Interpretation of λ-Additive Fuzzy Discrete Measures in the Problems of Fuzzy Measure Restoration from Corresponding Insufficient Data

G. Sirbiladze, N. Zaporozhets
Chair of Random Processes Theory, Iv. Javakhishvili Tbilisi State University

Abstract

In this paper we consider problems of fuzzy measure restoration from corresponding insufficient data on the finite set. An approach is located in the class of Choquet's second order capacities with nearest distance from λ-additive fuzzy measures, when the singleton's "fuzzy weights" are known. This essentially concerns certain frequency distributions, where the feature of additivity is doubtful. This follows from the fuzzy nature of data distribution, when the expert "appoints" data. This fact is an indisputable condition of fuzzy measure introduction, but insufficient for its construction.

Fuzzy measure of the optimal approximation is constructed in the class of Choquet's second order capacities. Measures of specificity, indices of uncertainty and estimations of approximations are calculated.

Some properties of the correctness of the approximation are proved.

INTRODUCTION

There are two classical approaches to data analysis. If experimental data is "sufficiently" exact then for their processing and estimation of general characteristics probabilistic-statistical methods can be used. If data is presented with sufficient "inaccuracy", then for their study the methods of theory of errors will be used. But there are cases when both methods of statistics and the theory of errors do not give satisfactory results.

When data is presented by intervals and their description is "vague" and characterized by overlapping and the receipt of data the expert is and in the intervened, it is clear that the nature of data are combined: parallel to probabilistic-statistical uncertainty there exists the possibilistic uncertainty, guarantees more or less adequate results.

Fuzzy statistics play an essential part in probability-possibility analysis and they are used very effectively in fuzzy expert decision-making systems. Non-additive but monotone measures (fuzzy measures) were first used in fuzzy statistics in 80S by M. Sugeno [3].

We consider problems of fuzzy measure restoration from corresponding insufficient data (the third section).

In [4] there is presented a problem of construction of the distance on fuzzy measures, which is reduced to the distance between probabilistic measures in the class of associated probabilities. This is the problem, considered in the second section. There are considered basic definitions with needed commentaries in the second section.

In the fourth section there is constructed the concrete example and its table interpretation.

1. PRELIMINARY CONCEPTS

Let $X=\{x_1,x_2,...,x_n\}$ is the finite reference set, $\mathcal{B}(X)$-algebra of all subsets of X, g-fuzzy measure $\mathcal{B}(X)$ in Sugeno's sense and $(X,\mathcal{B}(X),g)$-fuzzy measure space [3]

1°. Fuzzy measure $g_\lambda \in [0,1]^{\mathcal{B}(X)}$ $(\lambda > -1)$ is λ-additive fuzzy measure [2] if for $\forall A,B \in \mathcal{B}(X),\ A \cap B = \varnothing$,

$$g_\lambda(A\cup B)=g_\lambda(A)+g_\lambda(B)+\lambda g_\lambda(A)\cdot g_\lambda(B). \qquad (1)$$

It is easy to verify that $\forall A \in \mathcal{B}(X)$:

$$g_\lambda(A)=\frac{1}{\lambda}\left\{\prod_{x_i\in A}(1+\lambda \hat{g}_i)-1\right\}, \qquad (2)$$

where $0<\hat{g}_i \equiv g\{x_i\}<1;\quad \lambda>-1$ is a parameter with the following normalization condition: (2')

$$\frac{1}{\lambda}\left\{\prod_{x_i\in X}(1+\lambda\hat{g}_i)-1\right\}=1.$$

Note that g_0 is probabilistic measure if $\sum_{x_i\in X}\hat{g}_i=1$.

2^o. Dual fuzzy measures $g, g^* \in [0,1]^{\mathrm{B}(X)}$ are called respectively lower and upper Choquet's second order capacities [1], [6] if $\forall A,B\in \mathrm{B}(X)$:

$$\begin{gathered} g(A\cap B)+g(A\cup B)\geq g(A)+g(B), \\ g^*(A\cap B)+g^*(A\cup B)\leq g^*(A)+g^*(B), \end{gathered} \tag{3}$$

where $g^*(A)=1-g(\overline{A})$ (duality). Choquet's second order capacities are enough broad class of fuzzy measures. For example, λ-additive fuzzy measure g_λ is Choquet's second order capacity. It is easy verifiable that $g^*_\lambda = g_{-\lambda/(1+\lambda)}$. Let $\{\hat{g}_i\}$ *and* $\{\hat{g}^*_i\}, i=1,2,...,n$ denote "fuzzy weights" of singletons for g, g^* dual fuzzy measures respectively.

3^o. For each $\sigma=(\sigma(1),\sigma(2),...,\sigma(n))\in S_n$ permutation of the finite set {1,2,…,n} the probability functions [1], [8]:

$$\begin{gathered} P_\sigma(x_{\sigma(1)})=g(\{x_{\sigma(1)}\}), \\ P_\sigma(x_{\sigma(2)})=g(\{x_{\sigma(1)},x_{\sigma(2)}\})-g(\{x_{\sigma(1)}\}), \\ P_\sigma(x_{\sigma(i)})=g(\{x_{\sigma(1)},...,x_{\sigma(i)}\})-g(\{x_{\sigma(1)},...,x_{\sigma(i-1)}\})), \\ P_\sigma(x_{\sigma(n)})=1-g(\{x_{\sigma(1)},...,x_{\sigma(n-1)}\})) \end{gathered} \tag{4}$$

are called the associated probabilities to the fuzzy measure g, where S_n is the permutations group of all natural number from 1 to n. It is proved [1] that if $g, g^*\in[0,1]^{\mathrm{B}(X)}$ are dual fuzzy measures then they have common associated probabilities class:

$$\{P_\sigma(\cdot)\}_{\sigma\in S_n}=\{P_\sigma(\cdot)\}_{\sigma\in S_n} \text{ and } \forall\sigma\in S_n : P_\sigma(\cdot)=P^*_{\sigma^*}(\cdot),$$

where σ^* is dual permutation of $\sigma(\sigma(i)=\sigma^*(n-i+1), i=1,2,...,n)$.

By (2) and (4) we may write down associated probabilities class for λ-additive fuzzy measure g_λ. $\forall\sigma\in S_n$

$$P_\sigma(x_{\sigma(i)})=g_\lambda(\{x_{\sigma(i)}\})\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{x_{\sigma(i)}\})), \tag{5}$$

more suitable

$$P_\sigma(x_{\sigma(i)})=g_\lambda(\{x_{\sigma(i)}\})\prod_{j=1}^{i(\sigma)-1}(1+\lambda g_\lambda(\{x_{\sigma(i)}\})), \tag{5'}$$

where $i=1,2,...,n;\ \sigma\in S_n;\ i(\sigma)$ is the location of x_i in σ permutation. (If $i(\sigma)=1$ than $\prod_{j=1}^{0}\equiv 1$).

4^o. Introduce the following notations. $\mathfrak{m}(X)\subset[0,1]^{\mathrm{B}(X)}$-fuzzy measures on $\mathrm{B}(X)$; $\mathfrak{m}_2(X)$-Choquet's second order capacities on $\mathrm{B}(X)$; $\mathfrak{m}_\lambda(X)-\lambda$ additive fuzzy measures on $\mathrm{B}(X)$; $\mathrm{R}(X)$-probability measures on $\mathrm{B}(X)$. It is clear

$\mathrm{R}(X)\subset\mathfrak{m}_\lambda(X)\subset\mathfrak{m}_2(X)\subset\mathfrak{m}(X)$. We know [1] that if $g\in\mathfrak{m}_2(X)$ then $\forall A\subseteq X$

$$g(A)=\min_{\sigma\in S_n}P_\sigma(A),\quad g^*(A)=\max_{\sigma\in S_n}P_\sigma(A)$$

5^o. Let $T^m\equiv\{(y_1,y_2,...,y_m)\in R^m / y_i\geq 0, i=1,2,...,m\}$. Let f be a function $f: T^m\to R^+$. f is called a function generatrix of distance, if the following five properties are satisfied:

(1) $f(y_1,y_2,...,y_m)=0 \Leftrightarrow y_1=y_2=...=y_m=0$,

(2) $y_i \le z_i, \forall i \Rightarrow f(y_1, y_2, ..., y_m) \le f(z_1, z_2, ..., z_m)$. f is monotone non-decreasing,

(3) $f(y_1 + z_1, y_2 + z_2, ..., y_m + z_m) \le f(y_1, y_2, ..., y_m) + f(z_1, z_2, ..., z_m)$. f is sub additive,

(4) $f(y, y, ..., y) = y$. f is idempotent,

(5) $f(y_{\sigma(1)}, y_{\sigma(2)}, ..., y_{\sigma(m)}), \forall \sigma \in S_m$. f is symmetric.

We rank the $n! \equiv m$ permutations of S_n with some criterion to number them, and thus to represent the class $\{P_\sigma(\cdot)\}_{\sigma \in S_n}$ as an n!-tuple $(P_1, P_2, ..., P_m)$. Let d be some distance on $\mathcal{R}(X)$ [4]. It is proved [4] that the function $D: \mathfrak{m}(X) \times \mathfrak{m}(X) \Rightarrow R^+$ defined as

$$D(g, g') = f(d(P_1, P_1'), d(P_2, P_2'), ..., d(P_m, P_m'))$$

is distance on $\mathfrak{m}(X)$. The examples of function f:

$$f_m(y_1, y_2, ..., y_m) = \max_{1 \le i \le m}\{y_i\},$$

$$f_q(y_1, y_2, ..., y_m) \equiv \left(\sqrt{\frac{1}{m}\sum_{i=1}^{m} y_i^q}\right)^{1/q}, \quad q \ge 1.$$

The examples of distance d:

$$d_m(P, P') = \max_{1 \le i \le m}\left|P(x_i) - P'(x_i)\right|,$$

$$d_q(P, P') \equiv \left(\sqrt{\frac{1}{m}\left|P_\sigma(x_i) - P_\sigma'(x_i)\right|^q}\right)^{1/q}, \quad q \ge 1,$$

$$d_S(P, P') = \max_{A \subseteq X}\left|P(x_i) - P'(x_i)\right|.$$

Let $D_2 \equiv D_{22}$ denotes the distance $(q = 2)$

$$D_{22}(g, g') = \sqrt{\frac{1}{n!}\sum_{\sigma \in S_n}\sum_{i=1}^{n}(P_\sigma(x_i) - P_\sigma'(x_i))^2}$$

and

$$D_{mq}(g, g') = \max_{\sigma \in S_n}\left(\sqrt{\left|P_\sigma(x_i) - P_\sigma'(x_i)\right|^q}\right)^{1/q}$$

6^o. Given $g \in \mathfrak{m}(X)$. The probability measure $P_g \in \mathcal{R}(X)$ is called nearest from fuzzy measure g if

$$D(g, P_g) = \min_{P \in \mathcal{R}(X)} D(g, P) \qquad (6)$$

It is known [1] that if $P \in \mathcal{R}(X)$ then associated probabilities class contains single probability distribution $P \equiv P_\sigma, \sigma \in S_n$. So the problem of minimizing may be reduced to the problem of minimizing the function D with respect to P:

$$D_2(g, P) = \sqrt{\frac{1}{n!}\sum_{\sigma \in S_n}\sum_{i=1}^{n}(P_\sigma(x_i) - P(x_i))^2} \Rightarrow \min$$

$P \in \mathcal{R}(X)$. Applied well-known tools of analysis we receive the solution

$$P_g(x_i) = \frac{1}{n!}\sum_{\sigma \in S_n} P_\sigma(x_i) \qquad (7)$$

$i = 1, 2, ..., n$. If in (7) g is λ-additive fuzzy measure- g_λ and we'll foresee (5) then

$$P_{g_\lambda}(x_i) = \frac{\hat{g}_i}{n!}\sum_{\sigma \in S_n}\prod_{j=1}^{i(\sigma)-1}(1 + \lambda \hat{g}_{\sigma(j)}), \qquad (8)$$

$i = 1, 2, ..., n$; if in (8) $i(\sigma) = 1$ then addend is equal to 1. Here $\hat{g}_i \equiv g_\lambda(\{x_i\})$. The minimum distance D_2 between fuzzy measure g_λ and $\mathcal{R}(X)$ is

$$D_2(g_\lambda, \mathcal{R}(X)) = D_2(g_\lambda, P_{g_\lambda}) =$$
$$= \sqrt{\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{i=1}^{n}\hat{g}_i^{\,2}\left[\prod_{j=1}^{i(\sigma)-1}\{1+\lambda\hat{g}_{\sigma(j)}\} - \frac{1}{n!}\sum_{\tau\in S_n}\prod_{k=1}^{i(\tau)-1}\{1+\lambda\hat{g}_{\tau(k)}\}\right]^2} \quad (9)$$

If $\lambda = 0$, (when $\sum_{i=1}^{n} g_i = 1$) g_0 is probability measure then $D_2 \equiv 0$. This distance is called a degree of unspecific [4].

7^o. For given $g \in \mathcal{M}(X)$

$$C(g) = min\{D(g, Bel_0), D(g^*, Pl_0)\} \quad (10)$$

is called an induce of specificity [4], where Bel_0 and Pl_0 are dual fuzzy measures of the beliefe and plausibility of whole ignorance. $\forall A \subseteq X$

$$Bel_0(A) = \begin{cases} 0 & if \quad A \neq X \\ 1 & if \quad A = X \end{cases}, \qquad Pl_0(A) = \begin{cases} 0 & if \quad A \neq \varnothing \\ 1 & if \quad A = \varnothing \end{cases}.$$

If $C(g_\lambda) \approx 0$ then g_λ is near to Bel_0 or Pl_0 and g_λ hasn't the specificity. If $c >> 0$ then g_λ has a high degree of specificity. The associated probabilities class of Bel_0 is:

$$P_\sigma^{Bel_0}(x_{\sigma(i)}) = \begin{cases} 1 & if \quad i = n \\ 0 & if \quad i \neq n \end{cases},$$

$i = 1,2,...,n; \sigma \in S_n$. Then

$$P_{Bel_0}(x_i) = \frac{1}{n!}\sum_{\sigma\in S_n} P_\sigma^{Bel_0}(x_{\sigma(i)}) = \frac{1}{n!}(n-1)! = \frac{1}{n}$$

and we receive the uniform probability distribution. Hence

$$C(g_\lambda) = \sqrt{\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{i=1}^{n}\hat{g}_i^{\,2}\left[\prod_{j=1}^{i(\sigma)-1}\{1+\lambda\hat{g}_{\sigma(j)}\} - \frac{1}{n}\right]^2}. \quad (11)$$

2. THE PROBLEM OF FUZZY MEASURE RESTORING

In practice the subjective expert data is often performed only for singleton factors, because any measurements of multifactorial variants practically don't exist. For example: if four x_1, x_2, x_3, x_4 factors (symptoms) act on the illness y then by some expert (doctor) may be performed frequency distribution table (table1), where some "weights" are subjectively "appointed" but pair "fuzzy weights" almost don't exists.

Here we offer the method which restores dual fuzzy measures dual fuzzy measures (g, g^*) with best approach to $\mathcal{M}(X)$ from $\mathcal{B}(X)$ in the sense of distance D_2 though with complimented condition. Let it is only known "fuzzy weights" of singletons:

$$0 < \hat{g}_i \equiv g(\{x_i\}) < 1, \ i = 1,2,...,n. \quad (12)$$

Let

$$\mathcal{M}(X, \hat{g}_1, \hat{g}_2, ..., \hat{g}_n) = \{g \in \mathcal{M}(X) / g(\{x_i\}) = \hat{g}_i, \quad i = 1,2,...,n\}$$

is the class of fuzzy measures of $\mathcal{M}(X)$ with coinciding values of measures on singletons.

TABLE 1

$A \subseteq X = \{x_1, x_2, x_3, x_4\}$	g
$\{x_1\}$	0.2
$\{x_2\}^{*)}$	0.3subj
$\{x_3\}$	0.4
$\{x_4\}$	0.2subj
$\{x_1, x_2\}$	?
$\{x_1, x_3\}$	?
$\{x_1, x_4\}$	?
$\{x_2, x_3\}$	?
$\{x_2, x_4\}$	?
$\{x_3, x_4\}$	?
$\{x_1, x_2, x_4\}$	?
$\{x_2, x_3, x_4\}$	?
$\{x_1, x_2, x_3\}$	?
$\{x_1, x_3, x_4\}$	?
$\{x_1, x_2, x_3, x_4\}$	1

TABLE1: insufficient expert frequency distribution of some illness with respect only to 4 symptoms in terms of the fuzzy measure g

*) Data with notion "subj" is appointed by the expert.

Analogously

$$\mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n) = \mathfrak{m}(X) \cap \mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n)$$

is the class of second order Choquet's capacities with the same property and

$$\mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n) = \mathfrak{m}(X) \cap \mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n)$$

is the same class for λ-additive measures. It is clear that

$$\mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n) \subset \mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n)$$

$\lambda > -1$ is a free parameter of the distribution of λ-additive fuzzy measure $g \in \mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n)$ with normalization condition (3). If $\sum_{i=1}^{n} \hat{g}_i = 1$, then $\lambda_0 \equiv 0$ value is assumed (g_0 is probability measure), otherwise λ is the root of the following polinom:

$$\Pi(\lambda) = \left(\prod_{i=1}^{n} \hat{g}_i\right)\lambda^{n-1} + ... + \left(\sum_{i<j<k} \hat{g}_i \hat{g}_j \hat{g}_k\right)\lambda^2 + \left(\sum_{i<j} \hat{g}_i \hat{g}_j\right)\lambda + \sum_{i=1}^{n} \hat{g}_i - 1. \quad (13)$$

Let $L \equiv \{\lambda_1, \lambda_2, ..., \lambda_l\}$ is the set of real roots of the polinom (13) ($\lambda > -1$).

Let $L \neq \varnothing$. Introduce the following short notations:

$$\mathfrak{m}^L(X) = \{ g_{\lambda_i} \in \mathfrak{m}(X; \hat{g}_1, \hat{g}_2, ..., \hat{g}_n) / \lambda_i \in L, i = 1,2,...,l \}$$

It is clear, that $\hat{g}_i^{*}$ values are not "freedom" (for $\forall \lambda \in L$):

$$\hat{g}_i^{*} = 1 - \frac{1}{\lambda}\left\{\prod_{\substack{j=1 \\ j \neq i}}^{n} (1 + \lambda \hat{g}_j) - 1\right\}, \quad i = 1,2,...,n$$

and if $\lambda > 0$ then $\hat{g}_i \le \hat{g}_i^*$, $i = 1,...,n;$ if $-1 < \lambda < 0$ then $\hat{g}_i \ge \hat{g}_i^*$, $i = 1,...,n$.

Analogously, we may construct $L^* = \{\lambda^* > -1 / \lambda^* = -\frac{\lambda}{1+\lambda},\ \lambda \in L\}$ and $\mathfrak{m}^{L^*}(X) \subset \mathfrak{m}^*\left(X; \hat{g}_1^*, \hat{g}_2^*, ..., \hat{g}_n^*\right)$ classes.

The classes

$$\mathfrak{R}^L(X) = \{P_{g_\lambda} \in \mathfrak{R}(X) / \lambda \in L\}, \qquad \mathfrak{R}^{L^*}(X) = \{P_{g_\lambda^*} \in \mathfrak{R}(X) / \lambda^* \in L^*\}$$

are probability measures classes. We calculate the distances:

$$D_2(\mathfrak{R}^L(X), \mathfrak{m}^L(\mathrm{X})) = \min_{\lambda', \lambda'' \in L} D_2(P_{g_{\lambda'}}, g_{\lambda''}) = \min_{\lambda \in L} D_2(P_{g_\lambda}, g_\lambda) =$$

$$= \min_{\lambda \in L} \sqrt{\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^n \hat{g}_i^2 \left\{ \prod_{j=1}^{i(\sigma)-1} \{1 + \lambda \hat{g}_{\sigma(i)}\} - \frac{1}{n!} \sum_{\tau \in S_n} \prod_{k=1}^{i(\tau)-1} \{1 + \lambda \hat{g}_{\tau(k)}\} \right\}^2}, \quad (14)$$

$$D_2(\mathfrak{R}^{\mathrm{L}^*}(X), \mathfrak{m}^{\mathrm{L}^*}(\mathrm{X})) = \min_{\lambda^* \in L^*} D_2(g_{\lambda^*}^*, P_{g_{\lambda^*}^*}) =$$

$$= \min_{\lambda^* \in L^*} \sqrt{\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^n \hat{g}_i^{*2} \left\{ \prod_{j=1}^{i(\sigma)-1} \{1 + \lambda^* \hat{g}_{\sigma(j)}^*\} - \frac{1}{n!} \sum_{\tau \in S_n} \prod_{k=1}^{i(\tau)-1} \{1 + \lambda^* \hat{g}_{\tau(k)}^*\} \right\}^2}. \quad (14')$$

Let these distances are reached on the fuzzy measures $\mathrm{g}_{\hat{\lambda}}$ and $\mathrm{g}_{\hat{\lambda}'}^*$:

$$\mathrm{D}_2(\mathfrak{R}^{\mathrm{L}}(\mathrm{X}), \mathfrak{m}^{\mathrm{L}}(\mathrm{X})) = \mathrm{D}_2(\mathrm{g}_{\hat{\lambda}}, \mathrm{P}_{\mathrm{g}_{\hat{\lambda}}}), \qquad \mathrm{D}_2(\mathfrak{R}^{\mathrm{L}}(\mathrm{X}), \mathfrak{m}^{\mathrm{L}^*}(\mathrm{X})) = \mathrm{D}_2(\mathrm{g}_{\hat{\lambda}}^*, \mathrm{P}_{\mathrm{g}_{\hat{\lambda}}^*}).$$

<u>DEFINITION 1:</u> *$(g_{\hat{\lambda}}, g_{\hat{\lambda}}^*)$ pair fuzzy measures are called λ-additive fuzzy approximation to $\hat{g}_i$, $i = 1,...,n$ insufficient expert data.*

<u>DEFINITION 2</u>: $\left(P_{g_{\hat{\lambda}}}, P_{g_{\hat{\lambda}}^*}\right)$ *pair probability measures are called probability approximation to $\hat{g}_i$, $i = 1,...,n$ insufficient expert data.*

Notice that if $\sum_{i=1}^n \hat{g}_i = 1$ and the problem of restoring of measure doesn't exist. If we know that g_0 is not probability measure then suppose $\lambda \ne 0$, $\mathfrak{R}^{\mathrm{L}}(\mathrm{X}) \cap \mathfrak{m}^{\mathrm{L}}(\mathrm{X})) = \varnothing$. It is easily checked that $(\mathrm{g}_\lambda)^* = \mathrm{g} - \frac{\lambda}{1+\lambda} \equiv \mathrm{g}_{\lambda^*}^*$, $\lambda^* = -\frac{\lambda}{1+\lambda}$; When $\lambda \to 0$ in L then $\lambda^* \to 0$ in L^* and minima in (14), (14') are respectively reached on $\hat{\lambda}$, $\left|\hat{\lambda}\right| = \min_{\lambda \in L} |\lambda|$ and on $\hat{\lambda}^*$, because $\min_{\lambda^* \in L^*} \left|\lambda^*\right| = \min_{\lambda \in L} \frac{|\lambda|}{1+\lambda} = \frac{\left|\hat{\lambda}\right|}{1+\hat{\lambda}} = \left|\lambda^*\right|$. We receive $(\hat{\lambda})^* = \hat{\lambda}'$, $(g_{\hat{\lambda}})^* = g_{\hat{\lambda}^*}^* = g_{\hat{\lambda}'}^*$. Given result may be represented as a proposition:

<u>PROPOSITION 1:</u> *Probability approximation corresponds to dual fuzzy approximation measures and $D_2(\mathfrak{m}^L, \mathfrak{R}^L) = D_2(\mathfrak{m}^{L^*}, \mathfrak{R}^{L^*})$.*

<u>PROPOSITION 2:</u> The *probability approximation pair is equal probability measures.*

<u>PROOF:</u> It is clear that $\mathrm{g}_{\lambda_1} \le \mathrm{g}_{\lambda_2}$ if $\lambda_1 \le \lambda_2$ $(\lambda_1, \lambda_2 \in \mathrm{L})$ and

$$g_{\hat{\lambda}} \le P_{g_{\hat{\lambda}}}, \quad P_{g_{\hat{\lambda}^*}^*} \le g_{\hat{\lambda}^*}^*.$$

λ-fuzzy measure $g_{\hat{\lambda}}, g^*_{\hat{\lambda}^*}$ are nearest to probability measures in the sense of distance D_2. Thus only probability distribution corresponds to the case $\lambda = 0$. $P_{g_{\hat{\lambda}}} = P_{g^*_{\hat{\lambda}^*}}$ $D_2(m^L, R^L) = D_2(m^{L^*}, R^{L^*})$.

In reality insufficient (similar to table 1) Data of dual fuzzy measures may be not single, but given by some experts $E_X = \{I_1, I_2, ..., I_E\}$.

<u>Definition 3:</u> Data $\hat{g}_i^{\alpha}, i = 1,...,n;$ $\alpha \in E_X$ (defined as (12)) are called insufficient expert data of the fuzzy measure g given by experts E_X.

Insufficient expert data produce $\{m^{L_\alpha}, m^{L^*_\alpha}, R^{L_\alpha}, R^{L^*_\alpha}\}, \alpha \in E_x$ classes from where we'll build probability approximations class $\{\hat{P}_\alpha / \alpha \in E_X\}$ and λ-additive fuzzy approximations class $\{(\bar{g}_\alpha, \bar{g}^*_\alpha) / \alpha \in E_X\}$.

<u>Definition 4:</u> Dual fuzzy measures defined $\forall A \subseteq X$:

$$\tilde{g}(A) = \min_{\alpha \in E_X} \hat{P}_\alpha(A), \quad \tilde{g}^*(A) = \max_{\alpha \in E_X} \hat{P}_\alpha(A) \tag{15}$$

are called zero approach optimal approximation.

<u>Definition 5</u>: Pair fuzzy measures defined as $\forall A \subseteq X$:

$$\bar{g}(A) = \min_{\alpha \in E_X} \bar{g}_\alpha(A), \quad \bar{g}'(A) = \max_{\alpha \in E_X} \bar{g}^*_\alpha(A) \tag{16}$$

are called first approach optimal approximation.

<u>Definition 6:</u> Pair fuzzy measures defined as $\forall A \subseteq X$:

$$\bar{\bar{g}}(A) = \max_{\alpha \in E_X} \bar{g}_\alpha(A), \quad \bar{\bar{g}}^*(A) = \min_{\alpha \in E_X} \bar{g}^*_\alpha(A) \tag{17}$$

are called second approach optimal approximation.

<u>Proposition 3:</u> *Pairs measures first and second approach optimal approximations are respectively dual fuzzy measures.*

<u>Proof:</u> $\forall A \subseteq X$: $\bar{g}'(A) = \max_{\alpha \in E} \bar{g}^*_\alpha(A) = \max_{\alpha \in E}(1 - \bar{g}_\alpha(\bar{A})) = 1 - \min_{\alpha \in E} \bar{g}_\alpha(\bar{A}) = 1 - \bar{g}(\bar{A})$. i.e. $\bar{g}' = \bar{g}^*$. Analogously we'll receive $\bar{\bar{g}}' = \bar{\bar{g}}^*$.

<u>Proposition 4:</u> *Between zero, first and second approach optimal approximations exists the following inequalities* $\forall A \subseteq X$:

$$1.\ \bar{g}(A) \le \bar{\bar{g}}(A); \quad \bar{\bar{g}}^*(A) \le \bar{g}^*(A),$$

$$2.\ \bar{g}(A) \le \tilde{g}(A); \quad \tilde{g}^*(A) \le \bar{g}^*(A),$$

$$3.\ \bar{\bar{g}}(A) \le \tilde{g}^*(A); \quad \tilde{g}(A) \le \bar{\bar{g}}^*(A).$$

<u>Proof:</u> Consider $\forall A \subseteq X$. Then

$$1.\ \bar{g}(A) = \min_{\alpha \in E_X} \bar{g}_\alpha(A) \le \max_{\alpha \in E_X} \bar{g}_\alpha(A) = \bar{\bar{g}}(A);$$

$$\bar{\bar{g}}^*(A) = \min_{\alpha \in E_X} \bar{g}^*_\alpha(A) \le \max_{\alpha \in E_X} \bar{g}^*_\alpha(A) = \bar{g}^*(A),$$

$$2.\ \bar{g}(A) = \min_{\alpha \in E_X} \bar{g}_\alpha(A) \le \min_{\alpha \in E_X} P_{\bar{g}_\alpha}(A) = \tilde{g}(A);$$

$$\tilde{g}^*(A) = \max_{\alpha \in E_X} P_{\bar{g}^*_\alpha}(A) = \max_{\alpha \in E_X} P_{\bar{g}_\alpha}(A) \le \max_{\alpha \in E_X} \bar{g}^*_\alpha(A) = \tilde{g}^*(A),$$

$$3.\ \bar{\bar{g}}(A) = \max_{\alpha \in E_X} \bar{g}_\alpha(A) \le \min_{\alpha \in E_X} P_{\bar{g}_\alpha}(A) = \max_{\alpha \in E_X} P_{\bar{g}^*_\alpha}(A) = \tilde{g}^*(A),$$

$$\tilde{g}(A) = \min_{\alpha \in E_X} P_{\bar{g}_\alpha}(A) = \min_{\alpha \in E_X} P_{\bar{g}^*_\alpha}(A) \le \min_{\alpha \in E_X} \bar{g}^*_\alpha(A) = \bar{\bar{g}}^*(A).$$

<u>Definition 7:</u> Distances $D_2(g, \tilde{g}) = D_2(g^*, \tilde{g}^*)$, $D_2(g, \bar{g}) = D_2(g^*, \bar{g}^*)$ and $D_2(g^*, \bar{\bar{g}}^*) = D_2(g, \bar{\bar{g}})$ are respectively called first and second approach optimal approximation errors.

There variants of optimal approximations, which are received from insufficient data of unknown fuzzy measure g, perform their "restored" faces. For comparison we consider the example where fuzzy measure g, will be known, We'll restore them their "fuzzy weights" of single sets (insufficient data) and estimate errors.

3. THE EXAMPLE

Consider one example when fuzzy dual measures (g, g^*) are known (case for one expert). Let $X = \{x_1, x_2, x_3\}$ and dual fuzzy measure $g \le g^*$ are "almost" uniform probability measures (table2),

TABLE2

$A \subseteq X = \{x_1, x_2, x_3\}$	g	g^*
$\varnothing$	0	0
$\{x_1\}$	1/4	1/3
$\{x_2\}$	1/3	1/3
$\{x_3\}$	1/3	1/3
$\{x_1, x_2\}$	2/3	2/3
$\{x_1, x_3\}$	2/3	2/3
$\{x_2, x_3\}$	2/3	1/4
$\{x_1, x_2, x_3\}$	0	0

but their associated probabilities class are given in table 2'

TABLE 2'

σ	$P_\sigma(x_{\sigma(1)})$	$P_\sigma(x_{\sigma(2)})$	$P_\sigma(x_{\sigma(3)})$	$P^*_\sigma(x_{\sigma(1)})$	$P^*_\sigma(x_{\sigma(2)})$	$P^*_\sigma(x_{\sigma(3)})$
1,2,3	1/4	5/12	1/3	1/3	1/3	1/3
1,3,2	1/4	5/12	1/3	1/3	1/3	1/3
2,1,3	1/3	1/3	1/3	1/3	1/3	1/3
2,3,1	1/3	1/3	1/3	1/3	5/12	1/4
3,1,2	1/3	1/3	1/3	1/3	1/3	1/3
3,2,1	1/3	1/3	1/3	1/3	5/12	1/4

The equation (13) has the following face:

$$0.02(7)\lambda^2 + 0.27(7)\lambda - 0.008334 = 0$$

from where $\lambda = \hat{\lambda} = 0.029928$. Then $\lambda^* = -\hat{\lambda}/(1+\hat{\lambda}) = -0.0290583$. $g_{\hat{\lambda}} = g_{0.029928}$, $\overline{g}^* = \hat{g}_{0.029928}$; the associated probabilities class of $\overline{g}$ is performed in table 3:

TABLE 3

σ	$P_\sigma(x_{\sigma(1)})$	$P_\sigma(x_{\sigma(2)})$	$P_\sigma(x_{\sigma(3)})$
1,2,3	0.25	0.3360771	0.4139229

1,3,2	0.25	0.3360771	0.4139229
2,1,3	0.3333333	0.2527458	0.4139229
2,3,1	0.3333333	0.3366473	0.3300194
3,1,2	0.3333333	0.2527438	0.4139229
3,2,1	0.3333333	0.3366473	0.3300194

We'll set distribution table of fuzzy measures $g, g^*, \overline{g}, \overline{g}^*$ with errors (table 4): (here $\overline{g} = \overline{\overline{g}}, \quad \overline{g}^* = \overline{\overline{g}}^*$)

TABLE 4

$A \subseteq X = \{x_1, x_2, x_3\}$	$\overline{g} = g_{0.0299084}$	$\overline{g}^* = g_{-0.0290583}$ $\lvert g(A) - \overline{g}(A)\rvert$	$\lvert g^*(A) - \overline{g}^*(A)\rvert$
$\{x_1\}$	$1/4$	0.3300185	0.0000185
$\{x_2\}$	0.3333333	0.4139569~0	~0.0806
$\{x_3\}$	0.3333333	0.4139569~0	0.0806
$\{x_1, x_2\}$	0.5860771	0.6666678 0.0805	~0.0000078
$\{x_1, x_3\}$	0.5960771	0.6666678 0.0805	0.00000078
$\{x_2, x_3\}$	0.6699806	0.7500764 0.003319	0.0000764
$\{x_1, x_2, x_3\}$	≈ 1	≈ 0	≈ 0

The first approach optimal approximation error is

$$D_2(g, \overline{g}) = D_2(g^*, \overline{g}^*) = 0.0108278 .$$

For setting $\tilde{g}, \tilde{g}^*$ we have one expert, therefore $\tilde{g}$ and $\tilde{g}^*$ are probability measures and $\tilde{g} = P_{g_{\hat{\lambda}}} = P_{g^*_{\hat{\lambda}^*}} = \tilde{g}^* = P_{g_{0.029928}}$ of which distribution is (table 5):

TABLE 5:

σ/P	$P(x_1)$	$P(x_2)$	$P(x_3)$
	0.2775876	0.3612063	0.3612061

but the distribution table with errors is (table 6):

TABLE 6:

$A \subseteq X$	$g = g^* = P_{g_{0.029928}}$	$\lvert g(A) - \tilde{g}(A)\rvert$	$\lvert g^*(A) - \tilde{g}^*(A)\rvert$
$\{x_1\}$	0.25	0	0.08
$\{x_2\}$	0.31	0.02	0.02
$\{x_3\}$	0.42	0.09	0.09
$\{x_1, x_2\}$	0.56	0.1	0.1
$\{x_1, x_3\}$	0.67	0.01	0.01
$\{x_2, x_3\}$	0.73	0.07	0.07
$\{x_1, x_2, x_3\}$	1	0	0

The zero approach optimal approximation error is

$$D_2(g, \tilde{g}) = D_2(g^*, \tilde{g}^*) = 0.0046453$$

As calculations shows (tables 2-6) estimations given by the approximations are enough “high”, maintain precisions of approaches. The zero approach optimal approximation is more exact than first one, because here fuzzy measures (g, g^{*}) are “near” or “similar” with probability measure and given by one expert.

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