The Inverse Variational Kupradze-Brebbia Formulation for the Navier-Stokes Equations

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Abstract:
We would like to notice that field of velocities and pressures for bridge piers in high-mountain regions of basin of river Durudschy (Georgia) were calculated with help the inverse variation Kupradze-Brebbia formulation for the Navier-Stokes equations and the iteration Walker-Brebbia formula. Also the tasks of bypassing of metal plate by viscous incompressible fluid, were solvers for different angels.

Keywords: *velocity vector, pressure, inverse variation*

1. INTRODUCTION

In this article we consider the task of flowing of viscous incompressible fluid in compact Rieman variety G with the Lipschitz boundary ∂G. We scrutinize the Navier-Stokes equations in non-dimensional form:

$$-\frac{1}{\mathrm{Re}}V_{j,ii} + V_i V_{j,i} + P_{,j} = f_j - \dot{V}_j \text{ in G} \tag{1}$$

$$V_{i,i} = 0 \text{ in G} \tag{2}$$

$$V_j\big|_{t=0} = V_j^0 \text{ in G} \tag{3}$$

$$V_j\big|_{\partial G} = 0 \text{ on } \partial G \tag{4}$$

where V_j - components of velocity vector, P - pressure, f - force for volume unit, Re - the Reynolds number.
To obtain

$$-\frac{1}{\mathrm{Re}}V_{j,ii} + V_i V_{j,i} + P_{,j} = f_j - \frac{V_j^{\tau+\delta\tau} - V_j^{\tau}}{\delta\tau} \tag{5}$$

we use difference form for temporal derivative from the equation (1).
Further we apply the Kupradze-Brebbia method, multiply the equation (5) by $\forall W_j \in L_2(G)$ in the sense of $L_2(G)$ and use the Green formula.

As a result the weak Lions-Temam formulation is obtained [1-4]:

$$\begin{aligned}&\frac{1}{\mathrm{Re}}\left(\left(V^{\tau+\delta\tau}, W\right)\right) + b\left(V^{\tau+\delta\tau}, V^{\tau+\delta\tau}, W\right) = \\ &= \left(f - \frac{V^{\tau+\delta\tau} - V^{\tau}}{\delta\tau}, W\right)\end{aligned} \tag{6}$$

where $V_j \in D_{NS}(G)$ and $D_{NS}(G)$ is system of vector functions, their divergence equals zero and these functions satisfy the boundary conditions (4).

$$\left(f - \frac{V^{\tau+\delta\tau} - V^{\tau}}{\delta\tau}, W \right) = \int_G \left(f_j - \frac{V_j^{\tau+\delta\tau} - V_j^{\tau}}{\delta\tau} \right) W_j dx \tag{7}$$

=

$$b\left(V^{\tau+\delta\tau}, V^{\tau+\delta\tau}, W\right) = \int_G V_j^{\tau+\delta\tau} V_{j,i}^{\tau+\delta\tau} W_j dx \tag{8}$$

$$\left(\left(V^{\tau+\delta\tau}, W\right)\right) = \int_G V_{i,j}^{\tau+\delta\tau} W_{i,j} dx \tag{9}$$

Then the weak variation formulation of the task (1)-(4) have form: find $V_j \in D_{NS}(G)$ which satisfy the equation (6) for the conditions (3), (7)-(9) and $\forall W_j \in L_2(G)$.

That formulation is used often for a solution of the Navier-Stokes equations by the method of finite elements [2-4].

To obtain the inverse variation Kupradze-Brebbia formulation for the task (1)-(4) we apply once again the Green formula to the equation (9) and have following:

$$\left(\left(V^{\tau+\delta\tau}, W\right)\right) = \int_G V_i^{\tau+\delta\tau} W_{i,jj} dx \tag{10}$$

Thus the inverse variation Kupradze-Brebbia formulation [5-6] for the task (1)-(4) have form: find $V_j \in D_{NS}(G)$ which satisfy the equation (6) for the conditions (3), (7), (8), (10) and $\forall W_j \in L_2(G)$.

To solve the Navier-Stokes equations within the framework of the inverse variation Kupradze-Brebbia formulation we apply the iteration Walker-Brebbia procedure (7), i.e. we transfer linear terms of the equation (6) for V_j in the left part of equation and non-linear terms are transfer by us in the right part of equation.

Then we give out (n+1)th approach $V_{j(n+1)}$ for linear terms and nth approach $V_{j(n)}$ for non-linear terms and obtain:

$$\frac{1}{\mathrm{Re}}\left(\left(V_{n+1}^{\tau+\delta\tau}, W\right)\right) - \left(f - \frac{V_{n+1}^{\tau+\delta\tau} - V^{\tau}}{\delta\tau}, W \right) = -b\left(V_n^{\tau+\delta\tau}, V_n^{\tau+\delta\tau}, W\right) \tag{11}$$

To realize the iteration Walker-Brebbia procedure [7] we use the equations (11) and search for solution of the task (3), (6), (8), (10) by the RO-function method [8-11]. We seek a solution of the equation (11) in following form:

$$V_\tau^{\tau+\delta\tau} = F_{i_{\tau+\delta}}\left(x_1, x_2, x_3, \alpha_1^{\tau+\delta\tau}, \ldots, \alpha_n^{\tau+\delta\tau}\right) \tag{12}$$

Since $V_j \in D_{NS}(G)$, then we must seek a solution in form of a field of curl of some vector field, i.e.

$$v_i^{\tau+\delta\tau} = \varepsilon_{ijk}\, \Psi_{k,j} \tag{13}$$

where Ψ_k is vector potential of velocity field.

Obviously, that the relation (13) guarantees a solenoids of velocity field of viscous fluid.

To satisfy the requirements of boundary conditions of sticking of viscous fluid to walls (4) we use RO – functions [8-11].

Further we take into account the equation (13) and suppose that

$$\Psi_k = \frac{RO^2(x_1, x_2, x_3)}{3 \cdot RO_{\max}^2} \cdot \sum_{p=1}^{n} \alpha_{pk}^{\tau+\delta\tau} \cdot g(r_p^2) \qquad (14)$$

Clearly, that the equations (13), (14) permit to satisfy to condition

$$V_i^{\tau+\delta\tau} \in D_{NS}(G) \text{ for } \forall\, g(x) \in C_{[0;diam\,G]}.$$

Now we must find coefficients of decomposition $\alpha_{ik}^{\tau+\delta\tau}$ for each moment of time $\tau + \delta\tau$ from the equation (11). We choose the fundamental solutions of the Laplace operators as tentative functions W_i:

$$W_i = \frac{1}{r_i} \qquad (15)$$

where $r_i = \sqrt{(x_1 - \xi_i)^2 + (x_2 - \eta_i)^2 + (x_3 - \varsigma_i)^2}$ (16)

and (ξ_i, η_i, ζ_i) are points of disposition of regular sources OG [12].

As

$$W_{i,jj} = -4 \cdot \pi \cdot \Delta(\xi_i, x) \qquad (17)$$

Where $\Delta(\xi_i, x)$ is the Dirac delta-function, which does not equal zero only for point (ξ_i, η_i, ζ_i).

Then we substitute the equation (17) in the equation (10) and obtain:

$$((V^{\tau+\delta\tau}, W)) = -4 \cdot \pi (V_1^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i) + V_2^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i) + V_3^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i)) \qquad (18)$$

Further we take into consideration the (18) and obtain from the equation (11) following:

$$\frac{4 \cdot \pi}{\mathrm{Re}} (V_{1(n+1)}^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i) + V_{2(n+1)}^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i) + V_{3(n+1)}^{\tau+\delta\tau}(\xi_i, \eta_i, \zeta_i)) +$$
$$+ \int_G \left(f_j - \frac{V_{j(n+1)}^{\tau+\delta\tau} - V_j^{\tau}}{\delta\tau} \right) \cdot \frac{1}{r_j} dx = \int_G V_{k(n)}^{\tau+\delta\tau} \cdot V_{j,k(n)}^{\tau+\delta\tau} \cdot \frac{1}{r_j} dx \qquad (19)$$

In the formulas (19) expression (n+1) points to corresponding approximation in the Walker-Brebbia iteration.

For each i^{th} point of disposition of regular sources OG [12] we have three equations $(j = \overline{1,3})$ from (19). Thus, to find coefficients $\alpha_i^{\tau+\delta\tau}$ $(i = \overline{1,n})$ for each time layers we have $3 \cdot n$ equations which are related with single-stage iteration formulas (19).

We base on the equations (13), (14) and find coefficients $\alpha_{ik}^{\tau+\delta\tau}$ for each time layer. And vector field of flowing of viscous incompressible fluid is designed on computers screen.

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