New Technologies of Design of Some Boundary Value Problems For Ordinary Differential Equations

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Abstract
The technology of design of two-point boundary value problems for ordinary differential equations, containing also boundary layer effects is elaborated. The proposed methods essentially refine and enlarge a class of algorithms for solving aforesaid problems. From these methods there follow also classical methods, including methods of Collatz, Henrici, Marchuk, Schröder, Tikhonov-Samarskii, finite elements and exponential fitted methods. Then the program part is realized in the form of package of applied programs consisting of control program and modules. For fulfilling this work we followed the manual [Ben-Israel A., Gilbert R.P.] with its software that was kindly given to us by Gilbert. Some parts of this technology are systematically inculcated in teaching processes and not only in the basic courses and also in student's course and diploma works at Iv. Javakhishvili Tbilisi State University, Vekua Institute of Applied Mathematics, University of Delaware. Is created the program package on Turbo Pascal 7.0 for solving the boundary value problems for the second order ordinary linear differential equations (fourth issue).

The contents of the report besides the scientific side present an effective manual, realizing purposes, which are stipulated in teaching processes for high school and in practice.

INTRODUCTION

A purpose of the present paper is to suggest a new technology of design of a class of boundary value problems (BVPs) for the second order ordinary differential equations, introduced in educational processes of a number of universities. We will present here also the manuscript-manual as the enlarged version of this report, using essentially the structure of the books [2,6,12].

We note also, that a class of studied BVPs, presented below is chosen for an illustration of the methodology, however, a more general case is considered in [22,24,25].

Let us consider BVPs for the second order non-linear ordinary differential equations

$$\left(k(x)y'(x)\right)' = f(x,y(x),y'(x)),\ \ k(x)>0,\ \ 0<x<1, \qquad (1)$$

with the boundary conditions

$$\begin{aligned} y(0)-k_1y'(0)=\alpha, &\quad k_1\ge 0,\\ y(1)+k_2y'(1)=\beta, &\quad k_2\ge 0. \end{aligned} \qquad (2)$$

The problems connected with our technology of design are studied for the following subclasses of BVPs (1)-(2):

1.Picard-Banach type conditions are satisfied; 2.The maximum pronciple is fulfilled.

These subclasses of BVPs, having uniqueness solutions, are important for practice also. This problem of solvability is studied, in particular, in [3,16,17,19,20] and [1,3,5,6,10,11] correspondingly.

We note, that for constructiong Tikhonov-Samarskii schemes [18] it is necessary to compute multiply integrals while by the works of Volkov [28] for getting p-th order ($p>2$) of exactness with respect to h of three-point schemes it is necessary to compute the derivatives of $p-2$ order from the given data of BVP (1)-(2).

1. BVPS OF THE PICARD-BANACH TYPE

For this class the numerical methods, presented, for instance, in the works [3,17], are usually supported on a construction of a difference analogue of the Green function and the solution is found using the fixed-point theorem. The corresponding iterative scheme has the form:

$$y_i^{(m)} = \sum_{k=1}^{N} g_{ik} f_i^{(m-1)} + l_i(\alpha,\beta), \;\; i = 1,2,...,N, \tag{3}$$

where $h = 1/N$ is a step of the mesh, g_{ik} are values of the discrete Green function, $l_i(\alpha,\beta)$ are corresponding functionals, satisfying the boundary conditions (2), $y_i^{(m)}$ are values of te unknown function y in x_i-mesh point on the m-th iteration. An exactness if schemes according to [3,17] have second order, if the iteration process (3) is convergent and of $y \in C^4[0,1]$. From the expression (3) it evidently follows that for constucting approximate solution it is necessary to do $\asymp N^2 \ln N$ arithmetical operations (as is well known that $\asymp \ln N$ operations is the number of iterations).

In the works of Vashakmadze [19,20] there are considered problems of numerical solutions of the Picard-Banach type subclasses of BVPs. The remainder terms of the corresponding schemes are $O(N^{-p+2})$, if $y \in C^p[0,1], (p \geq 4)$. For $p = 4$ the result with respect to an order of convergence is similar to those of classical methods [3,17], but the order of an arithmetic operations is minimal $\asymp N \ln N$. This order for an estimate of arithmetical formulae are presented in details in [19,20,23] or in the aforesaid manuscript.

2. BVPS SATISFYING THE MAXIMUM PRINCIPLE

The constructions of an approximate solution of this subclass by the finite-difference or varational-difference (i.e., Finite Elemet) methods represents classical part of numerical analysis and are studied, for example, in [1,3,5,6,7,10,11,13,14,16,28, etc]. If for the most of these works the coresspondingly schemes have the second order of accuracy, in the monographs having the fourth order of approximation are also investigated.

We remind that for this subclass satisfying the maximum principle the following conditions are true: the function f is independent of y' and $f_y = \partial f / \partial y \geq 0$ are fulfiled.

In the most well known manuals referenced above this case is investigated, when the left hand side is approximated by the three-point scheme or the veriational difference method, giving the three point template. As is known these schemes have the second order of exactness on h, if $y \in C^4(0,1)$ or the fourth order with respect to h, if the unknown function $y(x)$ is continuously differentiable up to the sixth order. For the linear BVPs, when $k(x) \equiv 1$, are investigated in [18,28] the three-point schemes of the high degree of exactness for this subclass. In the works of Vashakmadze [19-21,22], when initial BVPs are linear and the weight $k(x)$ is a positive differentiable function on $(0,1)$, the corresponding schemes are constructed by the special class of spline-finctions named as (P) and (Q) formulae, different from the correspoinding systems of the coordinate functions constructed in [4,9,13,15,etc]. (P) and (Q) formulae have also an arbitrary order of exactness depending on the smoothness of $y(x)$ and requiring neither calculating the multiply integrals, nor computing derivatives from the given data of initial BVPs (1)-(2) unlike works [18,28]. For non-linear problems in case $f_y \leq \pi^2 - \varepsilon$ the Belman-Kalaba iterative scheme [1,25] is applied. The suggested schemes for $p = 2$ coincide with the results of Henrichi [10].

In [26], when BVPs (1)-(2) is linear with $k = const$ small positive parameter, using (P) and (Q) formulae, we investigated this problem. At first we considered this problem with a view of the theory of differential equations, according to the work of Viskik and Lusternik [27]. Then we constructed high accuracy multi-point schemes, created the corresponding software and did the numerical experiments. The process of comparision with methods from the monograph of Doolan, Miller and Schilders [7] had been done. The scheme of [26] is also true in a more general case, when $k^{-1}(x)$ non-negative function is integrable with (2) boundary conditions, using data of [21,25].

3. Description of Programm Complexes

There is created the programm package, written in the programming language Turbo Pascal 7.0 for the resolution of the second order ordinary differential equations.

The programm modules are written on the base of new, high accuracy algorithms developed in [19], [21], [23] or [25] and are intended to solve the following problems:

$$y''(x) = f(x, y(x)), \qquad (a)$$

$$y''(x) - q(x)y(x) = f(x), \quad 0 < x < 1, \qquad (b)$$

$$y''(x) = f(x, y(x), y'(x)), \ 0 < x < 1, \qquad (c)$$

$$\varepsilon y''(x) - q(x)y(x) = f(x), \quad 0 < x < 1, \qquad (d)$$

with boundary comditions (2) when $K_\alpha = 0$,

$$y''(x) - q(x)y(x) = f(x), \quad 0 < x < 1, \qquad (e)$$

with boundary conditions (2), where $y(x)$ is unknown function and $q(x), f(x), f(x, y(x)), f(x, y(x), y'(x))$ satisfy conditiones given in [25], ε is a small positive parameter.

Package consists of five units, each of them separately solving (*a*), (*b*), (*c*), (*d*) and (*e*) problems and one main program, called MAIN.PAS, which enables choice of which problem is to be solved.

Each module uses the program, written by T. Zarqua, for reading the function from screen and counting its value in requested value of arguments. Thus, it is possible to enter the prescribed $q(x), f(x), f(x, y(x))$, and $f(x, y(x), y'(x))$ functionsfrom the screen using the keyboard. The unit is caleed GAMOTVLA.TPU.

The first module PROGRAM1.TPU is soving problem (a) usinf (P) formulae from [25]. Procedure PRO1 is executing its numerical resolution.

The second module is PROGRAM2.TPU. It is counting approximate solution of the problem (b) by means of algorithms elaborated in [25]. The main procedure is PRO2.

The third one is solving problem (c), PROGRAM3.TPU, containing procedure PRO3.

The fourth is for the resolution (d) is named PROGRAM4.TPU, procedure is PRO4.

All these units require input data: boundary conditions $y0$ and $y11$ values; s numbers for the calculation of boundary knots, k, n - number of points, tt number of knots for the calculation of approximate value of integrals in formula for b_{ij} and c_{ij} coefficients ([25]); Output is value of approximate solution $y[i],\ i = 1,2,..2ks$.

The fifth program module PROGRAM5.TPU is solving (e) problem, the main procedure is MP_MET MP-Met calls the procedures LIJ, BIJ, FUN_Q_F, AIJ, SOLSYST, GRAPHIC. LIJ computes the Lagrange polynomials. BIJ computes coefficients b_{ij} as $b[i, j]$. FUN_Q_F computes $q(x)$ and $f(x)$. AIJ forms the matrice of coefficients a_{ij} of the multi-point method from [25] which has a tape structure. The matrix $a_{ij},\ i, j = 2...n,$ is written to memory of the computer in the rectangle form as $a[i]\hat{}[j],\ j = 2,..,2s+3, i = 2,..,n$. Beginning with row $s+3$ the elements of the matrix are stored in memory of the computer beginning with the first column, i.e. $a_{i,i-s},\ i = s+3,..,n = s+1$ and $a_{i,n-2s+2},\ i = n-s+2,..,n$ will be stored it the first column. SOLSYS solves the obtained algebraic system by the Gauss exception method. As a consequence values $f0[i], i = 2,..,n$ are obtained. GRAPHIC constructs graphics of the obtained solution $y[i], i = 2..n$ with the boundary conditions $y0$ and $y1$.

The corresponding programm package represents complete, independent product ready for users.

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