Necessary Conditions of Optimality for One Class Neutral Type Problems of Optimal Control

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Abstract

For the quasi-linear neutral type problem of optimal control, necessary conditions of optimality in the form of an integral maximum principle and the transversality conditions are obtained.

Let $J=[a,b]$ be a finite interval; $O\subset R^n$ be a open set; $M\subset O$ be a convex bounded set; $U\subset R^r$ be a compact set; $V\subset R^p$ be a convex bounded set; $\tau:R^1\to R^1$, $\eta:R^1\to R^1$ are an absolutely continuous and continuously differentiable functions, respectively, and satisfying the conditions:

$$\tau(t)\le t,\ \dot\tau(t)>0,\quad \eta(t)<t,\ \dot\eta(t)>0;$$

$\gamma(t)=\tau^{-1}(t),\ \ \sigma(t)=\eta^{-1}(t);\ \ q^i:J^2\times O^2\to R^1,\ i=0,\ldots,l,$ are continuously differentiable functions; $\Delta=\Delta(J_1,M)$ is a set of continuously differentiable functions $\varphi:J_1\to M,\ \ J_1=[\rho(a),b],$ $\rho(t)=\min\{\eta(t),\tau(t)\},\ \ t\in J,$ $\|\varphi\|=\sup\{|\varphi(a)|+|\dot\varphi(t)|:t\in J_1\};$ Ω_1 is a set of measurable functions $u:J\to U$; Ω_2 is a set of measurable functions $v:J\to V$; $A(t,v)$ is a $n\times n$ dimensional matrix function, continuous on $J\times V$ and continuously differentiable with respect to $v\in V$;

Next, the function $f:J\times O^2\times U\to R^n$ satisfies the following conditions:

1) for a fixed $t\in J$ the function $f(t,x_1,x_2,u)$ is continuous with respect to $(x_1,x_2,u)\in O^2\times U$ and continuously differentiable with respect to $(x_1,x_2)\in O^2$;

2) for a fixed $(x_1,x_2,u)\in O^2\times U$ the functions $f,\ \ f_{x_i},\ i=1,2,$ are measurable with respect to t; for an arbitrary compact $K\subset O$ there exist $m_K=const>0,\ \ L_K(t)\in L_1(J,R_0^+),\ \ R_0^+=[0,\infty)$ such that

$$|f(t,x_1,x_2,u)|\le m_K,\quad \sum_{i=1}^{2}|f_{x_i}(t,x_1,x_2,u)|\le L_K(t),\qquad \forall(t,x_1,x_2,u)\in J\times K^2\times U.$$

To every element $\mu=(t_0,t_1,x_0,\varphi,u,v)\in B=J^2\times O\times\Delta\times\Omega_1\times\Omega_2,\ \ t_0<t_1,$ corresponds the differential equation

$$\dot x(t)=A(t,v(t))\dot x(\eta(t))+f(t,x(t),x(\tau(t)),u(t))),\ \ t\in[t_0,t_1]\qquad(1)$$

with initial condition

$$x(t)=\varphi(t),\ \ t\in[\rho(t_0),t_0),\ \ x(t_0)=x_0.\qquad(2)$$

<u>DEFINITION 1</u>. The function $x(t)=x(t,\mu)\in O, t\in[\rho(t_0),t_1], t_0\in[a,t_1)$, is said to be solution corresponding to the element $\mu\in B$, if on $[\rho(t_0),t_0]$ it satisfies the condition (2), while on the interval $[t_0,t_1]$ is absolutely continuous and satisfies the equation (1) almost everywhere.

<u>DEFINITION 2.</u> The element $\mu\in B$ is said to be admissible, if the corresponding solution $x(t)$ satisfies the conditions $q^i(t_0,t_1,x_0,x(t_1))=0,\ i=1\ldots l.$

The set of admissible elements will denoted by B_0.

<u>DEFINITION 3.</u> The element $\tilde\mu=(\tilde t_0,\tilde t_1,\tilde x_0,\tilde u,\tilde v)\in B_0$ is said to be locally optimal, if there exist a number $\delta>0$ and compact set $X\subset O$ such that for an arbitrary element $\mu\in B_0$ satisfying

$\left|\tilde{t}_0 - t_0\right| + \left|\tilde{t}_1 - t_1\right| + \left|\tilde{x}_0 - x_0\right| + \left\|\tilde{\varphi} - \varphi\right\| + \left\|\tilde{f} - f\right\|_X + \sup_{t\in J}\left|\tilde{v}(t) - v(t)\right| \le \delta$, the inequality $q^0(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0,) \le q^0(t_0, t_1, x_0, x(t))$ is holds.

Here

$\left\|\tilde{f} - f\right\|_X = \int_J H(t; f, X)dt$,

$H(t; f, X) = \sup\{\left|\tilde{f}(t,x_1,x_2) - f(t,x_1,x_2)\right| + \sum_{i=1}^{2}\left|\tilde{f}_{x_i}(\cdot) - f_{x_i}(\cdot)\right| : (x_1,x_2)\in X^2\}$;

$\tilde{f}(t,x_1,x_2) = f(t,x_1,x_2,\tilde{u}(t)), \quad f(t,x_1,x_2) = f(t,x_1,x_2,u(t)), \quad \tilde{x}(t) = x(t,\tilde{\mu})$.

The problem of optimal control consists in finding a locally optimal element.

<u>THEOREM 1</u>. *Let $\tilde{\mu}\in B_0$, $\tilde{t}_i \in (a,b)$, $i = 0, 1$, be a locally optimal element; $\gamma_0 = \gamma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1)$, $\sigma_0 = \sigma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1)$, there exist integer numbers $m_i \ge 0, i = 1, 2,$ such that $\gamma_0 \in (\eta^{m_1+1}(\tilde{t}_1), \eta^{m_1}(\tilde{t}_1))$, $\sigma_0 \in (\eta^{m_2+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1))$; the function $\dot{\tau}(t)$ is continuous at point $\tilde{t}_0$; the function $\tilde{f}(\omega) = \tilde{f}(t,x_1,x_2)$ is continuous at points $\omega_0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0)))$, $\omega_1 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{x}_0)$, $\omega_2 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(\tilde{t}_0))$, $\omega_3 = (\tilde{t}_1, \tilde{x}(\tilde{t}_1), \tilde{x}(\tau(\tilde{t}_1)))$; the function $\tilde{A}(t) = A(t,\tilde{v}(t))$ is continuous at points $\tilde{t}_0, \tilde{t}_1, \sigma^i(\gamma_0)$, $i=1,\ldots,m_1$, $\sigma^i(\sigma_0)$, $i=0,\ldots,m_2$; the function $\dot{\tilde{x}}(\eta(t))$ is continuous at point $\tilde{t}_1$. Then there exists a non-zero vector $\pi = (\pi_0,\ldots,\pi_l)$, $\pi_0 \le 0$ and solutions $\psi(t)$, $\chi(t)$ of the system*

$$\begin{cases}\dot{\chi}(t) = -\psi(t)\tilde{f}_{x_1}[t] - \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t),\\ \dot{\psi}(t) = \chi(t) + \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t), \quad t\in[\tilde{t}_0,\tilde{t}_1], \quad \psi(t) = 0, \quad t > \tilde{t}_1,\end{cases}$$

such that following conditions are fulfilled:

$$\int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t)\tilde{\varphi}(t)dt + \int_{\eta(\tilde{t}_0)}^{\tilde{t}_0} \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t)\dot{\tilde{\varphi}}(t)dt \ge$$

$$\ge \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t)\varphi(t)dt + \int_{\eta(\tilde{t}_0)}^{\tilde{t}_0} \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t)\dot{\varphi}(t)dt, \quad \forall\varphi\in\Delta;$$

$$\int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)f(t,\tilde{x}(t),\tilde{x}(\tau(t)),\tilde{u}(t))dt \ge \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)f(t,\tilde{x}(t),\tilde{x}(\tau(t)),u(t))dt, \quad \forall u\in\Omega_1;$$

$$\int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)\tilde{A}_v(t)\dot{\tilde{x}}(\eta(t))\tilde{v}(t)dt \ge \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)\tilde{A}_v(t)\dot{\tilde{x}}(\eta(t))v(t)dt, \quad \forall v\in\Omega_2;$$

$$\pi\tilde{Q}_{t_0} = \chi(\tilde{t}_0)[\tilde{A}(\tilde{t}_0)\dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + \tilde{f}(\omega_0)] +$$

$$+\psi(\sigma_0)\tilde{A}(\sigma_0)[\tilde{A}(\tilde{t}_0)\dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + \tilde{f}(\omega_0) - \dot{\tilde{\varphi}}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0)\,\psi(\gamma_0)[\tilde{f}(\omega_1) - \tilde{f}(\omega_2)]\dot{\gamma}(\tilde{t}_0),$$

$$\pi\tilde{Q}_{t_1} = -\psi(\tilde{t}_1)[\tilde{A}(\tilde{t}_1)\dot{\tilde{x}}(\eta(\tilde{t}_1)) + \tilde{f}(\omega_3)],$$

$$\pi\tilde{Q}_{x_0} = -\chi(\tilde{t}_0), \qquad \pi\tilde{Q}_{x_1} = \chi(\tilde{t}_1).$$

Here $Q = (q^0,\ldots,q^l)^T$, $\tilde{Q}$ means that the corresponding gradient is calculated at the point $(\tilde{t}_0,\tilde{t}_1,\tilde{x}_0,\tilde{x}(\tilde{t}_1))$; $\tilde{f}_{x_i}[t] = \tilde{f}_{x_i}(t,\tilde{x}(t),\tilde{x}(\tau(t)))$.

Finally we note that the theorem formulated above are an analogue of theorem given [1]. This theorem is proved, using formula of the differential of solutions with respect to the initial data and the right-hand side [2], by the scheme described in [3]. The case, when $\tilde{v}(t)$ is the piecewise continuous function, is considered in [4].

REFERENCES

1. T. Tadumadze. *On new necessary condition of optimality of initial moment in control problems with delay*. Mem. Differential Equations Math. Phys. 17(1999), 157-159.
2. T. Tadumadze and N. Gorgodz. *Differentiability of solution of differential equations with deviating argument with respect to the initial data and the right-hand side*. Seminar of I. Vekua Institute of Applied Mathematics. Reports. 23(1997), 108-119.
3. R. V. Gamkrelidze, G. L. Kharatishvili. *Extremal problems in linear topological spaces*. (Russian), Izv. AN SSSR, Sem. Math. 1969(33), N4, 781-839.
4. N. Gorgodze. *Necessary conditions of optimality in neutral type optimal problems with non-fixed initial moment*. Mem. Differential Equations Math. Phys. 19(2000). (In print)