Supersymmetry in the Dirac Equation and the Coulomb Potential

Anzor Khelashvili
Tbilisi State University, Tbilisi, Georgia

Abstract:
It is shown that the requirement of invariance of the Dirac Hamiltonian under some kind of Witten's superalgebra picks out the Coulomb potential only. The problem in the arbitrary higher dimensions is also considered. It is derived that the traditional view on the Coulomb potential is to be changed in the context of N=2 supersymmetry.

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In **1916** Arnold **J.W. Sommerfeld** obtained the energy spectrum formula for the hydrogen atom. This formula reads as follows [1]:

$$E=m\left\{1+\frac{(Z\alpha)^2}{\left(n-|\kappa|+\sqrt{\kappa^2-(Z\alpha)^2}\right)}\right\}^{-1/2}$$

Here κ is the eigenvalue of the Dirac operator

$$K=\beta\left(\vec{\Sigma}\cdot\vec{l}+1\right), \qquad |\kappa|=j+1/2$$

which commutes with the Dirac Hamiltonian

$$H=\vec{\alpha}\cdot\vec{p}+\beta m-\frac{a}{r}, \qquad a\equiv Ze^2=Z\alpha$$

$\vec{l}$ is the angular momentum vector, $\vec{\alpha}$ and β are usual Dirac matrices and $e^2=\alpha$ is the fine structure constant, while $\vec{\Sigma}$ is the electron spin matrix $\vec{\Sigma}=diag(\vec{\sigma},\vec{\sigma})$.

Eigenvalue $|\kappa|=j+1/2$. Let us mention the degeneracy of spectrum with respect to **signs** of κ.

It is surprising that for other solvable potentials the degeneracy with respect to signs of κ does not take place.

Physically this degeneracy leads to the forbidden of the Lamb shift. Indeed, $\kappa=j+1/2$ is positive, when $j=l+1/2$, which corresponds to states $\left(S1/2, P3/2, etc.\right)$, while negative $\kappa=-\left(j+1/2\right)$,when $j=l-1/2$, corresponds to $\left(p1/2, d3/2, etc.\right)$. So the absence of the Lamb shift $\left(E_{S1/2}-E_{P1/2}\right)$ is a consequence of above mentioned degeneracy $\kappa\to-\kappa$.

Supersymmetry of the Dirac Hamiltonian

for General Central Potenetials

Let us consider the Dirac Hamiltonian again but now for arbitrary central potential $V(r)$:

$$H=\vec{\alpha}\cdot\vec{p}+\beta m+V(r)$$

In this form scalar function $V(r)$ is the fourth component of the Lorentz 4-vector. It is easy task to verify, that the K-operator commutes with the Dirac Hamiltonian for arbitrary $V(r)$:

$$[K,H]=0$$

Suppose , that there is some operator Q_1 , which anticommutes with K

$$\{Q_1,K\}=0$$

Then it is evident that the following operator

$$Q_2 = i\frac{Q_1K}{\sqrt{K^2}}$$

anticommutes with K and with Q_1 as well

$$\{Q_2,K\}=0$$

Moreover

$$\{Q_1,Q_2\} = 0, \qquad Q_1^2=Q_2^2\equiv\tilde{H}$$

One can construct new operators $Q_\pm=(Q_1\pm iQ_2)$ .They are nilpotent: $Q_\pm^2=0$ **and** $\{Q_+,Q_-\}=2\tilde{H}$. So we have $N=2$ superalgebra (Witten's algebra,[2]), where $\widetilde{H}$ plays the role of Witten's Hamiltonian.

Supersymmetry means

$$\left[Q_i, H\right]=0 \qquad (i=1,2)$$

So, we want to construct the operator, say Q_1 ,that commutes with H , and anticommutes with K :

$$\left[Q_1, H\right]=0, \quad \left\{Q_1, K\right\}=0$$

The γ^5 matrix has this property. What else?

<u>**Theorem**</u>[3]: ***Let*** $\vec{V}$ ***be a vector with respect to the angular momentum*** $\vec{l}$ ***, i.e.***

$$\left[l_i, V_j\right]=i\varepsilon_{ijk}V_k$$

In the vector product form it can be written as

$\vec{l}\times\vec{V}+\vec{V}\times\vec{l}=2i\vec{V}$.

Suppose also that this vector is perpendicular to $\vec{l}$

$$(\vec{l}\cdot\vec{V})=(\vec{V}\cdot\vec{l})=0$$

Then K ***anticommutes with operator*** $\left(\vec{\Sigma}\cdot\vec{V}\right)$***, which is scalar with respect of the total*** $\vec{J}=\vec{l}+\frac{1}{2}\vec{\Sigma}$ ***momentum.***

In general needed operator (so-called $K-odd$) is of kind $\hat{O}\left(\vec{\Sigma}\cdot\vec{V}\right)$, where $\hat{O}$ is commuting with K .

Useful relation for future is

$$K\left(\vec{\Sigma}\cdot\vec{V}\right)=-i\beta\left(\vec{\Sigma}, \frac{1}{2}\left[\vec{V}\times\vec{l}-\vec{l}\times\vec{V}\right]\right), \qquad (*)$$

We have the following physical vectors at hand

$\vec{V}=\hat{\vec{r}}$ (unit radial vector), $\vec{V}=\vec{p}$ (linear momentum) and $\vec{V}=\vec{A}$ (LRL vector) [4]

$$\vec{A}=\hat{\vec{r}}-\frac{i}{2ma}\left[\vec{p}\times\vec{l}-\vec{l}\times\vec{p}\right]$$

According to ($*$) there appears a relation between above three odd operators

$$\vec{\Sigma}\cdot\vec{A}=\vec{\Sigma}\cdot\hat{\vec{r}}+\frac{i}{ma}\beta K\left(\vec{\Sigma}\cdot\vec{p}\right)$$

Therefore our choice will be

$$\vec{\Sigma}\cdot\hat{\vec{r}} \quad \text{and} \quad K\left(\vec{\Sigma}\cdot\vec{p}\right)$$

We construct the most general expression from them

$$Q_1=x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)+ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right)+ix_3K\gamma^5 f(r)$$

Let's calculate

$$\left[Q_1,H\right]=\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left\{x_2V'(r)-x_3f'(r)\right\}+$$

$$+2i\beta K\gamma^5\left\{\frac{x_1}{r}-mf(r)x_3\right\}=0$$

It follows from these matrix equations,that

$$x_2V'(r)=x_3f'(r)$$

$$x_3mf(r)=\frac{x_1}{r}$$

$$V(r)=\frac{x_1}{x_2}\frac{1}{mr}$$

Therefore the ***only central potential for which the Dirac Hamiltonian has an additional symmetry (*** $N=2$ ***supersymmetry in the above mentioned sense) is a Coulomb potential.***

Physical Meaning and Some Applications of the Johnson – Lippmann Operator

In order to determine a physical meaning of Q_1 operator, note, that using $(*)$, this operator may be rewritten in the following form:

$$Q_1=\vec{\Sigma}\left\{\hat{\vec{r}}-\frac{i}{2ma}\beta\left(\vec{p}\times\vec{l}-\vec{l}\times\vec{p}\right)\right\}+\frac{i}{mr}K\gamma^5$$

In the nonrelativistic limit, when $\beta \to 1, \quad \gamma^5 \to 0$,

$$Q_1 \to \vec{\Sigma} \cdot \vec{A}$$

i.e. projection of LRL vector on Spin direction.

$A \equiv Q_1$ is called the **Johnson and Lippmann (JL)** operator, which was published only in the form of brief abstract in 1950 in Phys. Rev.[5], but not anywhere. One of the most curious fact of physics history of 20-th century.
The first publication about the explicit derivation of this operator appeared in 2005 , by us[6], see **T.Khachidze and A.Khelashvili**, Mod.Phys. Letters, 20,2277-2282(2005).

As for further application, let us calculate the square of JL operator. The result is as follows

$$A^2 = 1 + \left(\frac{K}{a}\right)^2 \left(\frac{H^2}{m^2} - 1\right)$$

Because all operators in this relation commute with each others, one can replace them by their eigenvalues. Therefore one obtains energy spectrum pure algebraically after specifying spectrum of A^2 . Since A^2 is positively defined (it is Witten's Hamiltonian) the minimal eigenvalue of A^2 is zero. For this eigenvalue upper relation gives precisely the ground state energy of hydrogen atom,

$$E_0 = m\left(1 - \frac{(Z\alpha)^2}{\kappa^2}\right)^{1/2}$$

Full spectrum can be easily derived by well–known ladder procedure which is known as a **shape invariance**. This cause to change

$$\sqrt{\kappa^2 - a^2} \to \sqrt{\kappa^2 - a^2} + n - |\kappa|$$

It leads to the familiar **Sommerfeld** formula displayed above.

Inclusion of the Lamb shift terms [7],

$$\Delta V_{Lamb} \approx \frac{4\alpha^2}{3m^2}\left(\ln\frac{m}{\mu} - \frac{1}{5}\right)\delta^3(\vec{r}) + \frac{\alpha^2}{2\pi m^2 r^3}(\vec{\Sigma} \cdot \vec{l})$$

found by calculating of radiative corrections to the photon propagator and photon – electron vertex function, into the Dirac Hamiltonian breaks commutativity of A with H . However it is evident that without radiative corrections, terms like ΔV_{Lamb} do not appear in the Dirac Hamiltonian, as in the one–electron theory, and as long as only Coulomb potential is considered, the appearance of the Lamb shift should be always forbidden.

Let us underline that the hidden (dynamical) symmetry, associated to the Coulomb potential, governs a wide range of physical phenomena from planetary motion till fine and hyperfine structure of atomic spectra.

The Lorentz – Scalar Potential in the Dirac Equation

Let us consider the full Hamiltonian

$$H=\vec{\alpha}\cdot\vec{p}+\beta m+V(r)+\beta S(r)$$

Repeating all above considerations, one can find the following general form of the symmetry operator

$$Q_1 \equiv \mathrm{X} = x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)+x_1'\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)H +$$

$$+ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right)+ix_3K\gamma_5 f_1(r)+ix_3'K\gamma_5\beta f_2(r)$$

Requiring supersymmetry ,as above, one obtains, that

$$S(r)=\frac{x_1'}{x_2 r}$$

S***o, the scalar potential must be Coulombic.***

Moreover solving for $V(r)$, one derives

$$V(r)=\frac{x_1}{r}\frac{1}{x_2(m+S)-\frac{x_1'}{r}},$$

At last, using here derived expression for $S(r)$, we find

$$V(r)=\frac{x_1}{x_2 mr}$$

Therefore we make sure that the $N=2$ supersymmetry in the above described content is the symmetry of the Dirac Hamiltonian only for Coulomb potential (for any general combination of Lorentz–scalar and 4th component of a Lorentz–vector).

Now if we take into account above obtained relations and use them into the general expression for X , one can reduce it to more compact form

$$\mathrm{X}=\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left(ma_V+Ha_S\right)-$$

$$-iK\gamma_5\left(H-\beta m\right)$$

It is not so evident the Physical Interpretation in terms of LRL operator !!

Algebraic Derivation of the Spectrum of the Dirac Hamiltonian for an Arbitrary Combination of the Lorentz-Scalar and Lorentz-Vector Coulomb Potential

We had

$$Q_1 = X, \quad Q_2 = i\frac{XK}{|\kappa|}$$

Then the anticommutativity $\{X, K\} = 0$, yields

$$\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2 \equiv \tilde{H}$$

Let us introduce the SUSY ground state $|0\rangle$:

$$\tilde{H}|0\rangle = X^2|0\rangle = 0 \quad \to X|0\rangle = 0$$

$$H = m\left[\left(\vec{\alpha}\cdot\hat{\vec{r}}\right)a_S + iK\right]^{-1}\left[iK\beta - a_V\left(\vec{\alpha}\cdot\hat{\vec{r}}\right)\right]$$

Diagonalization of this Hamiltonian may be achieved by using two Foldy-Wouthuysen [8] like transformations. This gives the following formula for the energy spectrum [9] :

$$E = m\left\{\frac{-a_S a_V}{a_V^2 + \left(n - |k| + \gamma\right)^2} \pm \sqrt{\left(\frac{a_S a_V}{a_V^2 + \left(n - |\kappa| + \gamma\right)^2}\right)^2 + \frac{\left(n - |\kappa| + \gamma\right)^2 - a_S^2}{a_V^2 + \left(n - |\kappa| + \gamma\right)^2}}\right\}$$

$$\gamma^2 = \kappa^2 - a_V^2 + a_S^2$$

which coincides with a correct expression, obtained by explicit solving of the Dirac equation [10]. So, this problem is ***totally integrable.***

Arbitrary dimensions

Generalization for Arbitrary Dimentions of the JL operator was done by Katsura G.H. and Aoki H. [11] , Journ. Math. Phys. 47,032302(2006).

They define a new operator

$$A = \gamma^{D+1}\gamma^0\gamma^i\frac{x^i}{r} - \frac{i}{Zm\alpha}K\gamma^{D+1}\left(H - \gamma^0 m\right)$$

Here they assumed that there is a matrix, γ^{D+1} a pseudo-scalar, generalization of γ^5 in (3+1)-dimensions, and which satisfies the following relations:

$$\left(\gamma^{D+1}\right)^{+} = \gamma^{D+1}, \qquad (\gamma^{D+1})^2 = 1,$$

$$\left\{\gamma^{D+1}, \gamma^{\mu}\right\} = 0$$

These relations coincide with that of ordinary γ^5 matrix.

γ^{D+1} is constructed from $\left\{\gamma^0, \gamma^1, ..., \gamma^D\right\}$, but its actual form depends on whether the spatial dimension is even or odd. They found the above derived relation

$$A^2 = 1 + \left(\frac{K}{Z\alpha}\right)^2 \left(\frac{H^2}{m^2} - 1\right)$$

and Witten's algebra.

It is evident from this consideration, that the algebra of gamma matrices is unchanged in arbitrary dimensions. Therefore, we can repeat all our arguments, given above, and prove, that our conclusions are valid also in this case, i.e. only for "Coulomb" $1/r$ potential Dirac Hamiltonian is supersymmetric among all central potentials in the abovementioned sense. But $1/r$ is a Coulomb potential only in 3-dimensions. $1/r$ behaviour of a potential is necessary for closeness of orbits in arbitrary dimensions, as was mentioned by Errenfest at the beginning of 20th century.

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