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An Approach for Solving Integer and Combinatorial Optimization Problems Based on Bilinear and Linear Parametrical Programming

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Abstract

A general scheme for studying and solving a bilinear programming problem based on linear parametrical programming is proposed. Using this scheme new algorithms are derived for solving linear Boolean and resource allocation problems. Computational complexity of the algorithms is also discussed.

Keywords: *integer programming, linear parametrical programming, computational complexity, Boolean programming, polynomial-time algorithms.*

1 Introduction and Problem Formulation

We consider the following bilinear programming problem (BPP) [1, 2, 3]:
to minimize

$$z = xCy + c'x + c''y \tag{1}$$

on subject

$$Ax \le a, \;\; x \ge 0; \tag{2}$$

$$By \le b, \;\; y \ge 0, \tag{3}$$

where C, A, B are matrices of size $n \times m$, $q \times n$, $l \times m$, respectively, and c', $x \in R^n$; c'', $y \in R^m$; $a \in R^q$, $b \in R^l$. In order to simplify the notations we will omit transposition sign for vectors.

This bilinear model generalizes a large class of integer and combinatorial optimization problems [4, 2]. An important particular case of BPP (1)-(3) represents the linear Boolean programming problem:
to minimize

$$z = \sum_{i=1}^{n} c_i x_i \tag{4}$$

on subject

$$\begin{cases} \sum_{i=1}^{n} a_{ji} x_i \le a_{j0}, \;\; j = \overline{1,q}; \\ x_i \in \{0,1\}, \;\; i = \overline{1,n}. \end{cases} \tag{5}$$

In [2] it is shown that this problem can be replaced by the following BPP:
to minimize

$$z = \sum_{i=1}^{n} c_i x_i + M \sum_{i=1}^{n} (x_i y_i + (1 - x_i)(1 - y_i)) \tag{6}$$

on subject

$$\begin{cases}\sum_{i=1}^{n} a_{ji}x_i \le a_{j0,} \;\; j=\overline{1,q}; \\ 0 \le x_i \le 1, \;\; i=\overline{1,n}, \end{cases} \tag{7}$$

$$0 \le y_i \le 1, \;\; i=\overline{1,n}, \tag{8}$$

where $M > \sum_{i=1}^{n} |c_i|$. Another important case of BPP (1)-(3) represents the piecewise linear concave programming problem:
to minimize

$$z = \sum_{i=1}^{l} \min\left\{c^{ik}x + c_0^{ik}, \;\; k=\overline{1,r_i}\right\} \tag{9}$$

on subject determined by (2), where $x \in R^n$, $c^{ik} \in R^n$, $c_0^{ik} \in R^n$. This problem arises as an auxiliary one when solve a class of resource allocation problems [4, 2]. In [2] it is shown that this problem can be replaced by the following BPP:
to minimize

$$z = \sum_{i=1}^{k} \; \sum_{k=1}^{r_i} (c^{ik}x + c_0^{ik}) y_{ik} \tag{10}$$

on subject

$$Ax \le a, \;\; x \ge 0; \tag{11}$$

$$\begin{cases}\sum_{k=1}^{r_i} y_{ik} = 1, \;\; i=\overline{1,l}; \\ y_{ik} \ge 0, \;\; l=\overline{1,r_i}, \;\; i=\overline{1,l}. \end{cases} \tag{12}$$

In this paper we propose an approach for solving BPP (1)-(3) which takes into account the particularity of the mentioned above cases of problems, i.e. when the matrix *B* is either identity one or step-diagonal one. The general scheme of the proposed approach is based on parametric linear programming. Using duality principle for the considered problem we show that it can be reduced in polynomial time to a problem of determining the consistency of the system of linear inequalities with right-hand members that depend on parameters, admissible values of which are defined by another system of linear inequalities. Then a specification of the proposed approach for the mentioned above linear Boolean and resource allocation problems are developed and new algorithms for solving these classes of problems are derived. Computational complexity aspects of the proposed approach are discussed and a class of problems for which polynomial-time algorithms exist is described.

2 Parametrical programming approach for BPP

Let *L* be the size of BPP (1)-(3) with integer coefficients of the matrices *C*, *A*, *B* and vectors *a*, *b*, *c'*, *c''*, i.e. *L* is the length of the input dates of BPP (1)-(3) [5, 4].

If BPP (1)-(3) has solution then it can be solved by varying the parameter $h \in \left[-2^L, 2^L\right]$ in the problem of determining the consistency of the system

$$\begin{cases} Ax \le a; \\ xCy + c'x + c''y \le h; \\ By \le b; \\ x \ge 0, \;\; y \ge 0. \end{cases} \tag{13}$$

In the following we will reduce the consistency problem for system (13) to the consistency problem for the system of linear inequalities with a right-hand member depending on parameters.

Theorem 1. *Let solution sets X and Y of systems (2) and (3) are nonempty. Then system (13) has no solution if and only if the following system of linear inequalities*

$$\begin{cases} -A^T u \le Cy + c'; \\ au < c''y - h; \\ u \ge 0 \end{cases} \tag{14}$$

is consistent with respect to u for every y satisfying (3).

Proof. $\Rightarrow$ Let us assume that system (13) has no solution. This means that for every $y \in Y$ the following system of linear inequalities

$$\begin{cases} Ax \le a, \\ x(Cy + c') \le h - c''y, \\ x \ge 0 \end{cases} \tag{15}$$

has no solution with respect to *x*. Then according to theorem 2.14 from [6] the inconsistency of system (15) involves the solvability with respect to *u* and *t* of the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \ge 0; \\ au + (h - c''y)t < 0; \\ u \ge 0,\ \ t \ge 0, \end{cases} \tag{16}$$

for every $y \in Y$. Note that for every fixed $y \in Y$ of system (16) for an arbitrary solution (u^*, t^*) the condition $t^* > 0$ holds. Indeed, if $t^* = 0$, then it means that the system

$$\begin{cases} A^T u \ge 0; \\ au < 0,\ \ u \ge 0, \end{cases}$$

has solution, what, according to theorem 2.14 from [6], involves the inconsistency of system (2) that is contrary to the initial assumption. Consequently, $t^* > 0$. Since $t > 0$ in (16) for every $y \in Y$, then, dividing every of inequalities of this system by *t* and denoting $z = (1/t)u$, we obtain the following system:

$$\begin{cases} -A^T z \le Cy + c'; \\ az < c''y - h; \\ z \ge 0, \end{cases}$$

Which has solution with respect to *z* for every $y \in Y$.

$\Leftarrow$ Let system (14) has solution with respect to *u* for every $y \in Y$. Then the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \ge 0; \\ au + (h - c''y)t < 0; \\ u \ge 0,\ \ t > 0, \end{cases}$$

is consistent with respect to *u* and *t* for every $y \in Y$. However this system is equivalent to system (16) as it was shown that for every solution (u, t) of system (16) the condition $t > 0$ holds. Again using theorem 2.14 from [6], we obtain from the solvability of system (16) with respect to *u* and *t* for every $y \in Y$ that system (15) is inconsistent with respect to *x* for every $y \in Y$. This means that system (13) has no solution. □

Theorem 2. *The minimal value of the object function in BPP (1)-(3) is equal to the maximal value h^* of the parameter h in the system*

$$\begin{cases} -A^T u \le Cy + c'; \\ au < c''y - h; \\ u \ge 0 \end{cases} \tag{17}$$

for which it is consistent with respect to u for every $y \in Y$. An arbitrary point $y^* \in Y$, for which system (14) with $h = h^*$ and $y = y^*$ has no solution with respect to u, corresponds to one of the optimal points for BPP (1)-(3).

Proof. Let h^* be a maximal value of parameter *h*, for which system (17) with $h = h^*$ has solution with respect to *u* for every $y \in Y$. Then system (14) with $h = h^*$ has solution with respect to *u* not for every $y \in Y$. From this on the basis of theorem 1 it results that system (13) with $h = h^*$ is consistent. Using theorem 1 we can see that if for every fixed $h < h^*$ system (14) has solution with respect to *u* for every $y \in Y$, then system (13) with $h < h^*$ has no solution. Consequently, the maximal value h^* of parameter *h*, for which system (17) has solution with respect to *u* for every $y \in Y$, is equal to the minimum value of the object function of BPP (1)-(3).

Now let us prove the second part of the theorem. Let $y^* \in R^m$ be an arbitrary point for which system (14) with $h = h^*$ and $y = y^*$ has no solution with respect to *u*. Then on the basis on the duality principle the following system

$$\begin{cases} Ax \le a; \\ x(Cy^* + c') \le h^* - c''y; \\ x \ge 0 \end{cases}$$

has solution with respect to *x*. So, system (13) with $h = h^*$ is consistent and point $y^* \in Y$ together with certain $x^* \in X$ represents the solution of BPP. $\sqcup$

Corollary 1. *Let* $\overline{Y}_h = \{ y \in R^m \mid U_h(y) \ne 0 \}$, *where* $U_h(y)$ *is the set of solutions of system (15) with respect to u for given* $y \in R^m$ *and fixed h. Assume that* y^0 *is an arbitrary basic solution of system (3) such that*

$$Z^0 = \min_{x \in X} (xCy^0 + c'x + c''y^0) > h^*.$$

Then

i) $y^0 \in$ *int* $\overline{Y}_{h^*}$, *i.e.* y^0 *is an interior point of set* $\overline{Y}_{h^*}$;

ii) $Y \subset$ *int* $\overline{Y}_h$ *if* $h < h^*$.

Note that in an analogous way the same mathematical tool for system (13) can be applied considering *x* as a vector of parameters. This allows us to replace the main problem by the problem of determining the consistency of the system

$$\begin{cases} -B^T v \le C^T x + c''; \\ bv \le c'x - h; \;\; v \ge 0 \end{cases} \tag{18}$$

with respect to *v* for every *x* satisfying (2). This means that for the linear parametric system the following duality principle holds (see [7]).

Theorem 3. *The system of linear inequalities (17) is consistent with respect to u for every y satisfying (3) if and only if the system of linear inequalities (18) is consistent with respect to v for every x satisfying (2).*

It is easy to observe that if Y is a bounded set then the consistency property in our auxiliary problem can be verified by checking the consistency of system (17) for every basic solution of system (3). This fact follows from the geometrical interpretation of the problem. Indeed, let $UY \subseteq R^{n+k}$ be a solution set of system (17) with respect to u and y. Then $\overline{Y}_h$ for given h represents the orthogonal projection on R^k of the set $UY \subseteq R^{n+k}$. Therefore $Y \subseteq \overline{Y}_h$ if and only if system (17) is consistent for every basic solution of (3). Of course such an approach for solving the auxiliary problem cannot be used for systems with big number of variables. The approach we propose allows us to avoid exhaustive search. Moreover, we can see that in the case of problems (4)-(8) and (9)-(12) our approach efficiently solves the auxiliary problem.

The results described above show that BPP (1)-(3) can be solved efficiently if there exists an efficient algorithm for solving the following problem: to determine the maximal value h^* of parameter h such that a basic solution y^* of system (3) belongs to $bd\overline{Y}_{h^*}$.

In the following we show how to verify the condition $Y \subset int\overline{Y}_h$ and propose an algorithm for solving BPP (1)-(3) in the case when (3) is determined by (8) or (12).

3 Some auxiliary results

In order to explain the main results we need some auxiliary results related to dependent inequalities of linear systems. An inequality

$$\sum_{j=1}^{m} s_j y_j \le s_0 \tag{19}$$

is called dependent [6] on the consistent system of linear inequalities

$$\sum_{j=1}^{m} d_{ij} y_j \le d_{i0}, \ i = \overline{1,p}, \tag{20}$$

if for an arbitrary solution of system (20) condition (19) holds.

The well-known Minkowski-Farkas theorem [6, 8] gives the necessary and sufficient condition of dependency (19) on (20) in the case of consistent system (20). We will extend this theorem for inconsistent systems and will use it in general form. In order to formulate this result we need the following definition.

Definition 1. *Assume that system (20) is inconsistent. Inequality (19) is called dependent on system (20) if there exists a consistent subsystem*

$$\sum_{j=1}^{m} d_{i_k j} y_j \le d_{i_k 0}, \ i = \overline{1,r}, \tag{21}$$

of system (20) such that inequality (19) is dependent on (21).

Theorem 4. *Let be given system (20) with rank $r \le m$ $(m < p)$. Inequality (19) is dependent on system of linear inequalities (20) if and only if there exist numbers $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_p$ such that*

$$\begin{cases} s_j = \sum_{i=1}^{p} d_{ij}\lambda_i, \ \ j = \overline{1,m}; \\ s_0 = \sum_{i=1}^{p} d_{i0}\lambda_i + \lambda_0; \\ \lambda_j \ge 0, \ \ j = \overline{1,m}, \end{cases} \tag{22}$$

where no more than r components among $\lambda_1, \lambda_2, ..., \lambda_p$ are nonzero.

Proof (Sketch). Necessary condition follows from [6] (Theorem 2.2). Indeed, if (19) is dependent on (20) then there exists nodal solution (21) such that (19) is dependent on (21) and necessary condition holds. Sufficient condition in the case of inconsistent system (21) can be proved in the following way. Assume that system (22) has solution $\lambda_0, \lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_r}, 0, 0, ..., 0$, where $r \le m$. Then system (21), corresponding to $\lambda_{i_k} > 0,\ k = \overline{1, r},$ has solution. This means that inequality (19) is dependent on consistent subsystem of linear inequalities (21). □

4 The main results

We consider the problem from section 2 and describe an algorithm for checking if $Y \subset int\overline{Y}_h$ in the case when *Y* is determined by system (20), which satisfies the following conditions:

a) system (20) has rank *m* ($m < p$) and *Y* is a bounded set with $int Y \ne 0$;

b) system (20) does not contain dependent inequalities;

c) if an arbitrary subsystem

$$\sum_{j=1}^{m} d_{i_k j} y_j \le d_{i_k 0},\ k = \overline{1, m}; \tag{23}$$

of system (20) has rank *m* then the solution of the system of linear inequalities

$$\sum_{j=1}^{m} d_{i_k j} y_j = d_{i_k 0},\ k = \overline{1, m}; \tag{24}$$

is a solution of system (20), i.e. system (20) contains all possible nodal solutions.

It is easy to observe that system (3) when *B* is an identity matrix and *B* is a step-diagonal matrix represents a particular case of system (20) with properties a)-c). Therefore the results we describe below can be referred to problems (6)-(8) and (9)-(12).

In order to guarantee $int\overline{Y}_h \ne \varnothing$ we will fix $h \in \left[-2^L, N\right)$, where $N = \min\left[h^0, 2^L\right]$, h^0 is the optimal value of the object function in the linear programming problem: to maximize *h* on subject (17), i.e. to maximize *h* on the set of solutions of the following system

$$\begin{cases} -\sum_{j=1}^{q} a_{ji} u_j - \sum_{j=1}^{m} c_{ij} y_j \le c'_i,\ i = \overline{1, n}; \\ \sum_{j=1}^{q} a_{j0} u_j - \sum_{j=1}^{m} c''_j y_j \le -h; \\ u_j \ge 0,\ \ j = \overline{1, q}. \end{cases} \tag{25}$$

Theorem 5. *Let be given set Y determined by system of linear inequalities (20) satisfying conditions a)-c). In addition assume that* $h \in \left[-2^L, N\right)$ *and set X of solutions of system (2) is bounded with* $int\ X \ne \varnothing$. *Then* $Y \not\subset int\overline{Y}_h$ *if and only if the following system of linear inequalities*

$$\begin{cases} \sum_{i=1}^{n} a_{ji} \lambda_i \le a_{j0},\ \ j = \overline{1, q}; \\ \sum_{i=1}^{p} d_{ij} \mu_i + \sum_{i=1}^{n} c_{ij} \lambda_i = -c''_j,\ \ j = \overline{1, m}; \\ -\sum_{i=1}^{p} d_{i0} \mu_i + \sum_{i=1}^{n} c'_i \lambda_i \le h; \\ \mu_i \ge 0,\ i = \overline{1, p};\ \lambda_i \ge 0,\ i = \overline{1, n}. \end{cases} \tag{26}$$

has such a solution that $\sum_{i=1}^{p} d_{ij}\mu_i \neq 0$ *at least for an index* $j \in \{1,2,...,m\}$ *and no more than m components among* $\mu_1, \mu_2, ..., \mu_p$ *are nonzero.*

Proof. $\Rightarrow$ Assume that system (20) satisfies conditions a)-c) and $Y \not\subset int\overline{Y}_h$ for given $h \in [-2^L, N)$. Then $int\overline{Y}_h \neq \varnothing$ and there exists a nodal solution $y^0 = (y_1^0, y_2^0, ..., y_m^0)$ of system (20) for which $y^0 \notin int\overline{Y}_h$, i.e. there exists subsystem (23) of system (20) such that for $y = y^0$ condition (24) holds and $y^0 \notin int\overline{Y}_h$. Note that an arbitrary nodal solution y^0 of system (20) can be regarded as a common vertex of two symmetrical cones one of which Y^0 is determined by system (23) and another one $\overline{Y}^0$ is determined by the following symmetric system

$$\sum_{j=1}^{m} d_{i_k j} y_j \geq d_{i_k 0}, \quad k = \overline{1,m}, \tag{27}$$

which is a subsystem of the following inconsistent system

$$\sum_{j=1}^{m} d_{ij} y_j \geq d_{i0}, \quad i = \overline{1,p}. \tag{28}$$

Based on properties a)-c) of system (20) we can show that there exists a nodal solution y^0 which determines the cone $\overline{Y}^0$ such that $\overline{Y}^0 \cap int\overline{Y}_h = \varnothing$.

This means that there exists a separating hyper plane $\sum_{j=1}^{m} s_j y_j = s_0$ [9] such that $\sum_{j=1}^{n} s_j y_j < s_0$ for $(y_1, y_2, ..., y_n) \in int\overline{Y}_h$ and

$$-\sum_{j=1}^{m} s_j y_j \leq -s_0 \tag{29}$$

for $(y_1, y_2, ..., y_n) \in \overline{Y}^0$. So, the inequality

$$\sum_{j=1}^{n} s_j y_j \leq s_0 \tag{30}$$

is dependent on system (25) with respect to variables $\mu_1, \mu_2, ..., \mu_p, y_1, y_2, ..., y_p$ and inequality (29) is dependent on system (27). If (29) is dependent on (27) then (29) is dependent on inconsistent system (28). Thus on the basis of Theorem 4, we obtain that the following systems

$$\begin{cases} 0 = -\sum_{i=1}^{n} a_{ji}\lambda_i + a_{j0}\lambda_0 - \lambda_{n+j}, \quad j = \overline{1,q}; \\ s_j = -\sum_{i=1}^{n} c_{ij}\lambda_i - c''_j\lambda_0, \quad j = \overline{1,q}; \\ s_0 = \sum_{i=1}^{n} c'_i\lambda_i - h\lambda_0 + \lambda_{n+q+1}; \\ \lambda_i \geq 0, \quad i = \overline{0, n+q+1}; \end{cases} \tag{31}$$

$$\begin{cases} -s_j = -\sum_{i=1}^{p} d_{ij}\mu_i; \\ -s_0 = -\sum_{i=1}^{n} d_{i0}\mu_i + \mu_0; \\ \mu_i \geq 0, \quad i = \overline{0,p}; \end{cases} \tag{32}$$

have solutions and no more than *m* components among $\mu_1, \mu_2, ..., \mu_p$ are nonzero.

Taking into account that we are seeking for a basic solution where $s_j \neq 0$ at least for an index $j \in \{1, 2, ..., m\}$ we obtain that $\lambda_0 \neq 0$. Therefore if we consider $\lambda = 1$ and introduce (32) in (31) then system (26) has a solution for which $\sum_{i=1}^{p} d_{ij}\mu_i \neq 0$ at least for an index $j \in \{1, 2, ..., m\}$ and no more than *m* components among $\mu_1, \mu_2, ..., \mu_p$ are nonzero.

$\Leftarrow$ Assume that problem (26) has solution with the properties mentioned in the theorem. This involves that systems (31), (32) have such a solution that $s_j \neq 0$ at least for an index $j \in \{1, 2, ..., m\}$ and there exist inequalities (29), (30) that (29) is dependent on (28) and (30) is dependent on a consistent subsystem (27) of inconsistent system (28). This means that there exists a nodal solution $y^0 = \left(y_1^0, y_2^0, ..., y_n^0\right)$ of system (20) for which $y^0 \notin int\overline{Y}_h$. □

Theorem 6. *Let* h^* *be the minimal value of parameter h for which system (26) has solution* $\mu_1^*, \mu_2^*, ..., \mu_p^*, \lambda_1^*, \lambda_2^*, ..., \lambda_n^*$ *such that* $\sum_{i=1}^{p} d_{ij}\mu_i^* \neq 0$ *at least for an index* $j \in \{1, 2, ..., m\}$ *and no more than m components among* $\mu_1^*, \mu_2^*, ..., \mu_p^*$ *are nonzero. Then* h^* *is equal to the optimal value of the object function in the following BPP: to minimize (1) on subject (2) and (20) with properties a)-c). An arbitrary solution* $y^* = \left(y_1^*, y_2^*, ..., y_m^*\right)$ *of the system of linear inequalities*

$$\begin{cases} \sum_{j=1}^{m} d_{ij} y_j \leq d_{i0}; \;\; i = \overline{1, p}; \\ \sum_{j=1}^{m} s_j^* y_j = s_0^*, \end{cases} \tag{33}$$

with $s_j^* = \sum_{i=1}^{p} d_{ij}\mu_i^*, \;\; j = \overline{0, m}$*, corresponds to a solution of BPP (1), (2), (20).*

Proof. Let h^* be the quantity which satisfies the condition of the theorem. Then for an arbitrary $h < h^*$ system (26) has no solution with the properties from Theorem 5. This means that $Y \subset int\ \overline{Y}_h$ for every $h < h^*$. So, h^* is the maximal value of parameter *h* for which system (17) is consistent with respect to *u* for every $y \in Y$. According to Theorems 1 and 2, the point y^* is a point for which system (14) with $h = h^*$ has no solution. Therefore y^* corresponds to a solution of BPP (1), (2), (20). Taking into account that equation $\sum_{j=1}^{m} s_j^* y_j = s_0^*$ determines a supporting plane for *Y* then a solution of system (32) is a solution of BPP (1), (2), (20). □

Now let us show how to find the solution of system (26) with the properties from Theorem 5.

Theorem 7. *Let be given system of linear inequalities (26) with fixed* $h \in \left[-2^L, N\right)$ *and consider the following 2m linear programming problems:*

$$\textit{to maximize } f_j = \sum_{i=1}^{p} d_{ij}\mu_i \textit{ on subject (26), } j = \overline{1, m}; \tag{34}$$

$$\textit{to minimize } f_j = \sum_{i=1}^{p} d_{ij}\mu_i \textit{ on subject (26), } j = \overline{1, m}. \tag{35}$$

Assume that $\overline{f}_1, \overline{f}_2, ..., \overline{f}_m$ *represent the corresponding optimal values of object functions of problems (34) and* $\overline{\overline{f}}_1, \overline{\overline{f}}_2, ..., \overline{\overline{f}}_m$ *represent the corresponding optimal values of object functions of problems (35). Then system (26) has a solution with the property from Theorem 5 if and only if*

1) *at least for an index* $j \in \{1, 2, ..., m\}$ *either* $\overline{f}_j \neq 0$ *or* $\overline{\overline{f}}_j \neq 0$;

2) *the corresponding basic solution for which 1) holds satisfies the condition that no more than m components among* $\mu_1^*, \mu_2^*, ..., \mu_p^*$ *are nonzero.*

Proof. The sufficient condition of the theorem is evident. Let us prove the necessary one. Assume that system (26) has solution $\mu_1^*, \mu_2^*, ..., \mu_p^*, \lambda_1^*, \lambda_2^*, ..., \lambda_n^*$ which satisfies conditions of Theorem 5. Then it is easy to observe that

$$\overline{f}_{j_0} \geq \sum_{i=1}^{p} d_{ij_0}\mu_i^* \text{ if } \sum_{i=1}^{p} d_{ij_0}\mu_i^* > 0$$

and

$$\overline{\overline{f}}_{j_0} \leq \sum_{i=1}^{p} d_{ij_0}\mu_i^* \text{ if } \sum_{i=1}^{p} d_{ij_0}\mu_i^* < 0. \; \square$$

Corollary 2. *For given* $h \in \left[-2^L, N\right)$ *a solution of system (25) with the properties from Theorem 5 can be found in polynomial time.*

Based on results described above we can propose the following algorithm.

Algorithm for solving BPP (1), (2), (20) with conditions a)-c)

We replace BPP (1), (2), (20) by system (25), where $h \in \left[-2^L, N\right)$. Then using the method of interval bisection after 2*L*+2 steps we find $\left[h_{k-1}, h_k\right]$ with $\varepsilon = h_k - h_{k-1} < 2^{-2L-2}$ (see [10, 11]), where for $h = h_k$ system (25) has a solution with the property from Theorem 5 and for $h = h_{k-1}$ system (25) does not have such a solution. Based on results from [10, 11] we can find the exact solution h^* in polynomial time by using a special approximate procedure. Note that for problem (6)-(8) it is sufficient to find h_k with precision $\varepsilon \in \left[0, 1/2\right)$, because h^* is integer and therefore it can be found from h_k by simple round off procedure.

If h^* and a solution $\mu_1^*, \mu_2^*, ..., \mu_p^*, \lambda_1^*, \lambda_2^*, ..., \lambda_n^*$ of system (26) which satisfies conditions of Theorem 5 are known, then find $s_j^* = \sum_{i=1}^{p} d_{ij}\mu_i$, $j = \overline{0, m}$. After that solve system (33) and find the solution $y^* \in Y$. Then fix $y = y^*$ in (1) and solve the linear program: to minimize $z = xCy^* + c'x + c''y^*$ on subject (2). In such a way we find $\left(x^*, y^*\right)$.

The proposed algorithm can be used for a large class of integer programming problems and some new results related to computational complexity of the considered problem can be obtained on the basis of such approach.

References

1. Altman M. Bilinear Programming. *Bul. Acad. Polan. Sci., Ser. Sci. Math. Astron. et Phis.*, (1968), 16(9), 741-746.
2. Lozovanu D. Extremal-Combinatorial Problems and Algorithms for their Solving (in Russian). Chisinau, Stiinta, (1991).
3. Lozovanu D. Parametrical approach for studying and solving bilinear programming problem. *Proceedings of the International Workshop on Global Optimization, San Jose, Almeria, Spain, September 18-22, (2005)*, 165-170.
4. Karp R. Reducibility among combinatorial problems. In Miler R. and Thatcher J. editors, Complexity of Computer Computations. (1972), 83-103.
5. Garey M., Johnson D. Computers and Intractability: A Guide to the Theory of NP-Complectness, Freeman, (1979).
6. Chernicov S.N. Linear Inequalities (in Russian). Moscow: Nauka, (1968).
7. Lozovanu D. Duality Principle for Systems of Linear Inequalities with a Right Part that Depends on Parameters (in Russian). *Izv. AN SSSR, ser. Techn. Cyb., (1987)*, 6, 3-13.
8. Farkas J. Uber die Theorie der einfachen Ungleichungen. *J. reine and angew. Math., (1901)*, 124, 1-24.
9. Karmanov V.G. Mathematical Programming (in Russian). Moscow: Nauka, (1990).
10. Khachian L.G. Polynomial time algorithm in linear programming. USSR, Computational Mathematics and Mathematical Physics, (1980), 20, 51-58.
11. Khachian L.G. On exact solution of the system of linear inequalities and linear programming problem. USSR, Computational Mathematics and Mathematical Physics, (1982), 22, 999-1002.

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