Insufficient Data and Weighted Fuzzy Expected Interval of Population

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Abstract
In the present work two new versions of Most Typical Value (MTV) are introduced – Generalized Weighted Means. The first WFEV g is generalized version of Weighted Fuzzy Expected Value (WFEV) for any fuzzy measure g and, of course it coincides with WFEV in case of uniform probability distribution. Most Interesting is the second one – Weighted Fuzzy Expected Interval (WFEI) – which is the generalization of WFEV g, when Fuzzy Expected Value (FEV) doesn't exist, but Fuzzy Expected Interval (FEI does. It is based on Friedman-Schneider-Kandel (FSK) principle and using the interval analysis of FEI constructs new MTV, called WFEI. Corresponding converging iteration process is constructed also, which in this case is the interval version of Newton's iteration process and for convergence Moor's MV-Extensions of function are used taking into account generalized interval arithmetic.

Let X be finite set, $(\mathrm{X},\mathrm{B},g)$ – space of fuzzy measure, χ_A – membership function of A fuzzy subset: $\chi_A : \mathrm{X} \to [0;1]$. Let X set be divided into k groups so that:

$$X_1 = \{x_1,...,x_{n_1}\},\ X_2 = \{x_{n_k},...x_{n_2}\},\dots X_k = \{x_{n_{k-1}+1},...,x_{n_k}\}.$$

On sets ($\mathrm{X} = \coprod_{i=1}^{k} X_i$) χ_A membership function is constant: $\chi_A(x) \equiv \chi_i$ for $\forall x \in X_i$. I.e. the fuzzy subset separates groups of population on X. For example, if X is n-dimensional sampling, from it we can get $k \le n$ dimensional variational sampling. If we take uniform discrimination in place of g, then fuzzy measure of group will equal to its frequency ($g(X_i) \equiv n_i/n$, $i = 1,2,...,k.$).

On the set $\mathrm{X} = \{x_1,...,x_n\}$ there exist n! permutations. Denote any permutations as $\sigma = (\sigma(1),\sigma(2),...,\sigma(n))$. The set of all possible permutations by - S_n.

<u>DEFINITION 1</u> [6]: If $\sigma \in S_n$ is any arrangement, then following probability distribution:

$$\begin{aligned} &P_\sigma(x_{\sigma(1)}) = g(\{x_{\sigma(1)}\}),\ P_\sigma(x_{\sigma(2)}) = g(\{x_{\sigma(1)},x_{\sigma(2)}\}) - g(\{x_{\sigma(1)}\}),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &P_\sigma(x_{\sigma(n)}) = 1 - g(\{x_{\sigma(1)},...,x_{\sigma(n-1)}\}) \end{aligned} \tag{1}$$

is called associated probability distribution with fuzzy measure g.

$\{\mathrm{P}_\sigma\}_{\sigma\in S_n} = \{P_\sigma(x_{\sigma(1)},...,P_\sigma(x_{\sigma(n)}\}_{\sigma\in S_n}$ is called the class of associated probabilities.

It's known, that for $\forall X_i \subset \mathrm{X}$ set, $\exists \tau_i \in S_n$ arrangement such, that class of associated probabilities uniquely determines fuzzy measure $g(X_i) = P_{\tau_i}(X_i)$ on finite set [9]. Practically this is probability interpretation of fuzzy measure, which is essential in probability representation of fuzzy means.

<u>DEFINITION 2:</u> Suppose w is nonnegative, strictly decreasing function and λ>1 is real number, then the solution of following equation:

$$S = \frac{\sum_{i=1}^{k} w(|\chi_i - S|)\cdot P_{\tau_i}{}^{\lambda}(X_i)\chi_i}{\sum_{i=1}^{k} w(|\chi_i - S|)\cdot P_{\tau_i}{}^{\lambda}(X_i)} \tag{2}$$

is called Weighted Fuzzy Expected Value of order λ with the attached weight function w. (MTV=WFEV g (λ, w)).

Lets recall two postulates of building WFEV[4], which we'll later call Friedman- Schneider -Kandel principle (FSK principle):

FSK PRINCIPLE: Mathematical expectation: $E^{g}_{w,\lambda}(\chi_A;\cdot)$ is invariant to the most typical value of population – MTV.

$$E^{g}_{w,\lambda}(\chi_A, MTV) = MTV . \qquad (3)$$

The estimation of FSK principle is called the solution of equation (3) and as we mentioned is called WFEV g. By analogy with [4] we create iteration process for WFEI:

$$S_i = E^{g}_{w,\lambda}(\chi_A, S_{i-1}), \qquad (4)$$

where S_0=FEV. If data on population groups isn't enough and FEV cannot be calculated, and values of membership compatibility are intervals, also it's possible interval representation of values of fuzzy measure $g(X_i)$ and this is more realistic. From statistic interval estimation of probabilities $g(X_i) = P_{\sigma_i}(X_i)$ we can estimate FEI and clearly (4) iteration process cannot work, because there doesn't exist S_0=FEV. In such conditions parameters in (2) are taken from intervals $\chi_i \in [\underline{\chi}_i; \overline{\chi}_i]$, $P^{\lambda}(\chi_i) \in [\underline{P}^{\lambda}_i; \overline{P}^{\lambda}_i]$. Also we can assume that solution of equation (2) is from interval FEI= $[\underline{fei}, \overline{fei}]$. (MTV$\in$FEI)

DEFINITION 3: If $\chi_i \in [\underline{\chi}_i; \overline{\chi}_i]$, $P^{\lambda}(\chi_i) \in [\underline{P}^{\lambda}_i; \overline{P}^{\lambda}_i]$, Suppose w is nonnegative, strictly decreasing function and λ>1 is real number, and if exists unique solution of (2) in FEI, its called Weighted Fuzzy Expected Interval of order λ with the attached weight function w. Its denoted by WFEI g (λ, w).

Evidently this principle takes into account FSK principle and for its estimation we'll use interval analysis [5]. Let's denote the right side function of equation (2) by $f_0(s)$. We'll introduce some interval extension of $f_0(s)$, where from analytical judgment $|\chi_i - s|$ will be changed by $|\chi_i - s|^2$.

$$F_0(s) = \frac{\sum_{i=1}^{k} w(|[\underline{\chi}_i, \overline{\chi}_i] - s|^2 \cdot [\underline{P}^{\lambda}_{\tau_i}; \overline{P}^{\lambda}_{\tau_i}] \cdot [\underline{\chi}_i; \overline{\chi}_i]}{\sum w(|[\underline{\chi}_i; \overline{\chi}_i] - s|^2)[\underline{P}^{\lambda}_{\tau_i}; \overline{P}^{\lambda}_{\tau_i}]}, \qquad (5)$$

where $s \subset$ FEI= $[\underline{fei}, \overline{fei}]$ is any interval.

Clearly WFEI is null of function

$$f(s) = s - f_0(s) \qquad (6)$$

Then interval extension of $f(s)$ will be: $F(s) = s - F_o(s), \forall s \subset$FEI.

Consider function:

$$f(s) = s - \frac{\sum_{i=1}^{k} w(|\chi_i - s|^3) P^{\lambda}_{\tau_i} \chi_i}{\sum_{i=1}^{k} w(|\chi_i - s|^3) P^{\lambda}_{\tau_i}}$$

MV-Extension [5] of which $F(S) = F_{MV}(S)$ is:

$$F_{MV}(s) = f(m(FEI)) + F^{'}_{s}(s)(s - m(FEI)). \qquad (7)$$

where F'_s is interval extension of:

$$f^{'}_{s}(s) = 1 - \frac{2\sum_{i=1}^{k} \frac{\partial w}{\partial s}[\chi_i - s] P^{\lambda}_{\tau_i} \rho_1 - 2\sum_{i=1}^{k} \frac{\partial w}{\partial s}[\chi_i - s] P^{\lambda}_{\tau_i} \chi_i \rho_0}{\{\sum_{i=1}^{k} w(|\chi_i - s|^2) P^{\lambda}_{\tau_i}\}^2}, \qquad (8)$$

where ρ_0 and ρ_1 are known numbers.

Clearly with interval extension its better to use generalized interval arithmetic. After everything develops according to Newton's method in [5,§5} with following denotions:

$$S_{i+1} = S_i \text{ I } N(S_i), \ \lambda im\lambda(S_i) = 0 \text{ and } \lambda im_{i \to \infty} S_i = WFEIg(\lambda, w) \qquad (9)$$

Clearly $S_0 \in FEI$. There exist sufficient conditions for convergence of (9).

Here we will state theorems without proofs, which is essential and unites all known and presented here weighted means, which retains correctness of generalization of statistical notions.

THEOREM 1*: If FEI=FEV, intervals of membership χ_i and associated probabilities $P_{\tau_i}^{\lambda}$ are intervals of one point then:*

$$WFEV\,g\,(\lambda, w) = WFEI\,g\,(\lambda, w)$$

Note, that for convergence of iteration process (9) the property of compression of $f_s^{'}$ on FEI is enough, which can be easily verified. E.g. when $w(t) = e^{-\beta t}$, $\lambda > 1$.

THEOREM 2: *If fuzzy measure g coincides with frequency distribution of population groups, then $WFEI\,g\,(\lambda, w) = WFEI\,(\lambda, w)$ and for one-point intervals $WFEI\,g\,(\lambda, w) = WFEV\,g\,(\lambda, w) = WFEV\,(\lambda, w)$.*

Can be stated, that in cases of insufficient data on population groups, process of fuzzy statistical estimation is distinguished by two stages: From little information follows generalization of fuzzy weighted estimator, which is formally formed by interval analysis, creates entropy growth of information, but mobile FSK principle enables entropy decrease of information, which is condensed in generalized fuzzy statistic, in new MTV of population, which is called Weighted Fuzzy Expected Interval (WFEI).

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