# Contrary Pairs in Pseudo-Boolean Algebra

G. Kashmadze
Chair of Cybernetics, Iv. Javakhishvili Tbilisi State University

***Abstract**.*

*The lattice operations for split subsets are considered when the values of membership function belong to the pseudo-boolean algebra.*

Let us consider a set of pairs of kind $(\mu I_A,(1-\mu)I_A)$, where $A\subseteq\Omega$, $\Omega$ is universal domain, $A$ - his fixed subset, $\mu$ - any mapping of $\Omega$ into [0,1] and $I_A$ - characteristic function or indicator of $A$ /1/.

Introduce $\wedge$ (greatest lower bound) and $\vee$ (least upper bound) operations traditionally componentwise:

$$(X,Y)\vee(Z,T)=(X\vee Z,Y\vee T)$$

$$(X,Y)\wedge(Z,T)=(X\wedge Z,Y\wedge T)$$

Take arbitrarily two pairs $(\mu I_A,(1-\mu)I_A)$, $(\nu I_A,(1-\nu)I_A)$, $\mu,\nu\in[0,1]^\Omega$ and consider their l.u.b. and g.l.b:

$$((\mu I_A\vee\nu I_A)(x),((1-\mu)I_A\vee(1-\nu)I_A)(x))= = \begin{cases}(0,0) & if\ \ x\notin A\\ (\nu(x),\ 1-\mu(x)) & if\ \ x\in A,\ \mu(x)\le\nu(x)\\ (\mu(x),\ 1-\nu(x)) & if\ \ x\in A,\ \mu(x)>\nu(x)\end{cases}$$

$$((\mu I_A\wedge\nu I_A)(x),\quad ((1-\mu)I_A\wedge(1-\nu)I_A)(x))= = \begin{cases}(0,0) & if\ \ x\notin A\\ (\mu(x),\ 1-\nu(x)) & if\ \ x\in A,\ \mu(x)\le\nu(x)\\ (\nu(x),\ 1-\mu(x)) & if\ \ x\in A,\ \mu(x)>\nu(x)\end{cases}$$

Thus the set of kind $(\mu I_A,(1-\mu)I_A)$ is not closed with respect to operations $\wedge$ and $\vee$. If we want to keep closety, then operations will be defined as follows: $(X,Y)\vee(Z,T)=(X\vee Z,Y\wedge T)$

In this case, we have:

$$((\mu I_A\vee\nu I_A)(x),\quad ((1-\mu)I_A\wedge(1-\nu)I_A)(x))= = \begin{cases}(0,0) & if\ \ x\notin A\\ (\nu(x),\ 1-\nu(x)) & if\ \ x\in A,\ \mu(x)\le\nu(x)\\ (\mu(x),\ 1-\mu(x)) & if\ \ x\in A,\ \mu(x)>\nu(x)\end{cases}$$

$$((\mu I_A\wedge\nu I_A)(x),\quad ((1-\mu)I_A\vee(1-\nu)I_A)(x))= = \begin{cases}(0,0) & if\ \ x\notin A\\ (\mu(x),\ 1-\mu(x)) & if\ \ x\in A,\ \mu(x)\le\nu(x)\\ (\nu(x),\ 1-\nu(x)) & if\ \ x\in A,\ \mu(x)>\nu(x)\end{cases}$$

$\wedge$ and $\vee$ operations induce a following partial ordering relation between the pairs

$$(X,Y)\le(Z,T)\Leftrightarrow(X,Y)\vee(Z,T)=(Z,T)\Leftrightarrow$$

$$\Leftrightarrow(X,Y)\wedge(Z,T)=(X,Y)\Leftrightarrow(X(u)\le Z(u))\,\&\,(Y(u)\ge T(u))$$

$$X,Y,Z,T\in[0,1]^A, u\in A.$$

The complement $(X,Y)^c$ of pair $(X,Y)$ is defined in this way: $(X,Y)=(I_A\setminus X,I_A\setminus Y)$

Denote by a symbol $L$ the lattice of fuzzy subsets of $\Omega$ and by symbols $L^r$ the lattice of same subsets with reverse order. It is fairly straitforward to show that the following theorem holds:

THEOREM 1. *Pairs of kind* $(\mu I_A, (1-\mu) I_A)$ *for fixed* $A \subseteq \Omega$ *and any* $\mu \in [0,1]^\Omega$ *form complemented distributive lattice, which is a sublattice of* $L \times L^r$*, satisfying de Morgan's law; the complement is involutory and order reversing.*

Now suppose, that the values of membership function $\mu$ belong to the pseudo-boolean algebra $B = \langle B, \leq \rangle$ /2/. As intuitionistic negation is not, in general, involutory, the unique splitting into contrary pair is impossible /3/. In this case, the construction defined below perhaps proved to be useful.

We introduce the "$\approx$" relation between elements of $B$ and between their pairs as follows:

DEFINITION 1. $(a \approx b) \underset{df}{=} (a^* = b^*)$, $a,b \in B$. Here $(\ )^*$ denotes pseudo-complement.

2. $((a,b) \approx (c,d)) \underset{df}{=} ((a \approx c) \text{ and } (b \approx d))$, $a,b,c,d \in B$.

3. $(a,b)^* \underset{df}{=} (a^*, b^*)$, $a,b \in B$.

It is evident that "$\approx$" is an equivalence relation.

THEOREM 2. *If* $a_1 \approx a_2$, $b_1 \approx b_2$ *then*

$$((a_1, a_1^*) \vee (b_1, b_1^*)) \approx ((a_2, a_2^*) \vee (b_2, b_2^*)), \quad (a_1, a_1^*)^* \approx (a_2, a_2^*)^*$$

PROOF.

$$(a_1, a_1^*) \vee (b_1, b_1^*) = (a_1 \vee b_1, a_1^* \wedge b_1^*) = (a_1 \vee b_1, a_2^* \wedge b_2^*)$$

$$(a_2, a_2^*) \vee (b_2, b_2^*) = (a_2 \vee b_2, a_2^* \wedge b_2^*), \qquad (a_1 \vee b_1)^* = a_1^* \wedge b_1^* = a_2^* \wedge b_2^* = (a_2 \vee b_2)^*.$$

2. follows immediately from definitions.

Let $B' \subset B$ be the set of elements satisfying the following condition:

$$(a \wedge b)^* = a^* \vee b^*$$

THEOREM 3. *If* $a,b,c,d \in B'$, $a_1 \approx a_2$, $b_1 \approx b_2$ *then*

$$((a_1, a_1^*) \wedge (b_1, b_1^*)) \approx ((a_2, a_2^*) \wedge (b_2, b_2^*))$$

Proof by analogy with case 1. of theorem given above. Thus, we can consider the factor set $B' \times B' / \approx$.

## REFERENCES


1. F. Criado, T. Gachechiladze. *Fuzzy random events and their corresponding conditional probability measures.* Publicado en la Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, de Madrid. Tomo LXXXIX, Cuaderno 1.º - 2.º
2. Е. Расева, Р. Сикорски. *Математика Метаматематики*. Москва. Наука. 1972.
3. R. Yager. *On the Measure of Fuzziness and Negation*. Part II. Inf. and Control. 44. 1980.