On one Common Generalization of Some Well-Known Analytical Constructions

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Abstract

The function, the private cases of which are Riemannian integral, the functions of type of multiplicative integral, the directional derivative, the total variation of function and some others, are defined.

1.THE BASIC DEFINITIONS

The present paper describes one situation of such a type, when some mathematical constructions are the private cases of other, more complicate construction, owing to selection of values of parameters. So, it can be useful for young mathematicians. Moreover, independently important is observation of relations of different mathematical constructions.

Let us define function $K(a,b,f(t,dt),*,Step,Ord)$ and its arguments. In what follows, we shall assume that: $a,b\in R; \{(t,s)\alpha \quad f(t,s)\}:V\to M$,where $V\subset R^2$, and the limiting structure in M is determined by means of directednesses, as in [1]. M is an abstract monoid whose algebraic structure is defined by a binary associative and continuous operation $*$ and by the unity $e.$ The admissible values of parameter $Step$ are: $ZeroStep=0,\quad FullStep=1$ and $RndStep$ -the rundom number from $[0,1]$ (note that $RndStep+RndStep$ can be equal to $\frac{1}{2}+\frac{1}{4}$). Σ denotes the set of all partitions of the form $\sigma=\{0=s_0<\Lambda<s_n=1\}$ and $\Delta s_i=s_i-s_{i-1}$, $|\sigma|=\max\{\Delta s_i \ |i=1,...,n\}$.

Now, $Ord\in\{NormOrd, InscrOrd\}$ and is the relation on the Σ:

$\sigma_1(NormOrd)\sigma_2 \Leftrightarrow |\sigma_1|\le|\sigma_2|$;

$\sigma_1(InscrOrd)\sigma_2 \Leftrightarrow$ (σ_1 is inscribed in σ_2).

Preliminarily define $K(a,b,f(t,dt),*,Step,\sigma)$, where $\sigma=\{0=s_0<\Lambda<s_n=1\}$. Take $g=e.$ If the following loop

$for \quad i:=1 \ to \quad n \quad do$

$g=g*f(a+(s_{i-1}+Step\cdot\Delta s_i)(b-a),\Delta s_i(b-a))$,

will be performed correctly, then we shall denote

$$K(a,b,f(t,dt),*,Step,\sigma)=g.$$

<u>DEFINITION 1</u>**.** Let there exist $\sigma_0\in\Sigma$ and $\tilde{g}\in M$ such that the directedness

$$\{K(a,b,f(t,dt),*,Step,\sigma)\}_{\sigma\in(\Sigma,Ord),\sigma(Ord)\sigma_0},$$

converges to $\tilde{g}$ .Then we denote

$$K(a,b,f(t,dt),*,Step,Ord)=\tilde{g}.$$

(Σ,Ord) is the directed set, and for every σ_1,σ_2 there exists their majorant. Therefore in Definition 1 the values of $K(a,b,f(t,dt),*,Step,Ord)$ do not depend on the choice of σ_0.

The order of co-factors in the right-hand side of (1) is important in the non-commutative case.

2. Some Examples

2.1. Reimannian integral in monoid. Let $t_1, t_2 \in R$, $\{(t,s) \mapsto f(t,s)\} : V \to M$, where $V \subset R^2$ and $(M, *)$ is monoid, endowed by a limiting structure. Then $(*)\int_{t_1}^{t_2} f(t,dt)$, determined in [2], is the same as $K(t_1, t_2, f(t,dt), *, RndStep, NormOrd)$, determined by Definition 1. Thus, private cases of $(*)\int_{t_1}^{t_2} f(t,dt)$ are also private cases of Definition 1 (see examples 2.2, 2.3, 2.4).

2.2. Riemannian integral. Let $f : [a,b] \to X$, where $[a,b] \in R$ and X is the Banach space and $t_1, t_2 \in R$. Obviously, $\{(t,s) \mapsto s \cdot f(t)\} : [a,b] \times R \to X$. It is easily to seen, that $K(a, b, f(t)dt, +, RndStep, NormOrd)$, determined by Definition 1, is the same as Riemannian integral $\int_{t_1}^{t_2} f(t)dt$, i. e. $K(t_1, t_2, f(t)dt, +, RndStep, NormOrd)$ exists then and only then, when exists $\int_{t_1}^{t_2} f(t)dt$ and $\int_{t_1}^{t_2} f(t)dt = K(t_1, t_2, f(t)dt, +, RndStep, NormOrd)$.

This fact and some others bellow (examples 2.3, 2.4) can be proved in the standard way (see [1] and [2]).

2.3. T-exponent. Let $A(t)$, $t \in [a,b]$, be a piecewise-continuous mapping in a noncommutative Banakh algebra. Then $K(a, b, \exp(A(t)dt, \cdot, UpStep, NormOrd)$ is the same, as T-exponent $Exp\int_a^b A(t)dt$; they exist simultaneously and are equal.

2.4. The multiplicative integral. Let $A(\cdot)$ be a continuous mapping from $[a,b]$ to $B(X)$ ($B(X)$ -the set of bounded linear operators in the Banach space X). Then $K(a, b, \exp(A(t)dt, \circ, UpStep, NormOrd)$ is the same as the multiplicative integral.

2.5. The total variation of function. Let $f : [a,b] \to X$, where X is a Banach space. Taking into account the simple equality

$$\sup_{\sigma \in \Sigma} \sum_{i=0}^{n-1} \| f(s_{i+1}) - f(s_i) \| = \lim_{\sigma \in (\Sigma(a,b), InscOrd)} \sum_{i=0}^{n-1} \| f(s_{i+1}) - f(s_i) \|$$

$\forall \sigma = \{ 0 = s_0 < \ldots < s_n = b \} \in \Sigma(a,b),$

we see that

$$K(a, b, \| f(t+dt) - f(t) \|, +, DownStep, InscOrd)$$

is the same as total variation of f on $[a,b]$, i.e. $V_a^b[f]$.

2.6. The directional derivative. Let X and Y be Banakh spaces, O is open subset in X, $f: O \to Y$, $x \in O$, $h \in X$ and there exist $f'(x;h)$ ($f'(x;h)$ denotes the derivative of f in x with direction h). Then there exists

$$K(0{,}1, f(x + dt \cdot h) - f(x), +, Step, NormOrd)$$

and

$$K(0{,}1, f(x + dt \cdot h) - f(x), +, Step, NormOrd) = f'(x;h),$$

$\forall Step \in \{DownStep, UpStep, RndStep\}$. The inverse result is also valid in certain assumptions (see [2]).

2.7. Representation of c_0-semigroups of operators. In [1] is proved an interesting result, which in terms of Definition 1 takes the following face:

THEOREM 1. *Let a linear operator A in the Banach space X generate the strongly continuous semigroup $\{U(s)\}_{s \ge 0}$. Then for sufficiently small s is determined*

$$(I_X - sA)^{-1} \in B(X)$$

and for each $s \ge 0$ takes place:

$$U(s) = K(0, s, (I_X - dt \cdot A)^{-1}, \circ, Step, NormOrd),$$

$\forall Step \in \{DownStep, UpStep, RndStep\}$*, where $B(X)$ is considered to have the unity I_X, operation of composition and strongly convergence.*

2.8. The integral representation of Cauchy's problem solution (see [4]). Consider the Cauchy's problem:

$$\dot{x} = f_t(x), \quad x(t_0) = x_0$$

and suppose that the field $f_t(x)$ has the following properties: $(t,x) \mapsto f_t(x)$ maps $[a,b] \times R^r$ into R^r, $-\infty < a < b < +\infty$, $f_t(x)$ is continuous with t for each $x \in R^r$ and there exists $k \ge 0$ such that

$$\left| f_t(x_1) - f_t(x_2) \right| \le k \left| x_1 - x_2 \right|, \quad \forall t \in [a,b], \quad \forall x_1, x_2 \in R^r.$$

$C_{Lip}(R^r)$ denotes the set of Lipschitz's mappings $g: R^r \to R^r$. The identical mapping $I: R^r \to R^r$ and the operation of composition $\circ$ create a structure of monoid on $C_{Lip}(R^r)$.

$C_{Lip}(R^r)$ is endowed by a limiting structure too, and the composition is continuous.

Under these constraints,

$$\{(t,s) \mapsto (I + sf_t)\} : [a,b] \times R^r \to C_{Lip}(R^r)$$

and in terms of Definition 1 the proved in [4] result takes the form:

THEOREM 2. *Let $(t_0, t, x) \in (a,b)^2 \times R^r$ be given arbitrarily . Then there exists*

$$K(t_0, t, (I + dtf_t), \circ, RndStep, NormOrd)$$

and

$$\varphi(t) = K(t_0, t, (I + dtf_t), \circ, RndStep, NormOrd)(x_0)$$

is the solution of $\dot{x} = f_t(x)$ with initial conditions $\varphi(t_0) = x_0$.

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