_2\right)$ for the first time were obtained in the work [11].

In the present work, there a symmetrized sequential-parallel method of the third degree precision is offered for the solution of Cauchy abstract problem in case of bounded operator.

The present scheme may be generalized for any finite number of addends $\left(A = A_1 + A_2 + ... + A_m, \quad m \geq 2\right)$.

Let us consider Cauchy abstract problem in Banach space X:

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi. \tag{1}$$

Here A is bounded linear operator, φ is a given element from X, $f(t) \in C([0,\infty); X)$.

The solution of the problem (1) is given by the following formula:

$$u(t) = U(t,A)\varphi + \int_0^t U(t-s,A) f(s) ds, \tag{2}$$

where

$$\mathrm{U(t,A) = exp(-tA)} = \sum_{\mathrm{k=0}}^{\infty} (-1)^{\mathrm{k}} \frac{t^k}{k!} A^k .$$

Let $A = A_1 + A_2$, where A_i, (i=1,2) are bounded linear operators in X.

Let us introduce difference net domain:

$$\varpi_\tau = \{ t_k = k\tau : \tau > 0, k = 1,2,..\}.$$

Along with problem (1) on each [t_{k-1}, t_k] interval we consider two sequences of the following problems:

$$\begin{cases} \dfrac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) = \dfrac{\alpha}{2} f(t_k) - 2\sigma_0 (t_k - t) f'(t_k), \\ v_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), \\ \dfrac{dv_k^2(t)}{dt} + A_2 v_k^2(t) = \dfrac{1}{2} f(t_k) - 2\sigma_1 (t_k - t) f'(t_k), \\ v_k^2(t_{k-1}) = v_k^1(t_k), \\ \dfrac{dv_k^3(t)}{dt} + \bar{\alpha} A_1 v_k^3(t) = \dfrac{\bar{\alpha}}{2} f(t_k) - 2\sigma_2 (t_k - t) f'(t_k) + \dfrac{(t_k - t)^2}{2} f''(t_k), \\ v_k^3(t_{k-1}) = v_k^2(t_k); \end{cases}$$

$$\begin{cases} \dfrac{dw_k^1(t)}{dt} + \alpha A_2 w_k^1(t) = \dfrac{\alpha}{2} f(t_k) - 2\sigma_0 (t_k - t) f'(t_k), \\ w_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), \\ \dfrac{dw_k^2(t)}{dt} + A_1 w_k^2(t) = \dfrac{1}{2} f(t_k) - 2\sigma_1 (t_k - t) f'(t_k), \\ w_k^2(t_{k-1}) = w_k^1(t_k), \\ \dfrac{dw_k^3(t)}{dt} + \bar{\alpha} A_2 w_k^3(t) = \dfrac{\bar{\alpha}}{2} f(t_k) - 2\sigma_2 (t_k - t) f'(t_k) + \dfrac{(t_k - t)^2}{2} f''(t_k), \\ w_k^3(t_{k-1}) = w_k^2(t_k). \end{cases}$$

Here $\alpha, \sigma_0, \sigma_1, \sigma_2$ are numerical complex parameters, which will be defined later, $u_0(0) = \varphi$. The function $u_k(t)$ on each $[t_{k-1}, t_k]$, $(k = 1,2,...)$ interval is defined as follows:

$$u_k(t) = \frac{1}{2}\left(v_k^3(t) + w_k^3(t)\right).$$

We declare the function $u_k(t)$ as the approached solution of the problem (1).
Above-stated scheme in case of homogenous equation is considered in [12].

<u>THEOREM.</u> *If* $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$, $f(t) \in C^3([0,\infty); X)$, *and the parameters* $\sigma_0, \sigma_1, \sigma_2$ *satisfy the following relations:*

$$\sigma_0 = \frac{2-\bar{\alpha}}{4+\alpha} - \frac{2+\bar{\alpha}}{4+\alpha}\sigma_1, \quad \sigma_2 = \frac{1+\bar{\alpha}}{2(4+\alpha)} - \frac{3-2\bar{\alpha}}{4+\alpha}\sigma_1$$

where σ_1 *is any complex number,then*

$$\begin{aligned} \|u(t_k) - u_k(t_k)\| \le c e^{\omega t_k} t_k \tau^3 \Bigg(\|\varphi\| + t_k \sup_{t\in[0,t_k]} \|f(t)\| + \\ + \sup_{t\in[0,t_k]} \|f'(t)\| + \sup_{t\in[0,t_k]} \|f''(t)\| + \sup_{t\in[0,t_k]} \|f'''(t)\| \Bigg), \end{aligned} \tag{3}$$

where c, ω *are positive constants.*

<u>SCHEME OF THE PROOF</u>:
According to the property of semigroup the formula (2) we can transform as follows:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A) F_i^{(1)}, \tag{4}$$

where

$$F_i^{(1)} = \int_{t_{i-1}}^{t_i} U(t_i - s, A) f(s) ds .$$

$u_k(t_k)$ can be written in the following expression:

$$u_k(t_k) = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau) F_i^{(2)}, \tag{5}$$

where

$$V(\tau) = \frac{1}{2}\left[U(\tau,\overline{\alpha}A_1)U(\tau,A_2)U(\tau,\alpha A_1) + U(\tau,\overline{\alpha}A_2)U(\tau,A_1)U(\tau,\alpha A_2)\right],$$

$$F_i^{(2)} = \int_{t_{i-1}}^{t_i} V_0(\tau,t_i - s)\left(\frac{\alpha}{2} f(t_i) - 2\sigma_0(t_i - t) f'(t_i)\right) ds +$$

$$+ \int_{t_{i-1}}^{t_i} V_1(\tau,t_i - s)\left[\frac{1}{2} f(t_i) - 2\sigma_1(t_i - t) f'(t_i)\right] ds +$$

$$+ \int_{t_{i-1}}^{t_i} V_2(t_i - s)\left[\frac{\overline{\alpha}}{2} f(t_i) - 2\sigma_2(t_i - t) f'(t_i) + \frac{(t_i - t)^2}{2} f''(t_i)\right] ds$$

and

$$V_o(\tau,t) = \frac{1}{2}\left[U(\tau,\overline{\alpha}A_1)U(\tau,A_2)U(t,\alpha A_1) + U(\tau,\overline{\alpha}A_2)U(\tau,A_1)U(t,\alpha A_2)\right],$$

$$V_1(\tau,t) = \frac{1}{2}\left[U(\tau,\overline{\alpha}A_1)U(t,A_2) + U(\tau,\overline{\alpha}A_2)U(t,A_1)\right],$$

$$V_0(t) = \frac{1}{2}\left[U(t,\overline{\alpha}A_1) + U(t,\overline{\alpha}A_2)\right].$$

From the equalities (4) and (5) we have:

$$\begin{aligned} u(t_k) - u_k(t_k) &= \left[U^k(\tau,A) - V^k(\tau)\right]\varphi + \\ &+ \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau,A) - V^{k-i}(\tau)\right)F_i^{(1)} + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right]. \end{aligned} \tag{6}$$

It is proved that, (see[12]):

$$\left\|U^k(\tau,A) - V^k(\tau)\right\| \le c e^{\omega t_k} t_k \tau^3 .$$

Also the following estimation takes place:

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \le c e^{\omega t_k} t_k \tau^4 \left(\left\|f(t_k)\right\| + \left\|f'(t_k)\right\| + \left\|f''(t_k)\right\| + \sup_{t\in[0,t_k]} \left\|f'''(t)\right\| \right).$$

According to this estimations and formula (6) we obtain estimation (3).
Scheme of the proof is finished.

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