The Optimal Control Problem for the Second Order Ordinary Differential Equation with Integral Boundary Condition and Quadratic Functional

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Abstract
In this paper the optimal control problem for the second order ordinary differential equation with non-local boundary conditions is considered. The necessary and sufficient condition for optimality has been obtained.

INTRODUCTION

Many processes in practice are controlled and it is important to find the optimal resolution for their realization. Besides, while mathematical modeling physical, biological and ecological processes we obtain non-local boundary problems.

Creado, Meladze and Odisehlidze considered the optimal control problem [1] for Helmholtz equation with Bitsadze-Samarski's type non-local boundary conditions [2] and quadratic functional. In the present paper is considered the optimal control problem for the second order ordinary differential equation with another type of non-locality – integral boundary conditions [3].

1. STATEMENT OF THE PROBLEM

Let V be an open subset of R and Ω be a set of control functions: $v:[0,1]\to V, v\in L_2([0,1])$, V is called domain of controls.

Let us consider following problem for each fixed $v\in\Omega$ in $[0,1]$ interval:

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} - qu(x) = f(x) + a(x)v(x), \\ u(0)=\alpha, \\ \displaystyle\int_0^1 u(x)dx = \beta, \end{cases} \tag{1}$$

where $x\in[0,1], \alpha, \beta\in R, a\in L_\infty([0,1]), f\in L_2([0,1]), 0<q=const$. It is known, that the solution of problem (1) exists, is unique and belongs to space $W_2^2([0,1])$ [6].

Let $I(v)$ be the following quadratic functional:

$$I(v) = \int_0^1 \left[b_1(x)u^2(x) + b_2(x)v^2(x)\right]dx, \tag{2}$$

where $b_1, b_2 \in L_\infty([0,1])$ are given functions.

Now let's state the following optimal control problem: find the function $v_0\in\Omega$, whose corresponding solution of problem (1) together with v_0 results in the minimal functional value.

2. ADJOINT EQUATION

To obtain conditions of optimality we follow the scheme developed in the works [4][5].

Assume, that $v_0\in\Omega$ is an optimal control, $v_\varepsilon\in\Omega$ is arbitrary admissible control and u_0, u_ε are corresponding solutions of problem (1). Let's take the following notations:

$$\tilde{v} \equiv v_\varepsilon - v_0, \qquad \tilde{u} \equiv u_\varepsilon - u_0. \tag{3}$$

If we'll consider problem (1) correspondingly to (u_0, v_0) and $(u_\varepsilon, v_\varepsilon)$, then we come to the following problem for $\tilde{u}$:

$$\begin{cases} \dfrac{d^2\tilde{u}(x)}{dx^2} - q\tilde{u}(x) = a(x)\tilde{v}(x), \quad x \in (0,1), \\ \tilde{u}(0) = 0, \\ \int_0^1 \tilde{u}(x)dx = 0. \end{cases} \tag{4}$$

For a certain v_0 and v_ε let us consider the following difference:

$$\begin{gathered} \tilde{I} \equiv I(v_\varepsilon) - I(v_0) = \int_0^1 b_1(x)u_\varepsilon^2 dx + \int_0^1 b_2(x)v_\varepsilon^2(x)dx - \\ \int_0^1 b_1(x)u_0^2(x)dx - \int_0^1 b_2(x)v_0^2(x)dx = \\ \int_0^1 \left[b_1(x)\tilde{u}^2(x) + b_2(x)\tilde{v}^2(x)\right]dx + 2\int_0^1 \left[b_1(x)u_0(x)\tilde{u}(x) + b_2(x)v_0(x)\tilde{v}(x)\right]dx. \end{gathered} \tag{5}$$

Let $\psi \in W_2^2([0,1])$ and $\psi \neq 0$. If we multiply (4) on ψ, then integrate obtained expression on the interval [0,1] and take into account (5) equality, we shall have:

$$\begin{gathered} \tilde{I} = \int_0^1 \psi(x)\left[\frac{d^2\tilde{u}}{dx^2} - q\tilde{u}\right]dx - \int_0^1 \psi(x)a(x)\tilde{v}(x)dx + \\ \int_0^1 \left[2b_1(x)u_0(x)\tilde{u}(x) + 2b_2(x)v_0(x)\tilde{v}(x)\right]dx + \int_0^1 \left[b_1(x)\tilde{u}^2(x) + b_2(x)\tilde{v}^2(x)\right]dx. \end{gathered} \tag{6}$$

Let us make the folowing transformation for constructing the adjoint equation: two times using partially integration formula and fact that $\tilde{u}(0) = 0$, the first member of equation (6) will transform into following:

$$\int_0^1 \psi(x)\frac{d^2\tilde{u}}{dx^2}dx = \int_0^1 \tilde{u}(x)\frac{d^2\psi}{dx^2}dx + \psi(1)\tilde{u}'(1) - \psi(0)\tilde{u}'(0) - \psi'(1)\tilde{u}(1). \tag{7}$$

Integrating (4) equation in [0,1] interval we obtain:

$$\tilde{u}'(1) = \tilde{u}'(0) + q\int_0^1 \tilde{u}(x)dx + \int_0^1 a(x)\tilde{v}(x)dx = \tilde{u}'(0) + \int_0^1 a(x)\tilde{v}(x)dx. \tag{8}$$

Placing (8) equation in (7) expression, we shall have:

$$\begin{gathered} \int_0^1 \psi(x)\frac{d^2\tilde{u}}{dx^2}dx = \int_0^1 \tilde{u}(x)\frac{d^2\psi}{dx^2}dx + \left(\psi(1) - \psi(0)\right)\tilde{u}'(0) - \psi'(1)\tilde{u}(1) + \\ + \psi(1)\int_0^1 a(x)\tilde{v}(x)dx. \end{gathered} \tag{9}$$

Replacing (6) into (9) we shall see, that if ψ_0 is the solution of the following problem:

$$\begin{cases} \dfrac{d^2\psi(x)}{dx^2} + q\psi(x) = -2b_1(x)u_0(x), \quad x \in (0,1), \\ \psi(0) = \psi(1), \\ \psi'(1) = 0, \end{cases} \tag{10}$$

then functional $\tilde{I}$ expressed in (6) will be as follows:

$$\tilde{I} = \int_0^1 \left[\left(\psi(1) - \psi(x)\right)a(x) + 2b_2(x)v_0(x)\right]\tilde{v}(x)dx + \int \left[b_1(x)\tilde{u}^2(x) + b_2(x)\tilde{v}^2(\cdot \tag{11}$$

THEOREM 1: Let $q = const > 0$ *and* $b_1(x) \in L_\infty([0,1])$, $u(x) \in W_2^2([0,1])$ *are given functions, then the solution of the problem (10) exisis, is unique, belongs to space* $W_2^2([0,1])$ *and could be written as follows:*

$$\psi(x) = c_0 e^{\sqrt{q}x} + c_1 e^{-\sqrt{q}x} + \frac{1}{2\sqrt{q}} \int_0^x \left(c_0 e^{-\sqrt{q}(t-x)} - c_0 e^{\sqrt{q}(t-x)} \right) f(t)dt + k \frac{e^{-\sqrt{q}x} - e^{-2\sqrt{q}} e^{\sqrt{q}x}}{1 - e^{\sqrt{q}}} +$$

$$+ \left(k \frac{\left(e^{\sqrt{q}} - e^{-\sqrt{q}}\right)^2}{2\left(e^{\sqrt{q}} + e^{-\sqrt{q}}\right)} - \frac{\left(c_0 e^{\sqrt{q}} - c_1 e^{-\sqrt{q}}\right)\left(e^{\sqrt{q}} - e^{-\sqrt{q}}\right)}{e^{\sqrt{q}} - e^{-\sqrt{q}}} + \frac{2}{e^{\sqrt{q}} + e^{-\sqrt{q}}} - \right.$$

$$\left. - \frac{\left(e^{\sqrt{q}} - e^{-\sqrt{q}}\right)^2}{2\sqrt{q}\left(e^{\sqrt{q}} - e^{-\sqrt{q}}\right)} \int_0^1 f(t)dt \right) \frac{e^{\sqrt{q}x} - e^{-\sqrt{q}x}}{e^{\sqrt{q}} - e^{-\sqrt{q}}},$$

where

$$f(x) = -2b_1(x)u(x),$$

$$k = \frac{4\left(c_0 e^{2\sqrt{q}} + c_1 e^{-2\sqrt{q}} - 1\right) + \frac{1}{\sqrt{q}}\left(e^{\sqrt{q}} + e^{-\sqrt{q}}\right) \int_0^1 \left(e^{-\sqrt{q}(t-1)} - e^{\sqrt{q}(t-1)}\right) f(t)dt}{2\left(1 + e^{\sqrt{q}} + e^{-\sqrt{q}}\right) - e^{2\sqrt{q}} - e^{-2\sqrt{q}}} -$$

$$- \frac{e^{2\sqrt{q}} + e^{-2\sqrt{q}} - 2}{2\left(\sqrt{q}\left(1 + e^{\sqrt{q}} + e^{-\sqrt{q}}\right) - e^{-2\sqrt{q}} - e^{2\sqrt{q}}\right)} \int_0^1 f(t)dt.$$

And c_0 *and* c_1 *are defined as follows:*

$$\begin{cases} c_0 = -c_1 \\ c_1 = \dfrac{1}{2\sqrt{q}\left(e^{\sqrt{q}} - e^{-\sqrt{q}}\right)} \displaystyle\int_0^1 \left(e^{\sqrt{q}(1-t)} - e^{-\sqrt{q}(1-t)}\right) f(t)dt. \end{cases}$$

3. Necessary and Sifficient Condision for optimality of Solution

THEOREM 2. *Let functional be given by the formula (2),* $b_2(x) > 0$ *and* ψ_0 *is solution of the problem (10), then the couple* (u_0, v_0) *is optimal if, and only if, the following condition is satisfied:*

$$\left(\psi_0(1) - \psi_0(x)\right)a(x) + 2b_2(x)v_0(x) = 0$$

almost everywhere in the [0,1] interval.

The confirmation of this theorem is the same as in [1] ,where the confirmation of necessary and sufficient condition for optimal control problem with another type of non-locality is presented.

4.Concequence

Let's consider the following optimal control problem:

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} - q(x)u(x) = f(x) + a(x)v(x), \\ u(0) = \alpha, \\ \displaystyle\int_0^1 q(x)u(x)dx = \beta, \end{cases} \qquad (12)$$

where $x \in [0,1], \alpha, \beta \in R, a \in L_\infty([0,1]), f \in L_2([0,1]), 0 < q \in L_\infty([0;1])$. It is known, that the solution of problem (1) exists, is unique and belongs to space $W_2^2([0,1])$ [6].

Let $I(v)$ be the following quadratic functional:

$$I(v)=\int_0^1 \left[b_1(x)u^2(x)+b_2\ (x)v^2(x)\right]dx, \qquad (13)$$

where $b_1, b_2 \in L_\infty([0,1])$ are given functions and (u, v) searched couple, satisfy (12) and will assign minimal value to (12) functional.

Above considered problem will transform to the following system:

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} - q(x)u(x) = f(x) + \dfrac{a^2(x)}{2b_2(x)}\big(\psi(x)-\psi(1)\big), \quad x \in (0,1), \\ u(0) = \alpha, \\ \int_0^1 q(x)u(x)dx = \beta; \end{cases} \qquad (14)$$

$$\begin{cases} \dfrac{d^2\psi(x)}{dx^2} + q(x)\psi(x) = -2b_1(x)u(x), \quad x \in (0,1), \\ \psi(0) = \psi(1), \\ \psi'(1) = 0. \end{cases}$$

The adjoint problem of (12)-(13) problem is corrected and also the already received necessary and sufficient condition of optimality is truthful for it.

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