On an Approximately Solution of Saint-Venant's Problems for a Beams with a Perturbed Lateral Surfaces

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Abstract

First works on an indicated problems for an isotropic beams were given by Panov D.I.,Riz P.M., Rukhadze A.K and various authors. These results were generalized on a composed bodies and anisotrpic medium by various authors. In these works indicated problems were studied with help of a transformation of a system of coordinates and differential operators and boundary conditions were approximated with accuracy up to first power of a small parameter v. As this takes place it is impossible to estimate a power of an approximation and give proof of this method for anisotropic medium is difficult, because a coefficients of elasticity in this method are varying. In this paper a solution of Saint-Ven ant's problems in a domain, occupying by a body similar to prismatic (cylindrical), with perturbed cylindrical surface, is represented as a series with respect of a small parameter v, characterized a perturbation of a cylindrical surface. For each terms of a series are obtained the recurrent boundary problems of elasticity of Almansi-Michel's type for a cylindrical body. A class of surface is indicated, for which later on may be studied a question of a convergence of a double series with respect of a small parameters. Also this way gives a possibility of a solution of a problem with a required exactness. A first results on this direction were given by author of this article in papers (1981 and 1983) used methods considered in articles of A.N.Guz (1962) and I.N.Nemish (1976), where a method of a perturbation of a cross section of a surfaces of a canonical form was considered. This way as a base was used in another direction for a construction of algorithms for a solution of some problems of an elasticity, for a bodies similar to cylindrical by an arbitrary cross section. These results are given in the book [7]. A.N. Guz, I.N. Nemish and N.M. Bloshko created the methods of a perturbation of bodies boundary's form by its further generalization (see A.N. Guz, I.N. Nemish 1987, I.N. Nemish, N.M. Bloshko 1987, I.N. Nemish 1989]).

1. Basic Equations

Let us consider a system of a cartesian coordinates $Ox_1x_2x_3$ and an elastic body occupying a domain Ω_*, bounded by planes

$$x_3 = 0,\ x_3 = l\ \ (l > 0) \tag{1}$$

and by lateral surface Γ_* given in a parametric form

$$x_J^* = f_j(t) + vP_j(z)q_j(t),\ (j = 1,2),\ \ x_3^* = z, \tag{2}$$

where v is a small parameter, $P_j(z)$ are a given polynomials with respect of a variable x_3 and t is a given parameter. We consider a particular case of an equations (2), when $q_j(t) = 1$ and $P_j(z) = -x_3^p(pml^p)^{-1}$. Thus, it will be considered a body bounded by planes (1) and by lateral surface

$$x_j^* = f_j(t) + vP(z),\quad (j = 1,2),\ x_3 = z, \tag{3}$$

where $0 < v < 1$ is a small parameter and m and p are integer numbers, which must be chosen so that it can be $\left|z^{p-1}(ml^p)^{-1}\right| < 1$; t is a natural coordinate taken on a curve γ of a boundary of a domain ω, wich is obtained by normal crossection of a cylindrical body, bounded by planes (1) and a surface

$$x_1 = f_1(t), \quad x_2 = f_2(t). \qquad (4)$$

It is obvious that surface is obtained from surface (3) for $v = 0$ (or $x_3 = 0$). It must be remember that $0 \le x_3 \equiv z \le l$.

A cosinus of a normal $n^0(n_1^0, n_2^0, n_3^0)$ of the surface (4) will be given as

$$n_1^0 = n_0^{-1} f_2'(t), \; n_2^0 = -n_0^{-1} f_1'(t), \quad n_3^0 = 0, \quad n_0^2 = \left(f_1'\right)^2 + \left(f_2'\right)^2. \qquad (5)$$

As is known, a cosinus of a normal $n(n_1, n_2, n_3)$ to the surface (2) will be given by equalities

$$n_j = B_j B^{-1} \; (j = 1,2,3); \; B_1 = (x_2^*)_t'(x_3^*)_z' - (x_2^*)_z'(x_3^*)_t',$$
$$B_2 = (x_3^*)_t'(x_1^*)_z' - (x_1^*)_t'(x_2^*)_z', \; B_3 = (x_1^*)_t'(x_2^*)_z' - (x_1^*)_z'(x_2^*)_t', \qquad (6)$$
$$B^2 = B_1^2 + B_2^2 + B_3^2 > 0.$$

According to equations (3), we get

$$\left(x_j^*\right)_t' = f_t'(t), \; \left(x_j^*\right)_z' = -v x_3^{p-1}\left(ml^p\right)^{-1} \; (j = 1,2); \; \left(x_3^*\right)_t' = 0, \; \left(x_j^*\right)_z' = 1. \qquad (7)$$

Substitute these values in the expressions (6), we get

$$n_0^{-1} B_1 = n_1^0, \; n_0^{-1} B_2 = n_2^0, \; n_0^{-1} B_3 = v(n_1^0 + n_2^0) z^{p-1} (ml^p)^{-1}$$
$$B = \left[n_0^2 + v^2 (n_1^0 + n_2^0)^2 z^{2p-2} (ml^p)^{-2}\right]^{1/2}, \qquad (8)$$

where $n_0 > 0$ is given by equality (5).

Thus, we consider Saint-Venant's problems for a body similar to a prismatic (cylindrical), bounded by planes (1) and by a lateral pertrubed cylindrical surface (3). The components of the stresses τ_{jk} must satisfy in a domain Ω_* the equations of an equilibrium of an elasticity and a Hook's law (for an isotropic or an anisotropic mediums) and also must be satisfaid an "ends conditions" (for a lower end, when $x_3 = 0$)

$$\iint_\omega \tau_{3j} d\omega = F_j \quad (j = 1,2,3); \; \iint_\omega x_2 \tau_{33} d\omega = M_1, \; \iint_\omega x_1 \tau_{33} d\omega = M_2,$$
$$\iint_\omega \left(x_1 \tau_{23} - x_2 \tau_{13}\right) d\omega = M_3, \qquad (9)$$

where F_j and M_j are the given numbers.

On an "upper" end a resultant force and a resultant moment of an exterior forces, in general, must be equal to F_j and M_j respectively, but with opposite sign.

On each points of a lateral surface (3) the components τ_{jk} must satisfy the following boundary conditions:

$$\tau_{1j} n_1 + \tau_{2j} n_2 + \tau_{3j} n_3 = 0, \; (j = 1,2,3), \qquad (10)$$

where a cosines n_j of a normal n are given by equalities (6).

Divide equalities (10) on n_0 and substitute values n_j, after a multiplication on B, by using of the equations (6), boundary conditions (10) takes the form

$$\tau_{1k} n_1^0 + \tau_{2k} n_2^0 - v \tau_{3k} \left(n_1^0 + n_2^0\right) P'(z) = 0 \quad (k = 1,2,3) \qquad (10v)$$

About of a fullfilment of an "ends conditions" (9) will be shown below.

2. An Approximately Solution

The solutions of the considered problems we seek as the series with respect of a small paramete v

$$\tau_{jk} = \sum_{b=0}^{\infty} v^b \tau_{jk}^{(b)}(x_1, x_2, x_3). \qquad (1)$$

Substitute these values in the condition (1.10v), taking x_j^* (see(1.3)) instead of x_j. Therefore, in expression (1) each term will be expanded on surface (1.3) in a Teylor's serie

$$\tau_{jk}^{(b)}(x_1+\nu P(z),x_2+\nu P(z),x_3)=\left(\tau_{jk}^{(b)}\right)_0+\sum_{\alpha=1}^{N}\frac{1}{\alpha!}[\nu P(z)]^{\alpha}\left(D^{\alpha}\tau_{jk}^{(b)}\right)_0+$$
$$+\frac{1}{(N+1)!}[\nu P(z)]^{N+1}\left(D^{N+1}\tau_{jk}^{(b)}\right)_{\xi_j},\quad \xi_j(\Theta)=x_j+\Theta\nu P(z)\quad (j=1,2),\ x_3=z, \tag{2}$$

where N is some integer positive number,

$$P(z)\equiv -z^{p}(mpl^{p})^{-1},\ D^{\alpha}\equiv(D_1+D_2)^{\alpha},\ 0<\Theta<1,\ (\tau_{jk}^{(b)})_0\equiv(\tau_{jk}^{(b)})_{\nu=0}. \tag{3}$$

Using these expressions, the conponents (1) in points of a surface (1.3) may be expressed by the values in points of a cylindrical surface (1.4), with accuracy up to ν^{N+1}, in a form

$$\tau_{jk}(x_1+\nu P(z),x_2+\nu P(z),x_3)=\sum_{b=0}^{N}\sum_{\alpha=1}^{N-b}\frac{1}{\alpha!}\nu^{b}[\nu P(z)]^{\alpha}\left(D^{\alpha}\tau_{jk}^{(b)}\right)_0.\quad (j,k=1,2,3).$$

For present purposes this expressions may be written in the form

$$\tau_{jk}(x_1+\nu P(z),x_2+\nu P(z),x_3)=\left(\tau_{jk}^{(0)}\right)_0+\sum_{\alpha=1}^{N}\nu^{\alpha}\left[\tau_{jk}^{(\alpha)}+\frac{1}{\alpha!}(P(z))^{\alpha}D^{\alpha}\tau_{jk}^{(0)}\right]_0+$$
$$+\sum_{b=1}^{N}\sum_{\alpha=1}^{N-b}\nu^{\alpha+b}\frac{1}{\alpha!}(P(z))^{\alpha}\left(D^{\alpha}\tau_{jk}^{(b)}\right)_0\Bigg]. \tag{4}$$

Substituting these expressions into the boundary conditions (1.10v), in each point of a cylindrical surface (1.4) for the values $\tau_{jk}^{(b)}$ the following boundary conditions are obtained:

$$\left(\tau_{1k}^{(0)}n_1^0+\tau_{2k}^{(0)}n_2^0\right)_0-\nu\left(n_1^0+n_2^0\right)P'(z)\left(\tau_{3k}^{(0)}\right)_0+\sum_{j=1,2}\left\{\sum_{\alpha=1}^{N}\nu^{\alpha}\left[\tau_{jk}^{(0)}+\frac{1}{\alpha!}(P(z))^{\alpha}D^{\alpha}\tau_{jk}^{(0)}\right]_0+\right.$$
$$\left.+\sum_{b=1}^{N}\sum_{\alpha=1}^{N-b}\nu^{\alpha+b}\frac{1}{\alpha!}(P(z))^{\alpha}\left(D^{\alpha}\tau_{jk}^{(b)}\right)_0\right\}n_j^0-$$
$$-P'(z)\left(n_1^0+n_2^0\right)\left\{\sum_{\alpha=1}^{N-1}\nu^{\alpha+1}\left[\tau_{3k}^{(\alpha)}+\frac{1}{\alpha!}(P(z))^{\alpha}D^{\alpha}\tau_{3k}^{(0)}\right]_0+\right. \tag{5}$$
$$\left.+\sum_{b=1}^{N-1}\sum_{\alpha=1}^{N-b-1}\nu^{\alpha+b+1}\frac{1}{\alpha!}(P(z))^{\alpha}\left(D^{\alpha}\tau_{3k}^{(b)}\right)_0\right\}=0,\quad (k=1,2,3).$$

In these expressions equating to zero the multipliers of the same power of a parameer ν, for the unknown components $\left(\tau_{jk}^{(b)}\right)_0$, in each points of a cylindrical surface (1.4) are obtained following boundary conditions:

$$\left(\tau_{1j}^{(0)}n_1^0+\tau_{2j}^{(0)}n_2^0\right)_0=0,\ \left(\tau_{1j}^{(1)}n_1^0+\tau_{2j}^{(1)}n_2^0\right)_0=-P(z)\left(n_1^0D^1\tau_{1j}^{(0)}+n_2^0D^1\tau_{2j}^{(0)}\right)_0+ \tag{6}$$
$$+\left(n_1^0+n_2^0\right)P'(z)\left(\tau_{3j}^{(0)}\right)_0,\ \left(\tau_{1j}^{(k)}n_1^0+\tau_{2j}^{(k)}n_2^0\right)_0=-\sum_{m=0}^{k-1}[(k-m)!]^{-1}\left\{P^{k-m}(z)\left(n_1^0D^{k-m}\tau_{1j}^{(m)}+\right.\right.$$
$$\left.\left.+n_2^0D^{k-m}\tau_{2j}^{(m)}\right)_0+(k-m)\left(n_1^0+n_2^0\right)P'(z)P^{k-m-1}(z)\left(D^{k-m-1}\tau_{3j}^{(m)}\right)_0\right\},\quad (k=2,3,...,N).$$

From these recurrent exprssions is seen that components $\left(\tau_{ij}^{(0)}\right)_0$ must be a solution of Saint-Venant's (SV) problems for a cylindrical body "*G*" with a lateral surface (1.4). Therefore, the "ends conditions" (1.9) will be satisfied. In this case, as is well-known, for a homogeneous beam, only the components $\left(\tau_{13}^{(0)}\right)_0, \left(\tau_{23}^{(0)}\right)_0$ and $\left(\tau_{33}^{(0)}\right)_0$ are not equal to zero, i.e. it must be taken $\left(\tau_{11}^{(0)}=\tau_{22}^{(0)}=\tau_{12}^{(0)}=0\right)$. Therefore, in conditions (6) it will be taken only $\left(\tau_{j3}^{(0)}\right)_0\neq 0,\ (j=1,2,3)$.

It must be noted that the problems of exstension by longitudinal force and bending due couples of forces only $\left(\tau_{33}^{(0)}\right)_0\neq 0$ and the expressions (6) greatly simplified.

In general an elementary analysis of the boundary conditions (6) shows that beginning from $k=2$ the components $\left(\tau_{ij}^{(k)}\right)_0$ must be the solutions of Almansi's and Michell's $(AM)_k$ problems studying in the book [8].

A solution of these three-dimensional problems are reduced to two-dimensional boundary problems in a plane Ox_1x_2 for the elliptic equations of a two, four and sixth powers, a solution of which may be estimated (include on a boundary of domain) in the way given in [1]. From the solutions of *(A-M)ₖ* problems for body "*G*" on the end $x_3 = 0$ may be araised a resultant forces $F_j^{(k)}$ and a resultant moments $M_j^{(k)}$ of a couple-forces, for neutralized of which to the cimponents $\left(\tau_{ij}^{(k)}\right)_0$ must be added a solution $\left(\tau_{ij}^{S}\right)_k$ of $(SV)_k$ problems, corresponding to the given resultant forces - $F_j^{(k)}$ and resultant moments - $M_j^{(k)}$.

It will be remarked that this algorithm may be used not only in an elasticity, but in another problems of mathematical physics. For example, it may be used also in an heat conduction problems (see [6,7]), for which on (6) must be taken only conditions for $j = 3$ and everywhere only $\left(\tau_{13}^{(0)}\right)_0 \neq 0$ and $\left(\tau_{23}^{(0)}\right)_0 \neq 0$.

In practice functions $P(z)$ are a linear, second or a third power polynomials of x_3. For instance, $P(z) = -x_3^2(4l)^{-2}$ (a slightly curved beam) and for a small parameter v we may take number $(tan\alpha)_{x_3=l}\, l^{-2}$, where α is a corner in a plane x_1Ox_3 composed between Ox_1 axis and a tangent to a curve obtained by crossing of indicated plane and surface (1.3). This algorithm give in our disposal two small paraeter v and $v^* = |P(z)| < 1$, what give a possibility to investigate a question of a double series, what will araise when $N \to \infty$.

It must be remarked, although an algorithm of the small parameter method for bodies similar to cylindrical has been constructed about 15 years ago [5,6], the idea of finding a perturbation function to estimate the remander of series with respect to he small parameter, made us come back to the issue.

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