

Hoph Bifurcation in Navier-Stokes Equations.

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Abstract :

We present a survey by a finite elements bidimensional code, of the bifurcation (of Hoph) of the Couette-Taylor flow in spiral waves. This transition appears in the survey of the Couette-Taylor problem whose equations are governed by those of incompressible Navier-Stokes. Remarks on the effect of the temporal discretization and the numerical integration are also presented.

keywords: Bifurcation, Couette-Taylor, Finite elements methods, Navier-Stokes.

1 Mathematical modelling of the problem of couette-Taylor

One designates by R_1 and R_2 ($R_1 < R_2$) radiuses of two coaxial cylinders in uniform rotation and by ω_1 and ω_2 their respective angular speed. In all the following, we are going to cut down to the case of the couette bifurcation in Spiral waves. What is the case if $\omega_1\omega_2 < 0$ with $|\omega_2|$ big enough [1, 1986].

1.1 Equations of the out-flow

The domain Ω is defined in cylindrical coordinates by :

$$\Omega = \{(r, \theta, z) \text{ with } R_1 < r < R_2, \theta \in S^1 = R/(2\pi Z) \text{ and } z \in R\} \quad (1)$$

Equations that govern the out-flow are those of Navier-Stokes

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p = \frac{1}{Re} \Delta v \quad (2)$$

with the condition of incompressibility

$$\nabla \cdot v = 0 \quad (3)$$

Conditions to limits of adhesion to the partition take the shape

$$\begin{cases} (u, v, w) = (0, 1, 0) & \text{in } r = 1 \\ (u, v, w) = (0, a\omega, 0) & \text{in } r = a \end{cases} \quad (4)$$

with $a = \frac{R_2}{R_1}$ and $\omega = \frac{\omega_2}{\omega_1}$. The parameters a and ω as the geometric parameters, the Reynolds number $Re = (R_1^2\omega_1/\nu)$ as the dynamical parameter.

NB : We reduced the number of parameters by a choice of scale, while defining the adimensional quantities

$$x = R_1\bar{x}, \quad t = \frac{\bar{t}}{\omega_1}, \quad p = \rho R_1^2\omega_1\bar{p} \text{ et } v = R_1\omega_1\bar{v}$$

and we suppressed bars on x, t, p and v .

2 Axial periodicity.

The hypothesis of an infinite length device not being realist, one considered a device of h height and one imposes conditions to the periodic limits in the axial direction in $z = \pm(h/2)$. In fact, this hypothesis introduces an "axial wave number"

$$\alpha = k \frac{2\pi}{h}$$

where, k is an integer, and we will look for solutions that are invariant by translations ($z \mapsto z + \frac{2\pi}{\alpha}$) along of the axis. One defines, then the "domain" of periodicity by

$$\Omega_p = \{(r, \theta, z) \text{ with } 1 < r < a, \theta \in S^1 \text{ and } -\frac{\pi}{\alpha} \leq z \leq \frac{\pi}{\alpha}\}$$

A solution of adimensional equations (3) – (4) is the one of couette

$$V_c = (0, v_c, 0) \text{ with } v_c = v_c(r) = Ar(1 + \frac{B}{r^2}) \text{ for } r \in [1, \alpha]$$

where $A = \frac{a^2\omega-1}{a^2-1}$ and $B = \frac{a^2(1-\omega)}{a^2\omega-1}$. The pressure is defined by

$$P_c = P_c(r) = C + A^2(\frac{r^2}{2} + 2B \log(r) - \frac{B^2}{2r^2})$$

2.1 Symmetries and equivariances of the problem.

One notes $GL(2)$ the linear group of order 2, and by $O(2)$ the orthogonal group of order 2

$$O(2) = \{R \in GL(2) \text{ as } R^t R = {}^t R R = I_2\}$$

$SO(2)$ designates the plane rotation group

$$SO(2) = \{R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ as } \theta \in S^1\}$$

According to the independence of mechanics laws in relation to the reference system. Our system is invariant by

- translations the long of the oz axis, i.e. by $\tau : z \mapsto z + s, s \in R$
- reflexion symmetry in relation to the xy plane i.e. by $S : z \mapsto -z$
- and rotations around oz i.e. $R_\phi : \theta \mapsto \theta + \phi, \phi \in S^1$.

Under the hypothesis of the axial periodicity, one shows that the group of symmetries of the problem (2) – (4) is isomorphous to the group $\Gamma = O(2) \times SO(2)$. However, the different observed states are more invariant by the action of the initial symmetry group $\Gamma = O(2) \times SO(2)$. So, with the following definition A vectorial space E on which operated a group G , the subgroup of isotropie of a vector $x \in E$ is

$$\{\gamma \in G \text{ tel que } \gamma.x = x\}$$

The subgroup of isotropie Σ_c of the out-flow of Couette is $\Gamma = O(2) \times SO(2)$, and the one of the out-flow in spirals is $\Sigma_S = SO(2)$.

2.2 Reduction of the couette bifurcation in spiral waves to a bidimensional problem.

One defines the number of wave azimuthal m as being the smallest integer such as $R_{\frac{2\pi}{m}}V = V$. Since the spiral waves possess the symmetry $SO(2)$, it is therefore at a time a rotary wave around oz and a wave of translation the long of oz . If one notes by ω the frequency of this out-flow (for a rotary wave of T , $\omega = \frac{2\pi}{T}$), one has

$$(V, q) = (V(r, m\theta + \alpha z + \omega t), (q(r, m\theta + \alpha z + \omega t)))$$

While putting $\phi = m\theta + \alpha z + \omega t$, one gets

$$m \frac{\partial}{\partial \phi} = \frac{\partial}{\partial \theta}, \quad \alpha \frac{\partial}{\partial \phi} = \frac{\partial}{\partial z} \text{ and } \omega \frac{\partial}{\partial \phi} = \frac{\partial}{\partial t}$$

so

$$\frac{\partial}{\partial \theta} = \frac{m}{\alpha} \frac{\partial}{\partial z} \tag{5}$$

Thus, the spiral wave problem amounts to a bidimensional problem i.e. that equations of Navier-Stokes, to calculate this type of out-flow are gotten in

$$\Omega_p = \{(r, z) \text{ with } r \in [1, a] \text{ and } z \in [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]\}$$

from (2) – (4) while taking account of the relation (5).

3 Methods and numerical results.

3.1 Finite elements methods.

3.1.1 Variational formulation

To get variational formulation associated to the problem of out-flow, one introduces bilinear shapes

$$a(u, v) = \frac{2}{Re}(D(u), D(v)), \quad u, v \in H^1(\Omega)$$

where $D(., .)$ is the tensor rate of distortion and

$$b(v, q) = -(q, \text{div}(v)), \quad v \in H^1(\Omega) \text{ and } q \in L^2(\Omega)$$

and the trilinear form

$$c(u, v, w) = ((u \cdot \nabla)v, w), \quad u, v, w \in H^1(\Omega)$$

The problem partner to the bidimensional out-flow models consists in determining, in the domain of periodicity,

$$\Omega_T = \Omega \times [0, T]$$

where $T > 0$ and Ω is defined by (1), the vectorial function u and the scalar function p solutions of the problem (2) – (4) of which a variational formulation is given by

$$\begin{cases} \text{searching } u \in H^1(\Omega) \text{ and } p \in L_0^2(\Omega) \text{ with } u - \hat{u} \in H_0^1(\Omega) \\ \left(\frac{\partial u}{\partial t}, v \right) + a(u, v) + c(u, u, v) + b(v, p) = 0 \\ b(u, q) = 0 \text{ for all } q \in L_0^2(\Omega) \\ u(x, 0) = u_0(x) \text{ for all } x \in \Omega \end{cases} \tag{6}$$

3.1.2 Discretisation

- Discretisation in time: The bifurcation of the couette in spiral waves is an instationnar out-flow, we discretize $\frac{\partial u}{\partial t}$ by a diagram of Gear of constant step time (an implicit diagram to two steps and of order two)

$$\frac{\partial u}{\partial t} = \frac{3u^{n+1} - 4u^n + u^{n-1}}{2 \Delta t}$$

- Discretisation in space: One discretize in space while replacing $H^1(\Omega)$ by V_h , $H_0^1(\Omega)$ and $L_0^2(\Omega)$ by Q_h . These approximations must satisfy a condition of compatibility inf – sup or of Brezzi-Babuska

$$\exists \beta > 0 \text{ such as } \inf_{q \in Q_h} \sup_{u \in V_h} \frac{b(u, q)}{|q|_0 \|u\|_1} \geq \beta$$

The retained element for our survey is the $Q_2 - P_1$. It satisfies the condition inf – sup and possessed a precision of the second order [5, 1981] i.e. if (u, p) is a solution to our problem, one has evaluations of errors

$$\begin{cases} \|u - u_h\|_{H^1(\Omega)} \leq c_1 h^2 \\ \|p - p_h\|_{L^2(\Omega)} \leq c_2 h^2 \end{cases}$$

V_h and Q_h form some conform approximations of $H^1(\Omega)$ and of $L^2(\Omega)$ and one considered the associated discreet problem to (6).

3.1.3 Algorithms of resolution

- Treatment of the nonlinear term: Newton method
- Treatment of the linear term: all linear systems are solved by the factorization LDU of the global matrix stocked under shape of profile.
- Algorithm of Uzawa generalized: One considered the condition of incompressibility like a linear constraint (that one will Penalize strongly) on the speed. The pressure appearing then like a Lagrange multiplier.
- Methods of Simultaneous Inverse Iterations: This method is used to calculate the smallest eigenvalue of a matrix A of large size. It permits to transform a problem of eigenvalues of big dimension in a problem of distinctly more reduced dimension. One shows that the convergence is improved if one uses q vectors with $q > p$. A good choice [2, 1976] seems to be $q = \min(2p, p + 8)$. The test of convergence only carries on p eigenvalues.

To determine the first eigenvalue that will cross the imaginary axis, it agrees to couple the method of Simultaneous Inverse Iterations to the algorithm of Uzawa-generalized.

3.2 Numerical results.

After some tests on the couette out-flow, we opted for a regular maillage 10×10 resulting in: 1323 degrees of liberty and 1140 unknown.

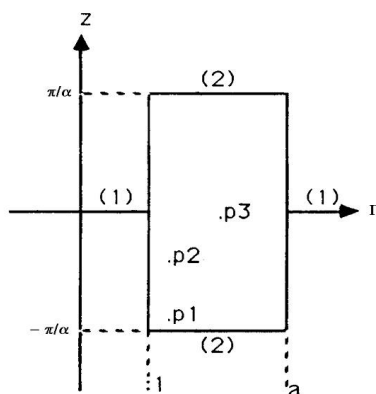


Figure 1 : It's indicates geometry and boundary conditions :
 (1) for Dirichlet knots, and (2) for periodic and free knots.

3.2.1 Effect of the utilization of a squareness method

If one uses a finite element $Q_2 - P_1$ to solve equations of Navier-stokes in Cartesian coordinates, the method of squareness of Gauss to 3 points by direction is adequate (recall that in Cartesian coordinates and for a rectangle, the method of squareness of Gauss to 4 points by direction is exact because the jacobien of the transformation is constant). However, the expression of equations of Navier-Stokes in cylindrical coordinates introduced a factor $\frac{1}{r}$ what makes that the method of squareness of Gauss to 4 points by direction does not integrated more precisely (even in a rectangle) a bilinear term $(D(u), D(v))$ where $D(\cdot)$ is tensor rate of distortion.

3.2.2 Determination of points of bifurcation

While using the linear theory of the stability, the Principle of stability exchange and the type of bifurcation, we show (for more details see [4, 1991]) that the bifurcation of the couette out-flow in spiral waves is a bifurcation of Hoph. Thus, the critical Reynolds number corresponds to a pair conjugated of eigenvalues that cross the imaginary axis. To determine this even eigenvalue, we opted for a method of simultaneous inverse iterations. The numerical results for different values of $a = \frac{R_2}{R_1}$ and of $\omega = \frac{\omega_2}{\omega_1}$ show that the critical number of Reynolds Re_c corresponds to a number of wave azimuthal $m \neq 0$ and the eigenvalue corresponding σ_0 is imaginary pure confirming so the nature of the bifurcation (a bifurcation of Hoph). Besides, our results agree very well with those of [3, 1984] and of [8, 1988]. The relative difference between the critical Reynolds numbers hardly passes them .1% [4, 1991].

3.2.3 Determination of the spiral wave branch

We are going to restrict to the case

$$a = 1.3300, \quad \omega = -0.6, \quad m = 1 \text{ and } \alpha = 10.98$$

which corresponds to $\sigma_0 = \omega_0 i = 0.346$ and to $Re_c \simeq 366.5$. To determine a point of the spiral wave branch for a number of Reynolds $Re_1, Re_1 \geq Re_c$, one takes for initial solution

$$u_0 = u_c + \varepsilon v$$

with $0.01 \leq \varepsilon \leq 0.1$, u_c is the solution of couette and v is the corresponding vector to the critical eigenvector. The figure 2 represent the signal of the radial component U_r in function to

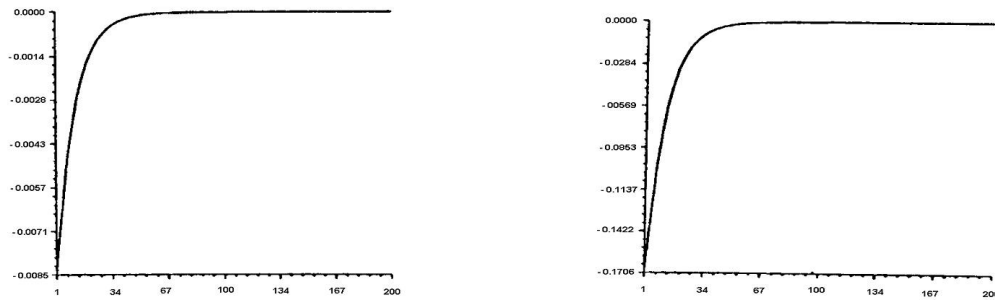


Figure 2.1 et 2.2

$n = \frac{t}{\Delta(t)}$, $\Delta(t) = .1$ at the point $P3$ for $Re = Re_c$ and $Re = 370$. It shows that if the number of points of Gauss by direction is equal to 4, the perturbation is absorbed in less a period of time and one gets the solution of couette even though one repeated the procedure with a number of Reynolds Re , $Re > Re_c$. If now, one uses a Method of squareness of Gauss with 5 points by direction, the perturbation gotten from a critical eigenvector is not absorbed anymore. Figure 3 represent in projection (r, z) the space of the vector current and equipotential lines of the current out-flow in spiral waves for numbers of Reynolds $Re \simeq 366.50$ and $Re = 390$.

3.2.4 Evolution of the out-flow in spiral waves.

According to the dynamical system theory, it is necessary, to have more information on the evolution of the out-flow, to use the signal of the speed and the bidimensional phase portrait, it consists in drawing the graph of $u(t + \tau)$ according to $u(t)$ where τ is an arbitrary constant; a " good " approximation of τ [7, 1987] and [, 1991] is $T/4$. The figure 4 represents the signal in $P3$ of the radial component and its phase portrait of the speed for Reynolds numbers $Re = Re_c = 366.50$ and $Re = 390$. They show that our numerical model, with a diagram of Gear to treat the derivative in relation to the time, confirm the simple periodicity of the out-flow in spiral waves.

Conclusion :

In this work, we presented a finite bidimensional elements code that permitted us to simulate the transition of the couette out-flow toward the out-flow in spiral waves (a solution instationnar and no-axisymetric). The reduction of the problem of couette-Taylor to a bidimensional problem has been made from subgroups of isotropic solution. The localization of points of bifurcation searched for, that amounts to the calculation of the eigenvalues of the tangent matrix, has been simplified considerably thanks to the identification of the bifurcation type in reason and made by a method of Simultaneous Inverse Iterations. We put in evidence the utilization of an implicit diagram of order 2 in time and the necessity of a very precise numerical integration. We showed the validity of our code while showing that the gotten results agree with the applied observations and other numerical results existing in the literature. As, we had, at the time of the transition of the couette out-flow toward the out-flow in spiral waves, difficulties with the numeric integration, it agrees to be careful at the time of a simulation of

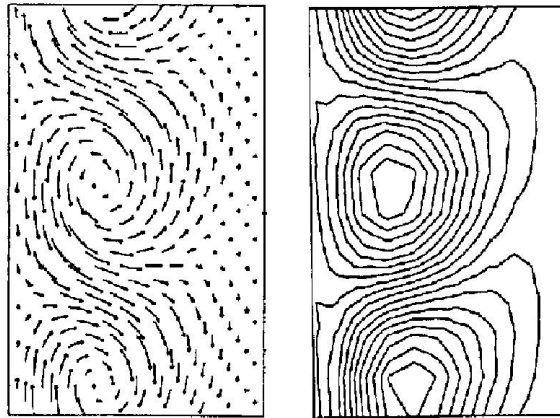


Figure 3.1 : Current vector and lines for $Re = 366, t = 100 * \Delta(t), \Delta(t) = .1$

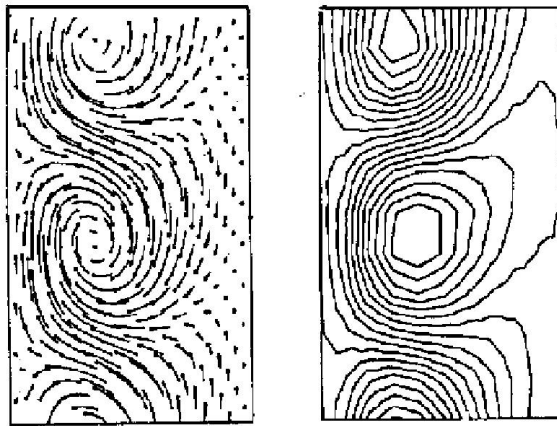


Figure 3.2 : Current vector and lines for $Re = 390, t = 100 * \Delta(t), \Delta(t) = .1$:

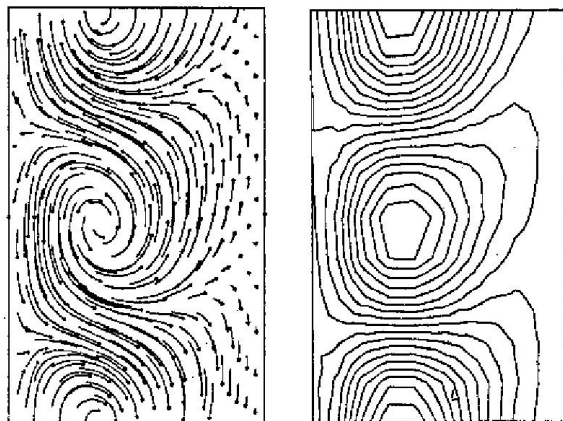


Figure 3.3 : Current vector and lines for $Re = 390, t = 200 * \Delta(t), \Delta(t) = .1$

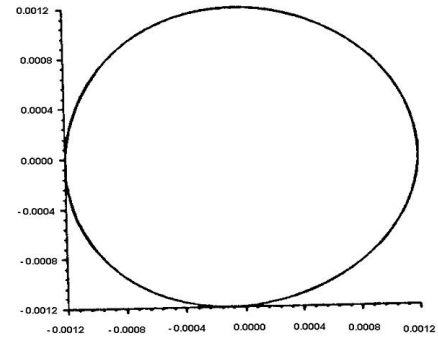
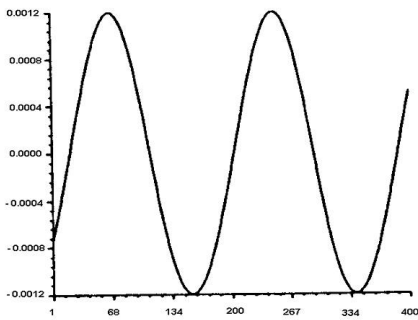


Figure 4.1 : Radial component for $Re = Re_c$, Figure 4.2 : Phase portrait for $Re = Re_c$

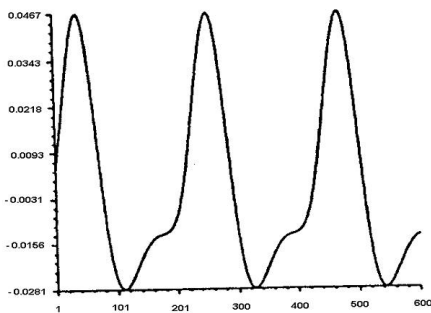


Figure 4.3 : Radial component for $Re = 390$, Figure 4.4 : Phase portrait for $Re = 390$

all tridimensional out-flow. The survey of the stability and transitions of the out-flow of Taylor and/or of the out-flow in spiral waves requires a tridimensional formulation. We think that the analysis that we developed can adjust to the case $3 - D$.

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