Analyzing Fuzzy Extrema

Angel Garrido
Department of Fundamental Mathematics
Faculty of Sciences, UNED University
Senda del Rey, 7, Madrid (SPAIN)
algbmv@telefonica.net http://www.uned.es

Abstract

Many techniques of Classical Analysis remain useful, but we need some new tools, adapted and improved, in essential questions, such as the optimization of crisp/fuzzy functions in fuzzy/crisp domains, for instance.

In this search we need to introduce two new fuzzy sets: the Maximizing Fuzzy Set and the minimizing Fuzzy Set, associated to the function (crisp or fuzzy). We define both sets reflecting the possibility of reaching the maximum / minimum value in each point of the universal set where f is defined. To obtain such values we use their degrees of membership. Furthermore, we show their application by some detailed examples.

Keywords: *Fuzzy Set Theory, Fuzzy Measure Theory, Fuzzy Analysis, Artificial Intelligence.*

1. Introduction

From the introduction of Fuzzy set concept and some related ideas, as the possibility distribution (for Zadeh), until fuzziness, their measure, and so on, in papers of people as Dubois and Prade, some new branch of mathematics are increasing, subsuming as a particular case the precedent analysis, for instance. We attempt study here the fuzzy extrema, with their relation and implications.

2. Maximizing and minimizing fuzzy sets

Let M_f be a real function defined on the universal set, U.

If we consider its supremum and infimum, *sup (f)* and *inf (f)*, it is possible to define a new set: M_f the maximizing fuzzy set.

Because taking the range of *f*, that is, the interval of the real line:

$$R_f = [\inf(f), \sup(f)]$$

We will assign to each $x \in U$, a numerical value in the unit interval: $\mu_{M_f}(x)$, and so, to introduce the new set:

$$M_f = \{x \mid \mu_{M_f}(x)\}_{x \in U}$$

With:

$$\mu_{M_f}(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)}$$

Obviously, it shows the proportion between the distance of the function value from the infimum and the width of the range. Therefore, the relative normalized position of f(x) in:

$$R_f \subset \mathrm{R}$$

So, we define the new fuzzy set, M_f, by the possibility that *x* takes the maximum value of *f*.

In a similar way, we can introduce the minimizing fuzzy set, denoted m_f. By symmetry, we say that it is the Maximizing fuzzy set of - *f*:

$$m_f = m(f) = M(-f) = M_{-f}$$

Equivalent to define the membership degree of each element of its domain, $D_f \subset U$, as:

$$\mu_{m_f}(x) = \frac{(-f)(x) - \inf(-f)}{\sup(-f) - \inf(-f)}$$

But according the known results:

sup(-f) = - inf(f)

-inf(-f) = sup(f)

(-f)(x) = - {f(x)}

we have:

$$\mu_{m_f}(x) = \frac{\sup(f) - f(x)}{\sup(f) - \inf(f)}$$

As in the precedent case, the sequence of steps will be:

$$x \to \mu_{m_f}(x) \to m_f = \{x \mid \mu_{m_f}(x)\}$$

Similarly, it shows the proportion between the distance from the function value to the supremum and the width of the range.

As an example, we can consider the sinus function in the domain [0, 2].

In such case, the Maximizing Fuzzy Set, M_f, and the minimizing Fuzzy Set, m_f, are defined by their membership function:

$$\mu_{M_f}(x) = \frac{\sin x - \inf(\sin x)}{\sup(\sin x) - \inf(\sin x)} = \frac{\sin x - (-1)}{1 - (-1)} = \frac{1}{2} + \frac{\sin x}{2}$$

$$\mu_{m_f}(x) = \frac{\sup(f) - f(x)}{\sup(f) - \inf(f)} = \frac{\sup(\sin x) - \sin x}{\sup(\sin x) - \inf(\sin x)} = \frac{1 - \sin x}{1 - (-1)} = \frac{1}{2} - \frac{\sin x}{2}$$

Where $\mu_{M_f}(x)$ and $\mu_{m_f}(x)$ denote the possibility of f taking in *x* its maximum/minimum value. So, for instance, if $x = \frac{\pi}{2}$, then:

$$f(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

with:

$$\mu_{M_f}\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{\sin\left(\frac{\pi}{2}\right)}{2} = 1$$

$$\mu_{m_f}\left(\frac{\pi}{2}\right)=\frac{1}{2}-\frac{\sin\left(\frac{\pi}{2}\right)}{2}=0$$

Therefore, the possibility of f taking its maximum value in $\frac{\pi}{2}$ is maximal, and its minimum is minimal.

Also there exist points in the domain with values of the membership degrees, $\mu_{M_f}(x)$ and $\mu_{m_f}(x)$ in the interior of the unit interval. So, if we take: $x=\pi$, then:

$$\mu_{M_f}(\pi)=\frac{1}{2}+\frac{\sin\pi}{2}=\frac{1}{2}$$

$$\mu_{m_f}(\pi)=\frac{1}{2}-\frac{\sin\pi}{2}=\frac{1}{2}$$

The possibilities of reaching the maximum and the minimum of f in $x=\pi$ are the same, 0.5, and precisely, then, both with the greatest uncertainty. This reflects their maximal distance to both situations.And other interesting example:

Let *f(x)* = x^2 be a crisp function with non-fuzzy domain:

$$D_f=[-1, 2]$$

Define its Maximizing and minimizing fuzzy sets and calculate the possibility of $x=0$.
The Maximizing fuzzy set is defined by:

$$\mu_{M_f}(x)=\frac{x^2-0}{4-0}=\left(\frac{x}{2}\right)^2$$

and the mimimizing fuzzy set by:

$$\mu_{m_f}(x)=\frac{4-x^2}{4-0}=1-\left(\frac{x}{2}\right)^2$$

Therefore, in $x=0$ we have:

$$\mu_{M_f}(0)=\left(\frac{0}{2}\right)^2=0$$

$$\mu_{m_f}(0)=1-\left(\frac{0}{2}\right)^2=1$$

This signifies that the possibility of $f(0)=0$ to be the maximum/minimum value of *f* is:

Null / Full, respectively

Turning to the relationship between both fuzzy sets, Maximizing and minimizing fuzzy set, we have:

$$\mu_{M_f}(x)+\mu_{m_f}(x)=\frac{f(x)-\inf(f)}{\sup(f)-\inf(f)}+\frac{\sup(f)-f(x)}{\sup(f)-\inf(f)}$$

that is:

$$\mu_{M_f}(x)+\mu_{m_f}(x)=1$$

or equivalently:

$$\mu_{M_f}(x)=1-\mu_{m_f}(x)$$

Therefore, if the value of μ_{M_f} increase, then decrease the associated value of μ_{M_f}, and respectively. Because the membership degree of each x to M_f is complementary to one with the membership degree of such x to m_f.

So, according we advance to the point of maximum value for *f*, we increment the distance until the point of minimum value, and with this, their possibility.

More connected results can be, for instance:

$$\mu_{c(M_f)}(x) = 1 - \mu_{M_f}(x)$$

$$\mu_{c(m_f)}(x) = 1 - \mu_{m_f}(x)$$

$c(M_f)$ and $c(m_f)$ are the complementary of the Maximizing and minimizing fuzzy sets. Then, adding both expressions:

$$\mu_{c(M_f)}(x) + \mu_{c(m_f)}(x) = \left[1 - \mu_{M_f}(x)\right] + \left[1 - \mu_{m_f}(x)\right] = 2 - \left[\mu_{M_f}(x) + \mu_{m_f}(x)\right] = 1$$

whereas:

$$c(M_f) \cup c(m_f) = c(M_f \cap m_f)$$

This implies:

$$\mu_{c(M_f)\cup c(m_f)} = 1 - \mu_{M_f \cap m_f}$$

And also:

$$c(M_f) \cap c(m_f) = c(M_f \cup m_f)$$

being then:

$$\mu_{c(M_f)\cap c(m_f)} = 1 - \mu_{M_f \cup m_f}$$

In consequence:

$$\mu_{c(M_f)\cup c(m_f)} + \mu_{c(M_f)\cap c(m_f)} = 2 - \left[\mu_{M_f \cup m_f} + \mu_{M_f \cap m_f}\right] = 1$$

Furthermore, as:

$$\mu_{m_f} = \mu_{M_{(-f)}}$$

we have that:

$$\mu_{M_f}(x) + \mu_{M_{(-f)}}(x) = 1$$

that is,

$$\mu_{m_f}(x) = 1 - \mu_{M_{(-f)}}(x)$$

And the similar result:

$$\mu_{M_{(-f)}} = 1 - \mu_{m_f}$$

Analogously:

$$\mu_{M_f}(x)-\mu_{m_f}(x)=\frac{f(x)-\inf(f)}{\sup(f)-\inf(f)}-\frac{\sup(f)-f(x)}{\sup(f)-\inf(f)}=\frac{2f(x)}{\sup(f)-\inf(f)}-1$$

The denominator:

sup(f) - inf(f)

is a fixed value, for each *f*. Therefore, only varies in this quotient *f(x)*.
So, in the difference between membership degrees to Maximizing and minimizing fuzzy sets of *f:*

$$\mu_{M_f}(x)-\mu_{m_f}(x)$$

we observe an almost linear dependence of the values of such function.
It is suffices to be linear establishing the new difference:

$$\mu_{M_f}(x)-\mu_{m_f}(x)+1=L[f]$$

Observe that also we have:

$$\mu_{M_f}(x)\cdot\mu_{m_f}(x)=-\left(\frac{f(x)-\inf(f)}{f(x)+\inf(-f)}\right)$$

This implies that:

$$\left[\mu_{M_f}(x)\cdot\mu_{m_f}(x)\right]'= f'(x)\left[\frac{\{\inf(f)+\inf(-f)\}}{-\{f(x)+\inf(-f)\}^2}\right]$$

where, once *f* established, only is varying *f(x)* and the value of its derivative in such point, *f ´(x).*

3. Final note

I hope to contribute with these remarks, and advancing through future papers, to the study of fuzzy extrema. Also, in general, working on Fuzzy Differentiation and Fuzzy Integration. Therefore, on Fuzzy Measure Theory.

References:

[1] Dubois and Prade (eds.), *Fundamentals of Fuzzy Sets*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, Netherlands, 2000.
[2] Dubois and Prade. *Possibility, Theory*, Plenum Press, New Jersey, 1988.
[3] Garrido, Heuristic Functions, Primitive Recursion and Recursivity ICCMSE, Loutraki-Khorintos. *Lecture Series on Computer Sciences and Computacional Science,* Vol. 4A, pp 234-238, VSP-Brill Publishing Co., Netherlands, 2005.
[4] Garrido, Special Functions in A. I. CDDE, Uni Gdansk (Poland), 2005, *Opuscula Mathematica.*
[5] Pedrick, *Fuzzy Control and Fuzzy Systems*, Wiley and Sons Co., New York, 1993.
[6] Wang, *A Course in Fuzzy Systems and Control*, Prentice-Hall, New Jersey, 1997.

Article received: 2008-06-05