On Theory of "Canonically Conjugated" Fuzzy Subsets

Magda Tsintsadze
Mathematical Cybernetics, Faculty of Applied Mathematics and Computer Sciences,
Iv. Javakhishvili Tbilisi State University, Georgia

***Abstract**:*
The work "Theory of "Canonically Conjugated" Fuzzy Subsets" offers the system attributes description with so-called "color" operators, which act in the Hilbert Space of "information functions". With help of "information functions" the "phase membership" function of attributes can be built, which represents the basis of the fuzzy membership function's probabilistic model

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Generally fuzzy subsets are built on the basis of an expert estimation of one of additional components. From this point of view thus constructed fuzzy subsets shortly characterize information unit. It is possible to construct membership function of the information unit on the basis of new concept (canonically conjugated fuzzy subset), where both components will be stipulated simultaneously, so it is possible to take into account the arranged information quite and optimally. Now we shall shortly consider those general reasons which lay in a basis of our model construction. In most cases there is uncountable (it is possible to tell unlimited) amount of assets of interaction of the expert with object.

Mostly this interaction is not full and is connected with small enough number of attributes (we offer the term "color"[5]), identification of which expert can and which correspond to his interests. Other "colors" are unattainable for direct supervision and thus make "uncontrollable" influences on object.

One of our main assumptions is that these uncontrollable influences are quantitatively characterized by random parameters, values which are not measured directly, occur estimations of these values on the basis of subjective decision.

On the basis of direct measurement and estimations is created an abstract image of objects, which we call variables. The expert will establish corresponding values of variables on the basis of interaction with objects, after this we say that the system is determined on object, so the system is an abstract image of the real object and it is characterized by the pair of canonically conjugate fuzzy subsets.

From the above mentioned it's clear that there is possibility to describe object using two approaches: if object is described in the basis of value uncertainty, expert is giving directly the membership function, but if we'll base on plausibility's uncertainty - first is constructed focal distribution, that gives us possibility to construct fuzzy measure, thus the membership function.

We have to mention here that these two possibilities are "complementary" of each other:

We are offering such description of appropriate uncertainty of the object, where some characteristic of first and second type joint uncertainties would be minimal; preciously we are offering to construct such membership function of fuzzy subset, which will provide minimization of above mentioned joint uncertainty [2].

Suppose, $\tilde{A}$ fuzzy subset of Ω Universal set corresponds to A concept and suppose this concept is characterized by numerical parameter ξ. Consider ξ is quantitatively characterizing some property of $\tilde{A}$ - let's call it "color" $\wp$.

Main definition: The numerical characteristic of color $\xi\,\wp_{\tilde{A}}[\omega]$ is a random quantity. Define appropriate distribution density of probabilities by $\rho_{\wp}(x;\omega)$.
Denote

$$x^*_{\omega} = M\xi(\wp_{\tilde{A}}[\omega]) = \int_{\Re} x\rho_{\wp}(x;\omega)dx \tag{1}$$

as Calculated value of $\tilde{A}$ fuzzy subset membership function's modal value.
Except of $M\xi$, presence of color to ω is characterized by dispersion also:

$$\sigma^2_{\wp}(\omega) = \int_{\Re} (x - x^*_{\omega})^2 \rho_{\wp}(x;\omega)dx \tag{2}$$

In our model exactly $\sigma^2_{\wp}(\omega)$ is connected with definition of presence $\wp$ color to ω. If $\sigma^2_{\wp}(\omega) \to 0$, we'll say $\wp$ has quite define value ω. The more $\sigma^2_{\wp}(\omega)$ is, the uncertain $\wp$ in ω. If $\sigma^2_{\wp}(\omega) \to \infty$ it means ω has no $\wp$ color.
suppose:

$$\begin{aligned} &1)\ \wp_1(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) = 0, \forall \omega \in \wp_1(\Omega) \\ &2)\ \wp_{\neq 1}(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) \neq 0, \forall \omega \in \wp_{\neq 1}(\Omega) \\ &3)\ \wp_0(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) = +\infty, \forall \omega \in \wp_0(\Omega) \end{aligned} \tag{3}$$

It means that:

1) for $\forall \omega \in \wp_1(\Omega)$, expression " ω has $\wp$ color" , $\wp[\omega]$ - is true;
2) for $\forall \omega \in \wp_{\neq 1}(\Omega)$, we say $\overline{\wp}[\omega]$ is true if $\wp[\omega]$ is false.
3) if for $\forall \omega \in \wp_0(\Omega)$, expression $\wp[\omega]$ is false, we say that in this case $\neg \wp[\omega]$ - is true.

The following is valid:

$$\neg \wp[\omega] \Rightarrow \overline{\wp}[\omega] \tag{4}$$

But the reverse implication is valid only on the following subset :
$A(\Omega) \equiv \wp_1(\Omega) \cup \wp_0(\Omega)$.
if $\wp_0(\Omega)$ is proper subset of $\wp_{\neq 1}(\Omega)$, than in Ω exist ω , that :

$$0 < \sigma^2_{\wp}(\omega) < +\infty$$

Now we may indicate easy way to separate $\wp_0(\Omega)$ subset from $\wp_{\neq 1}(\Omega)$. Assume Ω universal subset's every element is characterized $\wp$ color specified quantity. Formally it means that expert may give the reflection directly:

$$\mu_{\wp} : \Omega \to [0,1] \tag{5}$$

Which has the property:

$$\wp[\omega] \Leftrightarrow (\mu_{\wp}(\omega) > 0) \tag{6}$$

$\mu_{\wp}(\omega)$ is considered as measure (membership function) of $\wp$ color presence to ω. If $\mu_{\wp}(\omega)=1$, it's said: ω has $\wp$ color, but if $\mu_{\wp}(\omega)=0$, then ω has no $\wp$ color. $\wp_0(\Omega)$ is the set of such ω, which "are not colored in $\wp$ color": $\wp_0(\Omega)=\{\omega:\mu_{\wp}(\omega)=0,\omega\in\Omega\}$.

The elements of $\Omega\notin\wp_1(\Omega)\cup\wp_0(\Omega)$, which have $\wp$ color in some amount, are characterized by numbers from (0,1).

Proposition 1. There are defined just $\wp$ and $\neg\wp$ in Ω, so we have just one $\wp_0(\Omega)$ with $\wp_1(\Omega)$, as for other elements from Ω, which are not belonging this two subset , we are saying they have color "passing through" $\wp$ and $\neg\wp$:

$$\mu_{\neg\wp}(\omega)=1-\mu_{\wp}(\omega),\forall\omega\in\Omega \tag{7}$$

$\wp$ and $\neg\wp$ colors are not really complementary of each other ($\neg\wp\neq\overline{\wp}$), they are such just on appropriate $\mathrm{A}(\Omega)$.

when $\Omega\setminus\mathrm{A}(\Omega)\neq\varnothing$, $\wp[\omega]\vee\neg\wp[\omega]=\mathrm{T}$ is not correct ($\vee$-sign of dizunction, T-"always true proposition") and we should change it with (7).

Let's consider Ω universal set, some $\wp$ color and it's compatibility with points of Ω . It is clear that we can use normal indicator $\mathrm{I}_{\wp_1(\Omega)}(\omega)$ with values in $\{0,1\}$ as characteristic of $\wp$ color compatibility with points $\wp_1(\Omega)\cup\wp_0(\Omega)$, as for points $\wp_{\neq 1}(\Omega)\setminus\wp_0(\Omega)$, for them we'll use generalized indicator $\mu_{\wp}(\omega)$ with values from (0,1).

Proposition 2 . $\mu_{\wp}(\omega)$ is equal of fuzzy subset of $\tilde{\Omega}$, where $\sup\mathrm{p}\tilde{\Omega}=(\wp_{\neq 1}(\Omega)\setminus\wp_0(\Omega))\cup\wp_1(\Omega)$

Note 1. $\mu_{\wp}(\omega)$ and $(\wp_{\neq 1}(\Omega)\setminus\wp_0(\Omega))\cup\wp_1(\Omega)$ are defining $\tilde{\mathrm{A}}$ fuzzy subset, where $\sup\mathrm{p}\tilde{\mathrm{A}}=(\wp_{\neq 1}(\Omega)\setminus\wp_0(\Omega))\cup\wp_1(\Omega)$.

Definition 3. $\forall\omega\in\Omega$ introduce some $\text{Ш}_{\wp}\subseteq\Re$ interval on scale of $\wp$ color values (i.e. on $\Re$) by

$$\mu_{\wp}(\omega)=1-\int\limits_{\text{Ш}_{\wp}(\omega)}\rho(x;\omega)dx=1-\int\limits_{\Re}I_{\text{Ш}_{\wp}(\omega)}(x)\rho(x;\omega)dx \tag{8}$$

Main Definition: (1), (2), (7), (8) equations define the following set:

$$\tilde{\Omega}=\{\tilde{\omega}\equiv(\omega;\mu_{\wp}(\omega)):\omega\in\Omega\} \tag{9}$$

Call Ω universal set the probabilistic model of $\tilde{\Omega}$ fuzzy subset.

Theory of Information Functions Presence

suppose, $\chi_{\tilde{\mathrm{A}}}(\omega),\omega\in\Omega$ denotes $\tilde{\mathrm{A}}$ fuzzy subset's appropriate membership function.

note 4. Lets call expression

$$\sqrt{\rho_{\tilde{A}}(x;\omega)} \equiv \left\langle x, x^{*}_{\omega} \middle| \tilde{A} \right\rangle \tag{10}$$

Information function.

Here we are using Dicar's nomenclature [4]. We need this function to represent the information in $\tilde{A}$ concept. Information function's magnitude square determines membership function (precisely the appropriate density):

$$\rho_{\tilde{A}}(x;\omega) = \left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle^{+} \left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle \tag{11}$$

Any $\tilde{A}$ fuzzy subset might be defined separate from some value's hidden parameters. Call such values (by Dicar nomenclature) ket-vector and denote as $\left| \tilde{A} \right\rangle$.

We may sum ket-vectors, also product ket-vectors as on scalar also on complex values –and receive ket-vectors again.

suppose, $\left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle \in L^{2}(\Re)$ (Hilbert space) , consider Fourier transformation of this function:

$$F\left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle = \frac{1}{2\pi c} \int_{\Re} \left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_c} dx \tag{12}$$

where c is const. (12) expression is equal of information function at x_c presentation:

$$\hat{F}\left\langle x, x^{*}_{\omega} \middle| \tilde{A} \right\rangle = \left\langle x_c, x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle \tag{13}$$

where $\tilde{A}^{c}$ is canonically conjugate fuzzy subset:

$$\chi_{\wp c}(\omega) = \int_{I_{\wp c}(\omega)} \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle^{+} \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle dx_c =$$

$$= \int_{\Re} I_{I_{\wp c}(\omega)}(x_x) \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle^{+} \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle dx_c \tag{14}$$

At information function space $\left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle$, $\hat{\wp}$ operator is appropriate of $\wp$ color. If information about color is precise, than

$$\hat{\wp}\left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle = x\left\langle x; x^{*}_{\omega} \middle| \tilde{A} \right\rangle \tag{15}$$

analogically

$$\hat{\wp}_c \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle = x_c \left\langle x_c; x^{*}_{c\omega} \middle| \tilde{A}^{c} \right\rangle \tag{16}$$

Theorem 1. suppose $\left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle$ and $\frac{d}{dx}\left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \in L^2(\Re)$, $\left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle = \hat{F}\left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle$, than the following expression is valid for $\hat{\wp}$ and $\hat{\wp}^c$ operators:

$$\hat{\wp}_c \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle = -ic \frac{d}{dx} \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \tag{17}$$

And analogically

$$\hat{\wp} \left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle = ic \frac{d}{dx_c} \left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle \tag{18}$$

Note that by (15),(16) and (11),(1):

$$x_{\omega}^{*} = \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \hat{\wp}_c \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle dx \tag{19}$$

$$x_{c\omega}^{*} = \int_{\Re} \left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle^{+} \hat{\wp} \left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle dx_c \tag{20}$$

Consider (20) , lets show that the equality is true:

$$x_{c\omega}^{*} = \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left(-ic \frac{d}{dx} \right) \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle dx_c = \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \hat{\wp}_c \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle dx_c \tag{21}$$

We have:

$$x_{c\omega}^{*} = \int_{\Re} dx_c \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{\frac{i}{c} x_c x} \right] \hat{\wp}_c \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{-\frac{i}{c} x x_c} \right] =$$

$$= \int_{\Re} dx_c \left\langle x_c; x_{c\omega}^{*} \middle| \widetilde{A}^c \right\rangle x_c \frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{-\frac{i}{c} x x_c} =$$

$$= \frac{1}{2\pi c} \int_{\Re}\int_{\Re}\int_{\Re} dx_c dx dx' \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{\frac{i}{c} x_c x} \left(ic \frac{d}{dx'} e^{-\frac{i}{c} x_c x'} \right) =$$

$$= \frac{1}{2\pi c} \int_{\Re}\int_{\Re}\int_{\Re} dx_c dx dx \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle ic \frac{d}{dx'} e^{\frac{i}{c} x_c (x - x')} =$$

$$= \frac{i}{2\pi} \int_{\Re} dx_c \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left[\left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{\frac{i}{c} x_c (x - x')} \Big|_{+\infty}^{-\infty} - \int_{\Re} dx' \frac{d}{dx'} \left\langle x'; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle e^{\frac{i}{c} x_c (x - x')} \right] =$$

$$= -\frac{i}{2\pi} \int_{\Re} dx \int_{\Re} dx' \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \int_{\Re} dx_c e^{\frac{i}{c} x_c (x - x')} =$$

$$= \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left(-ic \frac{d}{dx} \right) \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle$$

note 2. The following are valid:

$$x^*_{\omega} = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\wp}\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right) = \left(\left\langle x_c, x^*_{c\omega} \middle| \tilde{A}^c \right\rangle, \hat{\wp}\left\langle x_c, x^*_{c\omega} \middle| \tilde{A}^c \right\rangle\right) \quad (22)$$

$$x^*_{c\omega} = \left(\left\langle x_c, x^*_{c\omega} \middle| \tilde{A}^c \right\rangle, \hat{\wp}_c\left\langle x_c, x^*_{c\omega} \middle| \tilde{A}^c \right\rangle\right) = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\wp}_c\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right) \quad (23)$$

The proofs of these equalities can be done by the same way , so we aren’t consider them here.
Theorem 2. Operators $\hat{\wp}$ and $\hat{\wp}_c$ are satisfying the following condition:

$$\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c\hat{\wp} = ic\hat{E} \quad (24)$$

where $\hat{E}$ - identity operator.

Proof. : Suppose $\hat{\wp} f(x) = xf(x),\ f(x),\ xf(x)$ and $f^{'}(x) \in L^2(\Re)$, then

$$(\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c\hat{\wp}) f(x) = \hat{\wp}(\hat{\wp}_c f(x)) - \hat{\wp}_c(\hat{\wp} f(x)) = -icx\frac{df(x)}{dx} + ic\frac{d}{dx}(xf(x)) = ic\hat{E}f(x)$$

Here we have used the following facts:

$\hat{\wp}(\hat{\wp}_c f) = x(\hat{\wp}_c f)$ and $\hat{\wp}_c(\hat{\wp} f) = -ic\frac{d}{dx}(\hat{\wp} f)$, $x \in \Re$

because of: $(f, \hat{\wp}\,\hat{\wp}_c f) = (\hat{\wp} f, \hat{\wp}_c f) = (xf, \hat{\wp}_c f) = (f, x\hat{\wp}_c f)$

Connection between canonically conjugated colors

Connection between canonically conjugated colors is given with the following theorem:
Theorem 3. if $\wp$ and $\wp_c$ are canonically conjugated colors, then

$$\sigma^2_{\wp}(x^*_{\omega})\sigma^2_{\wp_c}(x^*_{c\omega}) \geq \frac{c^2}{4} \quad (25)$$

Proof. Let's enter designations

$$\hat{\alpha} \equiv \hat{\wp} - x^*_{\omega}\hat{E}, \qquad \hat{\beta} \equiv \hat{\wp}_c - x^*_{c\omega}\hat{E} \quad (26)$$

appropriately

$$\sigma^2_{\wp}(x^*_{\omega}) = \left\langle \hat{\alpha}^2 \right\rangle = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\alpha}^2\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right) \quad (27)$$

$$\sigma^2_{\wp_c}\left(x^*_{c\omega}\right) = \left\langle \hat{\beta}^2 \right\rangle = \left(\left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle, \hat{\beta}^2 \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \right)$$

We have

$$\sigma^2_{\wp}\left(x^*_\omega\right)\sigma^2_{\wp_c}\left(x^*_{c\omega}\right) = \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \hat{\alpha}^2 \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \hat{\beta}^2 \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx =$$

$$= \int_{\Re} \hat{\alpha}^+ \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \hat{\alpha} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \int_{\Re} \hat{\beta}^+ \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \hat{\beta} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \quad (28)$$

Using Cauchy-Buniakovski inequality:

$$\int \left|f(x)\right|^2 dx \int \left|g(x)\right|^2 dx \geq \left| \int f(x) g(x) dx \right|^2 \quad (29)$$

And suppose that:

$$\hat{\alpha} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \equiv f(x) \quad \text{and} \quad \hat{\beta} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \equiv g(x) ,$$

We'll have:

$$\sigma^2_{\wp}\left(x^*_\omega\right)\sigma^2_{\wp_c}\left(x^*_{c\omega}\right) \geq \left| \int_{\Re} \hat{\alpha}^+ \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \hat{\beta} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \right|^2 = \left| \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \hat{\alpha}\hat{\beta} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \right|^2 =$$

$$= \left| \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \left[\frac{1}{2}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right) + \frac{1}{2}\left(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}\right) \right] \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \right|^2 =$$

$$= \frac{1}{4} \left| \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right) \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \right|^2 + \frac{1}{4} \left| \int_{\Re} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle^+ \left(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}\right) \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle dx \right|^2 \quad (30)$$

Missed member is equal to 0, because $\hat{\alpha}^+ = \hat{\alpha}$ and $\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} = ic\hat{E}$.
So using (26):

$$\left(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}\right) \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle = -ic \left[x \frac{d}{dx} \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle - \frac{d}{dx} \left(\left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \right) \right] = ic \left\langle x, x^*_\omega \middle| \tilde{A} \right\rangle \quad (31)$$

so, if at right side of equality (30) we'll ignore second summary (which ≥ 0) finally receive (25) .

Canonically conjugated colors joint distribution

Suppose $\hat{\wp}$ and $\hat{\wp}_c$ are canonically conjugated operators. Denote x and x_c their possible values. It is clear to count $e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)}$ to define phase distribution (joint distribution), the average value of which will be denoted as membership function

$$M(\tau,\theta)=\left\langle \tilde{A}\right|e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)}\left|\tilde{A}\right\rangle \tag{1}$$

Using well known opposite formula of Fourier transaction for discrete proper values we have:

$$F(x_i,x_{ck})=\lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T}\left\langle \tilde{A}\right|e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)}\left|\tilde{A}\right\rangle e^{-i(\tau x_i+\theta x_{ck})}d\theta d\tau \tag{2}$$

And for continuous proper values

$$F(x_i,x_c)=\frac{1}{4T^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\langle \tilde{A}\right|e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)}\left|\tilde{A}\right\rangle e^{-i(\tau x_i+\theta x_{ck})}d\theta d\tau \tag{3}$$

When $\hat{\wp}$ and $\hat{\wp}_c$ are canonically conjugated: $(\hat{\wp}\hat{\wp}_c-\hat{\wp}_c\hat{\wp})=-ic\hat{E}$, so the view of characteristic function would be very simple:

$$\hat{M}(\theta,\tau)=e^{-\frac{1}{2}i\tau\hat{\wp}_c}e^{i\theta\hat{\wp}}e^{+\frac{1}{2}i\tau\hat{\wp}_c} \tag{4}$$

If we consider expert functions as $\hat{\wp}$ operator proper value, then

$$M(\theta,\tau)=\left\langle \hat{M}(\theta,\tau)\right\rangle=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle \tilde{A}\middle|x^*-\frac{1}{2}c\tau\right\rangle e^{i\theta x}\left\langle x^*+\frac{1}{2}c\tau\middle|\tilde{A}\right\rangle d\tau \tag{5}$$

So,

$$F(x,x_c)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle \tilde{A}\middle|x^*-\frac{1}{2}c\tau\right\rangle e^{-ix_c\tau}\left\langle x^*+\frac{1}{2}c\tau\middle|\tilde{A}\right\rangle d\tau=\frac{1}{\sqrt{c}}e^{-\frac{1}{2}ic\frac{\partial^2}{\partial x\partial x_c}}\left\langle \tilde{A}\middle|x^*\right\rangle\left\langle \tilde{A}^c\middle|x_c^*\right\rangle e^{icxx_c} \tag{6}$$

Let's consider average value of complementary variable common function $G(x,x_c)$ to the phase distribution Function $F(x,x_c)$

$$\langle G\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(x,x_c)F(x,x_c)dxdx_c= \tag{7}$$

Hereinafter we'll consider $\hat{G}$ as "energetic function"

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, x_c) M(\tau,\theta) e^{-i(\tau x+\theta x_c)} dx dx_c d\tau d\theta = \left\langle \tilde{A} \left| \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \gamma(\tau,\theta) e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)} d\tau d\theta \right) \right| \tilde{A} \right\rangle \quad (8)$$

where

$$\gamma(\tau,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, x_c) e^{-i(\tau x+\theta x_c)} dx dx_c \quad (9)$$

Thus, the appropriate operator of canonically conjugated variables common function might be presented by the following way:

$$\hat{G} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \gamma(\tau,\theta) e^{i(\tau\hat{\wp}+\theta\hat{\wp}_c)} d\tau d\theta = e^{-\frac{1}{2}ic\frac{\partial^2}{\partial x \partial x_c}} \hat{G}_0(\hat{\wp},\hat{\wp}_c) \quad (10)$$

where $\hat{G}_0(\hat{\wp},\hat{\wp}_c)$ is received by $G(p,q)$ function in case of changing x and x_c with beforehand right order of appropriate operators $\hat{\wp}$ and $\hat{\wp}_c$. We should save this ordering in case of $e^{-\frac{1}{2}ic\frac{\partial^2}{\partial x \partial x_c}}$ operator's operating on G_0. We'll use "Energetic functions" of color in time derivation consideration.

If connection of information functions on time is presented by exponential phase items ($e^{i\alpha(x,x_c)t}$), what means that information about $\tilde{A}$ fuzzy subset is not depending on time, then

$$\frac{d\hat{\wp}}{dt} = -\frac{i}{c}\left(\hat{\wp}\hat{G} - \hat{G}\hat{\wp}\right) \quad (11)$$

We will consider case, when

$$\hat{G} = \alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2} \quad (12)$$

Now is easy to calculate following:

$$\left(\frac{d\xi}{dt}\right)^* = -ic\left(\int_{-\infty}^{+\infty} \left\langle \tilde{A} \middle| x^* \right\rangle \hat{\wp}\hat{G} \left\langle x^* \middle| \tilde{A} \right\rangle dx - \int_{-\infty}^{+\infty} \left\langle \tilde{A} \middle| x^* \right\rangle \hat{G}\hat{\wp} \left\langle x^* \middle| \tilde{A} \right\rangle dx\right)$$

using (11) and (12) we receive:

$$\frac{d\hat{\wp}}{dt} = -\frac{i}{c}\left(\hat{\wp}\left(\alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2}\right) - \left(\alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2}\right)\hat{\wp}\right) = -\frac{i\beta}{c}\left(\hat{\wp}\,\hat{\wp}_c^{\,2} - \hat{\wp}_c^2\hat{\wp}\right) = 2\beta\hat{\wp}_c \quad (13)$$

Thus,

$$\sigma\left(\frac{\mathrm{d}\hat{\wp}}{\mathrm{dt}}\right) = 2\beta\sigma(\hat{\wp}_c) \tag{14}$$

and

$$\mu_{\hat{p}\times\hat{p}_c}\left(x^*, x_c{}^*\right) = 1 - \iint_{I\left(x^*\right)\times I_c\left(x_c{}^*\right)} F\left(x, x_c\right) dx dx_c$$

It’s known that except of stochastic uncertainty exists uncertainties of other types, there is an opportunity to expand an area of use for differential equations even in those cases when parameters determining differential equations contain uncertainty of new types and therefore there is no opportunity to pose the classical problem - We are offering modeling of these situations with help of new concept - canonically conjugated fuzzy subset [3].

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