Modeling of Autocorrelation Functions Using Weighted Non-linear least squares

Altyeb Altaher ,*Sami M Sharif and Iman Abumaaly
Department of Computer Science,International University of Africa
altypaltaher@yahoo.com
* Department of Electrical and Electronic Engineering,University of Khartoum

Abstract
Because autocorrelation functions play an important role in stochastic processes and can be used to model the traffic data practically, it is significant to study how to find a function that best fits the autocorrelation sequence of a real-traffic trace. This paper presents a asymptotic model for autocorrelation functions using Weighted non-linear least squares.

Keywords: traffic modeling, long-range dependence, autocorrelation functions

1.INTRODUCTION

Recent researches have shown that the behaviors of the traffic on LAN and WAN are well modeled by second-order self-similar processes with long-range dependence (LRD) [1-2].
Second-order self-similar processes are classified into two classes [l-2]. One is exactly second-order self-similar model and the other asymptotically second-order self-similar model. [1] pointed out that exactly second-order self-similar model is not enough to model real trafic. Hence, asymptotically second- order self-similar processes are considered in the paper. Throughout the paper, the term LRD processes means second-order self-similar processes unless otherwise stated. LRD processes are defined by autocorrelation functions (ACFs). As ACFs can be used to study queuing systems [3], it is significant to study how to find a function that best fits the autocorrelation sequence of a real-traffic trace (target ACF). Because the ACFs of LRD processes are characterized by a single parameter H [2], the estimation of H is paid attention to [l-2], [5-7].
However, for a specific traffic trace, the various estimation methods might yield different values of H's substantially for the same trace as remarked in [7]. This makes it difficult to model traffic with ACFs. Thus, an optimal representation of ACFs of LRD processes is well worth discussing.
Mathematically,optimally modeling ACFs of real-traffic traces can be abstracted as follows. For a given r indicating a target ACF of a real traffic, find a function of a LRD process which best fits it in the sense of F(e) = min, where e = ($\hat{r}$ -r).

2. Definitions of ACFs

Definition 1 [4]: Let *X= (Xt:* ***t*** = 1, 2, ...)
be a stochastic process. If there exist the mean *E(X,)*
and the variance $Var(x_t)\forall t \in l_1(=1,2,...)$, *X* is
called a second-order process, where *E* is the expectation operator, ***Var*** is the variance operator.

Definition 2 [8]: **A** process *X* is called exactly
second-order self-similar with parameter *H* $\in$ (0.5, l), if its autocorrelation function is

$$r(k)=\frac{1}{2}[(k+1)^{2H}-2k^{2H}+(k-1)^{2H}]\},k\in l \quad (1\text{-}2)$$

The following properties PI and P2 have to be satisfied by all types of autocorrelation functions [4].
P1: ***r(k)*** is an even function.
P2: $r(0)\geq|r(k)|$.

Definition 3 [5],[8]: **A** process ***X*** is called asymptotically second-order self-similar with parameter ***H***$\in$ *(0.5,* **l),** if its autocorrelation function
is with the form

$$r(k)\sim ck^{2H-2}(k\longrightarrow\infty) \quad (2\text{-}2)$$

where *c* > **0** is a constant.

According to Definition *3,* the constraints for selfsimilar processes are summarized as **C 1** and **C2.**

$C1:\sum_k r(k)=\infty$

C2: *r(k)* decays hyperbolically.

It can be easily verified that ***r(k)*** of **(2-2)** satisfies P1,
P2, C1 and **C2.** Define the set ***S*** as
S1=
$\{r; r(k)=0.5[(k+1)^{2H}-2k^{2H}+(k-1)^{2H}]\}$
(3-2)
Then, autocorrelation functions of exactly second order self-similar processes are elements of S. The advantage of exactly self-similar model is that the whole correlation structure is specified and it is easy to use. Unfortunately, the exactly self-similar model is too narrow to model real traffic [7]. Attention has been paid to the asymptotically self-similar model.
Define

$S=\{r; r(k)\sim ck^{2H-2}(k\longrightarrow\infty)\}$

Then, $S1\subseteq S$ and an element of ***S*** is an autocorrelation function of self-similar process.

3.The derivation of Simple form of autocorrelation functions

This section focuses **on** the derivation of a form of autocorrelation functions with fixed finite lag specifying for asymptotically self-similar processes.
Statement **1:**
The autocorrelation function
$(|k|)^{2H-2}$
is asymptotically equivalent to k^{2H-2} under the limit
$k\longrightarrow\infty$:
$k^{2H-2}\sim(|k|)^{2H-2}(k\longrightarrow\infty), k\in l.$ (1-3)

Statement **2:** **A** process X is called asymptotically
second-order self-similar with parameter ***H***∈ (0.5, I), if its autocorrelation function is
$r(k)\sim c(|k|)^{2H-2}(k\longrightarrow\infty),$ (2-3)
where c > 0 is a constant.
we construct an interesting form of autocorrelation functions with fixed finite lag.
Statement 3: The autocorrelation function
$(|k|^{\alpha})^{2H-2}, \alpha\in(0,1], H\in(0.5,1)$ (3-3)
is nonsummable at infinity and decays
Proof
As $0<\alpha\leq 1, (|k|)^{2H-2}\leq(|k|^{\alpha})^{2H-2}$.
Then
$\sum_k(|k|^{\alpha})^{2H-2}=\infty.$
on the other hand as
$\left(|k|^{\alpha}\right)^{2H-2}, (k\longrightarrow\infty),$
$\left(|k|^{\alpha}\right)^{2H-2}$ decays hyperbolically.
According to Statement 3
The ***r(k)*** in (3-3) satisfies PI, P2, C1 and C2.
Define
the set ***A*** as
$A=\{r; r(k)=(|k|^{\alpha})^{2H-2}\}$ (2-4)
clearly $A\subseteq S$.

4.Optimization Method

Our aim is to solve the issue as: for a given r(k) to find
$\hat{r}(k)=(|k|^{\alpha})^{2H-2}, \alpha\in(0,1], H\in(0.5,1), k\in l$
which best fits it in the sense of F(e) = min, where e = ($\hat{r}$ -r) using Weighted non-linear least squares , The purpose of weighted fitting is to obtain best estimates of the parameters by forcing the line close to the data that are known to high precision, while giving much less weight to those data that exhibit large scatter.
In order to get $\hat{r}$ using weighted non linear least squares, we developed the following algorithm :
1- calculate a sum of squares of residuals that takes into account the standard deviation in the *k* values as following.

$$x^2=\sum\left(\frac{r_i-\hat{r}_i}{\sigma_i}\right)^2 \qquad (1\text{-}4)$$

where x^2 is the weighted sum of squares of residuals. σ_i is the standard deviation in the *i*th

r-value. The purpose of the weighted fitting is to find best estimates of parameters (H, α) that minimize x^2 in equation (1-4).

2- Construct the matrix of partial derivatives D,

construct the matrix of partial derivatives by calculate

$$\left(\frac{\partial r_i}{\partial H}\right)_{\alpha,k_i} \approx \frac{r[H(1+\delta),\alpha,k_i]-r[H,\alpha,k_i]}{H(1+\delta)-H} \quad (2\text{-}4)$$

and

$$\left(\frac{\partial r_i}{\partial \alpha}\right)_{H,k_i} \approx \frac{r[\alpha,H(1+\delta,k_i]-r[H,\alpha,k_i]}{\alpha(1+\delta)-\alpha} \quad (3\text{-}4)$$

δ is chosen to be 10^{-6}.

3- **Construct The weight matrix, W**

The weight matrix is a square matrix with diagonal elements proportional to $\frac{1}{\sigma_i^2}$

And other elements equal to zero. Take σ_i to be equal to ri .

4- Calculate the weighted standard deviation σ_w

the standard weighted deviation given by equation $\sigma_w = \left(\frac{x^2}{n-p}\right)^{\frac{1}{2}}$ (4-4)

the square root of x^2 is given by equation (1-4), n is the number of data points and p is the number of parameters in the equation to be fitted to the data.

5- **Calculate** $(D^T WD)^{-1}$

To obtain standard errors in estimates H and α , we must determine $(D^T WD)^{-1}$.there are everal steps required to determine $(D^T WD)^{-1}$. The steps consist of:

a) Calculation of the matrix **WD**.

b) Calculation of the matrix $D^T WD$.

c) Inversion of the matrix, $D^T WD$.

6 - Calculate the standard errors of the parameters (H , α) estimate.

standard errors in H, α given by

$$\sigma(B) = \sigma_w \left[\left(D^T WD\right)^{-1}\right]^{\frac{1}{2}} \quad \textbf{(5-4)}$$

B is the matrix containing elements equal to the best estimates of the parameters. σw is the weighted standard deviation, given by equation (4-4).Thus the standard error of the parameter H given by

$$\sigma_H = \sigma_w \left[\left(D^T WD\right)^{-1}\right]^{\frac{1}{2}}$$

and the standard error of the parameter H given by

$$\sigma_\alpha = \sigma_w \left[\left(D^T WD\right)^{-1}\right]^{\frac{1}{2}}$$

7- Calculate the confidence interval

Calculate the confidence interval for each parameter (H, α) at a specified level of confidence (usually 95%).

The 95% confidence intervals for H and α are,

H $\longleftarrow$ H ± *t95%,* ν σ_H

$\alpha \longleftarrow \alpha$ ± *t95%,* ν σ_α

t95%, ν is t value corresponding to the 95 % level of confidence and ν is the number of degrees of freedom.

5-Model Observation

A data file called pAug.TL [9] is a sequence of real-traffic data. Fig. **1** (a) illustrates its time sequence. Fig. **1** (b) illustrates the autocorrelation function of pAug.TL using equation (2-2).TL. Fig. **1** (c) illustrates the estimation of the autocorrelation function of pAug.TL.

To evaluate the result of modeling, the Sum of Squares of the Residuals.

$$SSR= E[(r - \hat{r})] \quad (5\text{-}1)$$

is used as a criterion, where $\hat{r}$ is the estimation of r.

The values of a and b which minimise SSR in equation (5-1).are founded and used in the following closed form of autocorrelation function for pAug.TL

$$\hat{r}(k, 0.75, 0.2) = \left(|k|^{0.2}\right)^{0.75} \quad (5\text{-}2)$$

where $\hat{r}$ (k , H , α) stands for the parameter estimation of the autocorrelation function of pAug.TL. Fig. **1** (c) shows the result of Equation **(5-2).**

To evaluate the benefit of our model of (3-7), the approximation with exactly self-similar model (2- **1)** is given in Fig. 2.2), the benefit of our model,

$$r(k) = (|k|^{\alpha})^{2H-2}, \alpha \in (0,1], H \in (0.5,1),$$

is obvious.

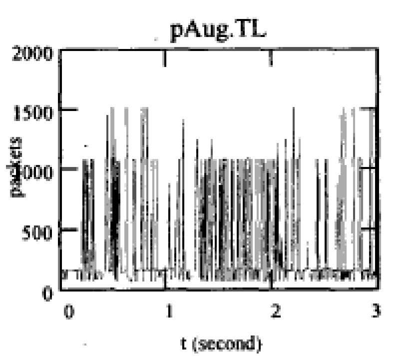
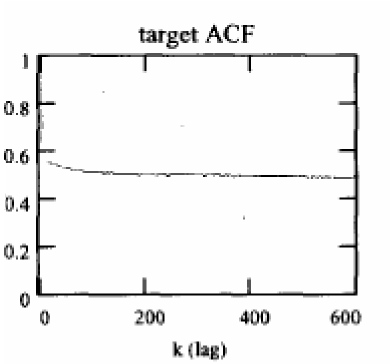


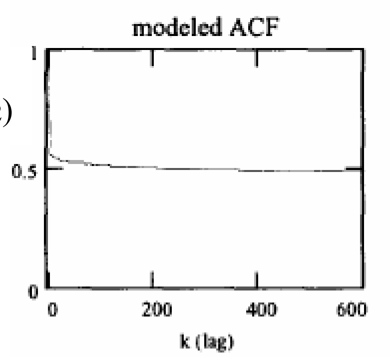
Fig.1 (a) An illustration of pAug.TL

r(k)



(b) Target autocorrelation function sequence of pAug.TL

$\hat{r}$ (k, 0.75, 0.2)



(c) Modeled autocorrelation function

r(k)

$\hat{r}$ (k, 0.75, 0.2)

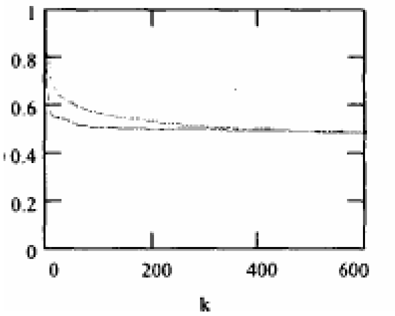


Fig. 2 Matching $\hat{r}$ ***(k,0.75,0.2)*** with exactly self-similar Model.

6. Conclusions

Autocorrelation functions play an important role in stochastic processes and can be used to model the traffic data practically.This paper presents a asymptotic model for autocorrelation functions using Weighted non-linear least squares.

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