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On one a Matrix of Fundamental Solutions of the Equation of Dynamic Elasticity Theory

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Abstract:
A matrix of fundamental solutions of the equation of dynamic elasticity theory is constructed for a homogeneous isotropic medium in the case, when a concentrated force changes at time exponentialy-periodicaly. Properties of the indicated matrix and its corresponding stress tensor are investigated. Besides, a way of calculation of their elements is given.

Keywords: *Matrix of fundamental solutions; Dynamic elasticity theory; Stress tensor.*

1. Introduction

Fundamental solutions play important role not only at solving boundary problems by method of fundamental solutions and at studying boundary problems with the help of integral equations, but also at solving some particular problems of mathematical physics [1, 2].

The main restriction at using the method of fundamental solutions from the point of view of its practical realization consists in necessity of knowledge of the form of the fundamental solution. This drawback essentially restricts the range of application of the method of fundamental solutions. Essentially this method is restricted only by a class of well known differential operators, for which due to the effort of mathematicians of many generations fundamental solutions are obtained.

In mathematical physics often arise problems, in which differential equations are "very close" to classical equations whose fundamental solutions are known. And in spite of closeness of these equations, generally, fundamental solutions of one of them cannot be used for solution of the another one (see [3]). Therefore it is natural to try firstly to get fundamental solution of a concrete differential equation and then use it for approximate solution of boundary problems.

Finding of the fundamental solution represents one of the most important and difficult problem of theory of equations with partial derivatives. With its help the analytically of the solution to the equation with analytic coefficients can be proved (see [4]). Moreover, fundamental solutions are essentially used in investigation of character of boundary problems, which may be stated for this equation. Therefore finding of the fundamental solutions attracted attention of mathematicians long ago.

Hilbert and Hedrik [4], also Picard and Holmgren [4] dealt with these questions for second order linear elliptic equations with analytic coefficients. For elliptic equations of higher order, when the equation has constant coefficients and contains only derivatives of higher order, the problem of finding corresponding fundamental solutions is fully solved in the Somigliana's work [4].

In Levi's work [4] the existence of fundamental solutions of high order equations of elliptic type is proved. This proof is constructive in the sense that it gives algorithm of construction of fundamental solution also. In this work the case of equations with a large number of variables and equations system is considered as well.

Further Picard [3], Giraud [3], Miranda [5], Bitsadze [3] and others dealt with general questions of fundamental solutions.

The method of fundamental solutions gets special elegancy and simplicity if the fundamental solutions are constructed explicitly (in elementary functions). This is the reason of interest in such constructions of fundamental solutions.

The theory of elasticity represents scientific basis and mathematical apparatus of dynamic seismology, studying displacements at sufficiently quick processes inside the Earth. It gives possibility of strict statement of the problem of seismology and study of their correctness.

2. On a Matrix of Fundamental Solutions

It is well-known [1,6] that the basic motion equation of a homogeneous isotropic and elastic body $D\ (D \subset E_3)$ has the form (in the vector form)

$$\mu\Delta U(x,t) + (\lambda+\mu)\mathrm{graddiv}U(x,t) + \Phi(x,t) = \rho\frac{\partial^2 U(x,t)}{\partial t^2}, \tag{2.1}$$

$$x \in D, \quad t \in]-\infty;\infty[,$$

and in projections:

$$\begin{aligned} &\mu\Delta U_1(x,t) + (\lambda+\mu)\frac{\partial\Theta(x,t)}{\partial x^1} + \Phi_1(x,t) = \rho\frac{\partial^2 U_1(x,t)}{\partial t^2}, \\ &\mu\Delta U_2(x,t) + (\lambda+\mu)\frac{\partial\Theta(x,t)}{\partial x^2} + \Phi_2(x,t) = \rho\frac{\partial^2 U_2(x,t)}{\partial t^2}, \\ &\mu\Delta U_3(x,t) + (\lambda+\mu)\frac{\partial\Theta(x,t)}{\partial x^3} + \Phi_3(x,t) = \rho\frac{\partial^2 U_3(x,t)}{\partial t^2}. \end{aligned} \tag{2.2}$$

In (2.1) and (2.2): λ and μ are the Lame elastic constants, which characterize elastic properties of the given body, they are defined experimentally and for all real bodies $\lambda > 0, \mu > 0$; ρ is the density of the body D; x^1, x^2, x^3 are the coordinates of the point $x \in D$ (before its deformation); t is a time; $U(x,t) = (U_1, U_2, U_3)$ is the displacement vector of the point $x \in D$ at moment t; $\Phi(x,t) = (\Phi_1, \Phi_2, \Phi_3)$ is the vector of volume force in time t (calculated on the unit of a volume of the body D); Δ is the Laplace operator

$$\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2},$$

and the function $\Theta(x,t)$ is defined by the formula

$$\Theta(x,t) = divU(x,t) = \frac{\partial U_1(x,t)}{\partial x^1} + \frac{\partial U_2(x,t)}{\partial x^2} + \frac{\partial U_3(x,t)}{\partial x^3}.$$

Let a homogeneous isotropic and elastic body with parameters λ, μ, ρ fill the Euclidean space E_3 and the concentrated (point) unit force

$$E^j(z,t) = E^j(z)f(t) = (\delta_{1j}, \delta_{2j}, \delta_{3j})f(t) \tag{2.3}$$

be applied at the point $z(z^1, z^2, z^3) \in E_3$, which is directed along the axis $Ox^j\ (j = 1,2,3)$, changes at time by the law $f(t)$ and begins action at moment $t = 0^+$. It is evident, in this case, that the force $\Phi^j(x,t)$, acting on E_3 will be

$$\Phi^j(x,z,t) = E^j(z,t)\delta(x-z) = (\delta_{1j}, \delta_{2j}, \delta_{3j})f(t)\delta(x-z), \quad x, z \in E_3. \tag{2.4}$$

In (2.3) and (2.4) $f(t)$ is known function and $f(t) = 0$ for $t \le 0$; $\delta_{kj}\ (k, j = 1,2,3)$ is the Kronecker symbol

$$\delta_{kj} = \begin{cases} 1 & for \quad k = j, \\ 0 & for \quad k \ne j; \end{cases}$$

$\delta(x-z)$ is the Dirac function and is defined with the following conditions:

$$\int_D f(x)\delta(x-z)dx = \begin{cases} 0 & for \;\; z \notin D, \\ f(z) & for \;\; z \in D, \end{cases}$$

$$\int_{E_3} f(x)\delta(x-z)dx = f(z),$$

where $D \subset E_3$, and $f(x)$ is an arbitrary continuous function at the point $x = z$.

By means of the matrix differential operator $A(\frac{\partial}{\partial x}, t)$ the system (2.2) (or the equation (2.1)) for the force (2.4) has the form [1]

$$A(\frac{\partial}{\partial x}, t)U^j(x,z,t) + \Phi^j(x,z,t) = \Theta, \;\; t \in]-\infty,\infty[, \quad x,z \in E_3, \tag{2.5}$$

where

$$A(\frac{\partial}{\partial x}, t) = \left\| A_{kj}(\frac{\partial}{\partial x}, t) \right\|_{3\times 3} \quad (k, j = 1,2,3),$$

$$A_{kj}(\frac{\partial}{\partial x}, t) = \delta_{kj}[\mu\Delta(\frac{\partial}{\partial x}) - \rho\frac{\partial^2}{\partial t^2}] + (\lambda+\mu)\frac{\partial^2}{\partial x^k \partial x^j}.$$

In (2.5) $U^j(x,z,t) = (U_1^j, U_2^j, U_3^j)$ is the displacement of the point $x \in E_3$ (at moment t), caused by action of the force $E^j(z,t)$; $\Theta = (0,0,0)$ is the zero vector.

From (2.5), it is seen that $U^j(x,z,t)$ represents the fundamental solution of the homogeneous equation

$$A(\frac{\partial}{\partial x}, t)U(x,t) = \Theta, \quad x \in E_3, \; t \in]-\infty,\infty[, \tag{2.6}$$

corresponding to the force $\Phi^j(x,z,t)$.

The matrix of fundamental solutions of the equations (2.1), (2.2), (2.6) or of the operator $A(\frac{\partial}{\partial x}, t)$ is called the matrix (see [1])

$$\Psi(x,z,t) \equiv (\Psi^1, \Psi^2, \Psi^3) = \left\| \Psi_{kj}(x,z,t) \right\|_{3\times 3}, \tag{2.7}$$

where

$$\Psi^j(x,z,t) \equiv U^j(x,z,t) = (\Psi_{1j}, \Psi_{2j}, \Psi_{3j}) \quad (k, j = 1,2,3).$$

In designation of $\Psi_{kj}(x,z,t)$ the index k denotes a component of a displacement vector of the point x. The index j indicates the cause of a displacement, namely indicates the point force $E^j(z,t)$, directed along the axis Ox^j $(j = 1,2,3)$. x denotes a moving point, and z denotes the point at which a point force is applied.

By the Cauchy-Binet formulae it is easy to see that if each column $\Psi^j(x,z,t)$ $(j = 1,2,3)$ of matrix (2.7) represents a solution of equation (2.5), then

$$A(\frac{\partial}{\partial x}, t)\Psi(x,z,t) + f(t)\delta(x-z)E = O, \quad t \in]-\infty,\infty[, \quad x,z \in E_3, \tag{2.8}$$

where O and E, respectively, are null and unit quadratic matrices, i.e.,

$$O = \left\| O_{kj} \right\|, \quad E = \left\| E_{kj} \right\|_{3\times 3}, \quad O_{kj} = 0, \quad E_{kj} = \delta_{kj}. \tag{2.9}$$

3. A Matrix of Fundamental Solutions for a Concentrated Force

Changing Exponentially-Periodically

Suppose a homogeneous isotropic elastic medium fills the Euclidean space E_3 and the unit concentrated force

$$E^j(z,t) = Re[e^{\tau t} E^j(z)] \tag{3.1}$$

is applied at a point $z = (z^1, z^2, z^3)$, which is directed along the axis Ox^j $(j = 1,2,3)$, i.e., $E^j(z) = (\delta_{1j}, \delta_{2j}, \delta_{3j})$, and begins action at a moment $t = 0^+$. In (3.1) $\tau = \alpha - i\omega$, where i is the imaginary unit, and α and ω are some real numbers $(\omega \geq 0)$. From (3.1) it is seen that the force $E^j(z,t)$ acts in time by the law $f(t) = Re[e^{\tau t}] = e^{\alpha t}\cos\omega t$.

It is clear that for $\omega = 0\ (\alpha \neq 0)$ the force (3.1) will act exponentially, and for $\alpha = 0\ (\omega \neq 0)$ − periodically. For $\alpha \neq 0, \omega \neq 0$ it will be oscillating and will be defferent from pure harmonic oscillation by an amplitude of oscillation changing in time by the law $e^{\alpha t}$ (for $\alpha < 0$ the oscillations will be damping, and for $\alpha > 0$ − reinforcing).

It is evident, under action of the force (3.1), that the force, acting on E_3 will have the form

$$\Phi^j(x,z,t) = Re[e^{\tau t} E^j(z)]\delta(x - z),\ (x, z \in E_3). \tag{3.2}$$

From the point of view of seismology the case of force (3.2) is interesting for investigation of isolated regions by artificial vibrators, which change in time periodically and represent sources for seismic waves. The law (3.2) is not suitable, since it assumes a steady-state of oscillation, when a seismogram of any earthquake gives us a picture of a transient process.

If force (3.2) acts sufficiently long, then the displacement vector $U^j(x,z,t)$ takes the form [1]

$$U^j(x,z,t) = Re[e^{\tau t} U^j(x,z)], \tag{3.3}$$

where $U^j(x,z) = U1^j(x,z) + iU2^j(x,z)$ in general case represents a complex vector function.

Thus, we are interested in such a solution of the equation (2.5), which has form (3.3).

Then from (2.5) we get

$$Re\{e^{\tau t}[A(\frac{\partial}{\partial x}, \tau)U^j(x,z) + (\delta_{1j}, \delta_{2j}, \delta_{3j})\delta(x - z)]\} = \Theta, \tag{3.4}$$

where the matrix differential operator $A(\frac{\partial}{\partial x}, \tau)$ is given by the formula

$$A(\frac{\partial}{\partial x}, \tau) = \left\| A_{kj}(\frac{\partial}{\partial x}, \tau) \right\|_{3\times3} \quad (k, j = 1,2,3), \tag{3.4$_0$}$$

$$A_{kj}(\frac{\partial}{\partial x}, \tau) = \delta_{kj}[\mu\Delta(\frac{\partial}{\partial x}) - \rho\tau^2] + (\lambda + \mu)\frac{\partial^2}{\partial x^k \partial x^j}.$$

For determination the $U^j(x,z)$ from (3.4) we get the equation

$$A(\frac{\partial}{\partial x}, \tau)U^j(x,z) + (\delta_{1j}, \delta_{2j}, \delta_{3j})\delta(x - z) = \Theta. \tag{3.5}$$

By virtue of (3.5) the complex vector function $U^j(x,z)$ represents a fundamental solution of the equation

$$A(\frac{\partial}{\partial x}, \tau)U(x) = \Theta, \quad x \in E_3, \tag{3.6}$$

where $U(x) = (U_1, U_2, U_3)$ is a complex vector function.

Analogously to (2.7) the matrix of fundamental solutions of equation (3.6) or of the operator $A(\frac{\partial}{\partial x},\tau)$ will be the matrix

$$\Psi(x,z,\tau)=\left\|\Psi_{kj}(x,z,\tau)\right\|_{3\times 3}\quad (k,j=1,2,3), \tag{3.7}$$

with complex elements, which satisfies the equation

$$A(\frac{\partial}{\partial x},\tau)\Psi(x,z,\tau)+\delta(x-z)E=O,\ \ (x,z\in E_3), \tag{3.8}$$

where the matrices O and E have form (2.9).

It should be noted that for $\tau=0$ and $\tau=-i\omega$ the equation (3.6) coincides, respectively, with the homogeneous equations of statics and steady oscillation of homogeneous isotropic elastic body [1,2].

The matrix of fundamental solutions of the equation of statics in explicit form is received by Kelvin and for the equation of the steady oscillation is received by V. D. Kupradze and systematically is applied to the theory of boundary problems of oscillation [1].

By virtue of (3.3) and (3.4) it is evident that after construction of the matrix $\Psi(x,z,\tau)$, the matrix $\Psi(x,z,t)$ of fundamental solutions of the dynamic equation (2.6) or of the operator $A(\frac{\partial}{\partial x},t)$ will be given by the formula

$$\Psi(x,z,t)=Re[e^{\tau t}\Psi(x,z,\tau)], \tag{3.9}$$

or

$$\Psi_{kj}(x,z,t)=Re[e^{\tau t}\Psi_{kj}(x,z,\tau)]\ \ (k,j=1,2,3). \tag{3.10}$$

We shall construct the matrix of fundamental solutions of the equation (3.6) by a simple method. With the help of this method at first constructed the matrix of fundamental solutions for the equation of the steady oscillation [1]. The mentioned matrix is called the matrix of Kupradze [1].

In this method, firstly it is necessary that system (3.6) to be reduced to one equation (see [1]). For this, the matrix $\Psi(x,z,\tau)$ is sought in the form

$$\Psi(x,z,\tau)=D(\frac{\partial}{\partial x},\tau)\varphi(x), \tag{3.11}$$

where $D(\frac{\partial}{\partial x},\tau)$ represents the adjoint matrix of $A(\frac{\partial}{\partial x},\tau)$ [1], and $\varphi(x)$ is some unknown scalar function.

It is easy to see that the matrix $D(\frac{\partial}{\partial x},\tau)$ has the form

$$D(\frac{\partial}{\partial x},\tau)=\mu^2(\lambda+2\mu)[\Delta(\frac{\partial}{\partial x})-\tau_2^2]B(\frac{\partial}{\partial x},\tau), \tag{3.12}$$

where the elements of the matrix $B(\frac{\partial}{\partial x},\tau)$ are given by the formula

$$B_{kj}(\frac{\partial}{\partial x},\tau)=\delta_{kj}[\frac{1}{\mu}\Delta(\frac{\partial}{\partial x})-\frac{\tau_1^2}{\mu}]-\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\frac{\partial^2}{\partial x^k\partial x^j}\quad (k,j=1,2,3), \tag{3.13}$$

$$\tau_1=\tau\sqrt{\frac{\rho}{\lambda+2\mu}},\ \ \tau_2=\tau\sqrt{\frac{\rho}{\mu}}. \tag{3.14}$$

Since all elements of the matrix $D(\frac{\partial}{\partial x},\tau)$ contain the factor $\mu^2(\lambda+2\mu)[\Delta(\frac{\partial}{\partial x})-\tau_2^2]$, therefore we can seek the matrix $\Psi(x,z,\tau)$ in the form

$$\Psi(x,z,\tau)=B(\frac{\partial}{\partial x},\tau)\varphi^*(x), \tag{3.15}$$

where

$$\varphi^*(x)=\mu^2(\lambda+2\mu)[\Delta(\frac{\partial}{\partial x})-\tau_2^2]\varphi(x).$$

If we substitute (3.15) in (3.6) and take into account that symbolic

$$\det A(\frac{\partial}{\partial x},\tau)=\mu^2(\lambda+2\mu)[\Delta(\frac{\partial}{\partial x})-\tau_2^2]^2[\Delta(\frac{\partial}{\partial x})-\tau_1^2], \tag{3.16}$$

and

$$\sum_{r=1}^{3}A_{kr}(\frac{\partial}{\partial x},\tau)D_{rj}(\frac{\partial}{\partial x},\tau)=\begin{cases}0 & \text{when } \ k\neq j,\\ \det A(\frac{\partial}{\partial x},\tau) & \text{when } \ k=j,\end{cases} \tag{3.17}$$

then we get the matrix equation

$$[\Delta(\frac{\partial}{\partial x})-\tau_2^2][\Delta(\frac{\partial}{\partial x})-\tau_1^2]E\varphi^*(x)=0. \tag{3.18}$$

Thus, for satisfaction (3.18) we must choose the function $\overset{*}{\varphi}(x)$ so that the following scalar equation

$$[\Delta(\frac{\partial}{\partial x})-\tau_2^2][\Delta(\frac{\partial}{\partial x})-\tau_1^2]\varphi^*(x)=0 \tag{3.19}$$

should be satisfied.

For fulfilment the condition (3.19) in the role of the function $\varphi^*(x)$ we can take a solution of the equations $[\Delta(\frac{\partial}{\partial x})-\tau_2^2]\varphi^*(x)=0$ or $[\Delta(\frac{\partial}{\partial x})-\tau_1^2]\varphi^*(x)=0$. But from (3.8), (3.16), (3.17) it seems that we are interested in such a particular (fundamental) solution of the equation (3.19) for which the condition

$$L(\varphi^*(x))\equiv[\Delta(\frac{\partial}{\partial x})-\tau_2^2][\Delta(\frac{\partial}{\partial x})-\tau_1^2]\varphi^*(x)=-\delta(x-z) \tag{3.20}$$

will be fulfilled, i.e., the solution, which depends on the parameter $z(z^1,z^2,z^3)$, for $x\neq z$ satisfies the equation (3.19) and $|L(\varphi^*(x)|\to\infty$ for $x\to z$.

Remember that a fundamental solution of the Helmholtz equation [7]

$$[\Delta(\frac{\partial}{\partial x})+k^2]h=0$$

is given by the formula

$$h(x,z)=\frac{\exp(\pm ikr)}{4\pi r},\ r=\left\{\sum_{j=1}^{3}(x^j-z^j)^2\right\}^{1/2},\ x\neq z, \tag{3.21}$$

i.e.,

$$[\Delta(\frac{\partial}{\partial x})+k^2]h(x,z)=-\delta(x-z),\ (x,z\in E_3),$$

where k, in general case is a complex number.

On the basis of a pure physical conception from solutions the (3.21) we are interested in such a solution, which is regular at the infinity, i.e., in (3.21) we must take a sign according to the condition

$$\lim_{r\to\infty} h(x,z) = 0. \tag{3.22}$$

Thus it is necessary to take fundamental solution $\varphi^*(x)$ of the equation (3.19) according to the conditions

$$[\Delta(\frac{\partial}{\partial x}) + (i\tau_1)^2]\varphi^*(x,z) = \frac{1}{4\pi}\frac{\exp(\pm\tau_2 r)}{r},$$

$$[\Delta(\frac{\partial}{\partial x}) + (i\tau_2)^2]\varphi^*(x,z) = \frac{1}{4\pi}\frac{\exp(\pm\tau_1 r)}{r}, \tag{3.23}$$

$$x \neq z.$$

From (3.23) we have

$$\varphi^*(x,z) = \frac{1}{4\pi(\tau_1^2 - \tau_2^2)}\frac{\exp(\pm\tau_1 r) - \exp(\pm\tau_2 r)}{r}, \tag{3.24}$$

where by virtue of (3.22) signs before of τ_1 and τ_2 are selected in the following way: if $Re\,\tau < 0$, then we take the upper signs; if $Re\,\tau \geq 0$ $-$ lower.

For a simplicity we rewrite the formula (3.24) as follows

$$\varphi^*(x,z) = \frac{1}{4\pi(\tau_1^2 - \tau_2^2)}\frac{\exp(\tau_1 r) - \exp(\tau_2 r)}{r}, \tag{3.25}$$

where

$$\tau_1 = \tau^*\sqrt{\frac{\rho}{\lambda + 2\mu}},$$

$$\tau_2 = \tau^*\sqrt{\frac{\rho}{\mu}}, \tag{3.26}$$

and

$$\tau^* = -\tau\,\mathrm{sgn}\,\alpha\,, \quad \mathrm{sgn}\,\alpha = \begin{cases} 1 & \text{when} \quad \alpha \geq 0, \\ -1 & \text{when} \quad \alpha < 0. \end{cases}$$

It is clear that the constructed function $\overset{*}{\varphi}(x,z)$ represents the fundamental solution of the equation (3.19), satisfies the equation (3.18) for $x \neq z$, and for $x, z \in E_3$ --- the equation (3.20).

From above mentioned it follows that by virtue of (3.15),(3.16),(3.17) ,(3.20) the matrix $\Psi(x,z,\tau)$ satisfies the matrix equation (3.6) for $x \neq z$, and for $x, z \in E_3$ $-$ the equation (3.8).

If we take into account (3.25),(3.23),(3.13) and the condition

$$\tau_1^2 - \tau_2^2 = -\frac{\rho(\lambda+\mu)\tau^2}{\mu(\lambda+2\mu)},$$

then from (3.15) we get that the elements of the matrix $\Psi(x,z,\tau)$ of the equation (3.6) are given by the formula

$$\Psi_{kj}(x,z,\tau) = \frac{\delta_{kj}}{4\pi\mu}\frac{e^{\tau_2 r}}{r} + \frac{1}{4\pi\rho\tau^2}\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_1 r} - e^{\tau_2 r}}{r} =$$

$$= \frac{\delta_{kj}}{4\pi\mu}\frac{e^{\tau_2 r}}{r} - \frac{(x^k - z^k)(x^j - z^j)}{4\pi\rho\tau^2 r^3}[\tau_2^2 e^{\tau_2 r} - \tau_1^2 e^{\tau_1 r}] + \tag{3.27}$$

$$+\frac{1}{4\pi\rho\tau^2 r^3}[\frac{3(x^k - z^k)(x^j - z^j)}{r^2} - \delta_{kj}][(1-\tau_1 r)e^{\tau_1 r} - (1-\tau_2 r)e^{\tau_2 r}],$$

$$(k, j = 1,2,3),$$

where τ_1, τ_2 are defined by the formulae (3.26), and $\tau = \alpha - i\omega$ (α and $\omega\,(\omega \ge 0)$ are real numbers).

The constructed matrix $\Psi(x,z,\tau)$ of fundamental solutions has the following properties.

Theorem I. *The matrix* $\Psi(x,z,\tau)$ *is symmetric, i.e.,*

$$\Psi_{kj}(x,z,\tau) = \Psi_{jk}(x,z,\tau), \quad (k, j = 1,2,3)$$

and every column (row) $\Psi^j(x,z,\tau)$ *of this matrix represents a solution of the equation*

$$A(\frac{\partial}{\partial x},\tau)\Psi^j(x,z,\tau) + (\delta_{1j},\delta_{2j},\delta_{3j})\delta(x-z) = \Theta \tag{3.28}$$

for $x,z \in E_3$.

A symmetry of the matrix $\Psi(x,z,\tau)$ directly follows from the form of the elements $\Psi_{kj}(x,z,\tau)$ of this matrix (see (3.27)). As for the equality (3.28), its correctness is seen from the matrix equation (3.8), but we can show it directly.

Indeed, let

$$C = A(\frac{\partial}{\partial x},\tau)\Psi(x,z,\tau),$$

then on the basis of (3.4_0),(3.23),(3.26) and the Cauchy-Binet formulae

$$C_{kj} = \sum_{i=1}^{3} A_{ki}(\frac{\partial}{\partial x},\tau)\Psi_{ij}(x,z,\tau), \quad (k, j = 1,2,3)$$

we have

$$C_{kj} = \sum_{i=1}^{3}\{\delta_{ki}[\mu\Delta(\frac{\partial}{\partial x}) - \rho\tau^2] + (\lambda+\mu)\frac{\partial^2}{\partial x^k \partial x^i}\}[\frac{\delta_{ij}}{4\pi\mu}\frac{e^{\tau_2 r}}{r} + \frac{1}{4\pi\rho\tau^2}\frac{\partial^2}{\partial x^i \partial x^j}\frac{e^{\tau_1 r} - e^{\tau_2 r}}{r}] =$$

$$= \frac{\delta_{kj}}{4\pi}[\Delta(\frac{\partial}{\partial x}) - \tau_2^2]\frac{e^{\tau_2 r}}{r} + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_2 r}}{r} + \frac{\mu}{4\pi\rho\tau^2}[\Delta(\frac{\partial}{\partial x}) - \tau_2^2]\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_1 r} - e^{\tau_2 r}}{r} +$$

$$+\frac{\lambda+\mu}{4\pi\rho\tau^2}\frac{\partial^2}{\partial x^k \partial x^j}\Delta(\frac{\partial}{\partial x})\frac{e^{\tau_1 r} - e^{\tau_2 r}}{r} = -\delta_{kj}\delta(x-z) + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_2 r}}{r} +$$

$$+\frac{\mu(\tau_1^2 - \tau_2^2)}{4\pi\rho\tau^2}\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_1 r}}{r} + \frac{\lambda+\mu}{4\pi\rho\tau^2}\frac{\partial^2}{\partial x^k \partial x^j}\frac{\tau_1^2 e^{\tau_1 r} - \tau_2^2 e^{\tau_2 r}}{r} = -\delta_{kj}\delta(x-z) +$$

$$[\frac{\lambda+\mu}{4\pi\mu} - \frac{(\lambda+\mu)\tau_2^2}{4\pi\rho\tau^2}]\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_2 r}}{r} + [\frac{\mu(\tau_1^2 - \tau_2^2)}{4\pi\rho\tau^2} + \frac{(\lambda+\mu)\tau_1^2}{4\pi\rho\tau^2}]\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_1 r}}{r} =$$

$$-\delta_{kj}\delta(x-z) + 0.\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_2 r}}{r} + 0.\frac{\partial^2}{\partial x^k \partial x^j}\frac{e^{\tau_1 r}}{r} = -\delta_{kj}\delta(x-z)$$

Thus, $C = -\delta(x-z)E$. Q.E.D.

Theorem II. *For every point* $x \in E_3 \setminus \{z\}$

$$\lim_{\tau\to 0}\Psi(x,z,\tau)=\Psi(x,z),$$

where $\Psi(x,z)$ *is Kelvin's matrix of fundamental solutions [1,2].*

Indeed, if we take into account the following equalities

$$\lim_{\tau\to 0}\frac{e^{\tau_k r}}{r}=1/r,\quad (k=1,2),$$

$$\lim_{\tau\to 0}\frac{\tau_2^2 e^{\tau_2 r}-\tau_1^2 e^{\tau_1 r}}{\tau^2}=\frac{\rho}{\mu}-\frac{\rho}{\lambda+2\mu}=\frac{\rho(\lambda+\mu)}{\mu(\lambda+2\mu)},\tag{3.29}$$

$$\lim_{\tau\to 0}\frac{(1-\tau_1 r)e^{\tau_1 r}-(1-\tau_2 r)e^{\tau_2 r}}{\tau^2}=\frac{\rho(\lambda+\mu)r^2}{2\mu(\lambda+2\mu)},$$

we shall obtain

$$\lim_{\tau\to 0}\Psi_{kj}(x,z,\tau)=\frac{\delta_{kj}}{4\pi\mu}\frac{1}{r}-\frac{(x^k-z^k)(x^J-z^j)}{4\pi\rho r^3}\frac{\rho(\lambda+\mu)}{2\mu(\lambda+2\mu)}+$$

$$+\frac{1}{4\pi\rho r^3}\left[\frac{3(x^k-z^k)(x^j-z^j)}{r^2}-\delta_{kj}\right]\frac{\rho(\lambda+\mu)r^2}{2\mu(\lambda+2\mu)}=$$

$$=\frac{\delta_{kj}}{4\pi\mu}\frac{1}{r}+\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{(x^k-z^k)(x^j-z^j)}{r^3}-\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{\delta_{kj}}{r}=$$

$$=\lambda'\frac{\delta_{kj}}{r}+\mu'\frac{(x^k-z^k)(x^j-z^j)}{r^3}=\Psi_{kj}(x,z),$$

where

$$\lambda'=(\lambda+3\mu)[8\pi\mu(\lambda+2\mu)]^{-1},\ \ \mu'=(\lambda+\mu)[8\pi\mu(\lambda+2\mu)]^{-1}.$$

Thus, the theorem is proved.

Theorem III. *For every point* $x\in E_3\setminus\{z\}$

$$\Psi(x,z,\tau)\big|_{\tau=-i\omega}=\Psi(x,z,-i\omega)\equiv\Psi(x,z,\omega),\tag{3.30}$$

$$\overline{\Psi}(x,z,\tau)=\Psi(x,z,\overline{\tau}),\tag{3.31}$$

where $\Psi(x,z,\omega)$ *is Kupradze's matrix or a matrix of fundamental solutions for the steady oscillation equation of a homogeneous isotropic elastic medium [1], and* $\overline{\Psi}$ *and* $\overline{\tau}$ *are complex conjugate, respectively, to* Ψ *and* τ.

The equality (3.30) is directly obtained from the expression of $\Psi_{kj}(x,z,\tau)$, if we substitute τ by $-i\omega$, and (3.31) − if we replace τ by $\overline{\tau}$ in expression of $\Psi_{kj}(x,z,\tau)$.

Theorem IV. *For every point* $x\in E_3\setminus\{z\}$ *elements of the stress tensor*

$$\Psi(x,z,n,\tau)=T\left(\frac{\partial}{\partial x},n\right)\Psi(x,z,\tau),$$

$$\Psi(x,z,n,\tau)=\left\|\Psi_{kj}(x,z,n,\tau)\right\|_{3\times 3}\quad (k,j=1,2,3)\,,\tag{3.32}$$

corresponding to the matrix $\Psi(x,z,\tau)$ *are given by the formula*

$$4\pi\Psi_{kj}(x,z,n,\tau)=-\frac{\delta_{kj}}{r^3}\sum_{l=1}^{3}n_l(x)(x^l-z^l)*$$

$$*\{(1-\tau_2 r)e^{\tau_2 r}-\frac{2\mu}{\rho\tau^2}[3\frac{\tau_2 e^{\tau_2 r}-\tau_1 e^{\tau_1 r}}{r}+3\frac{e^{\tau_1 r}-e^{\tau_2 r}}{r^2}+\tau_1^2 e^{\tau_1 r}-\tau_2^2 e^{\tau_2 r}]\}-$$

$$-\frac{2\mu}{\rho\tau^2}\sum_{l=1}^{3}n_l(x)(x^l-z^l)\frac{(x^k-z^k)(x^j-z^j)}{r^3}* \tag{3.33}$$

$$*[15\frac{e^{\tau_1 r}-e^{\tau_2 r}}{r^4}+\frac{\tau_2^3 e^{\tau_2 r}-\tau_1^3 e^{\tau_1 r}}{r}+15\frac{\tau_2 e^{\tau_2 r}-\tau_1 e^{\tau_1 r}}{r^3}+6\frac{\tau_1^2 e^{\tau_1 r}-\tau_2^2 e^{\tau_2 r}}{r^2}]+$$

$$+\frac{2\mu}{\rho\tau^2}\frac{n_k(x)(x^j-z^j)+n_j(x)(x^k-z^k)}{r^3}[3\frac{\tau_2 e^{\tau_2 r}-\tau_1 e^{\tau_1 r}}{r}+3\frac{e^{\tau_1 r}-e^{\tau_2 r}}{r^2}+\tau_1^2 e^{\tau_1 r}-\tau_2^2 e^{\tau_2 r}]-$$

$$-\frac{n_j(x)(x^k-z^k)}{r^3}(1-\tau_2 r)e^{\tau_2 r}-\frac{\lambda}{\lambda+2\mu}\frac{n_k(x)(x^j-z^j)}{r^3}(1-\tau_1 r)e^{\tau_1 r},$$

where $n(x)=(n_1(x),n_2(x),n_3(x))$ *is a arbitrary unit vector, applied to a point* x.

From(3.32) with the help of the Cauchy-Binet formulae

$$\Psi_{kj}(x,z,n,\tau)=\sum_{i=1}^{3}T_{ki}(\frac{\partial}{\partial x},n)\Psi_{ij}(x,z,\tau)\quad (k,j=1,2,3),$$

we shall easily obtain the expression (3.33), if we take into account the expressions of $\Psi_{ij}(x,z,\tau)$ and $T_{ki}(\frac{\partial}{\partial x},n)$ (see [1]),

$$T_{ki}(\frac{\partial}{\partial x},n)=\lambda n_k\frac{\partial}{\partial x^i}+\mu n_i\frac{\partial}{\partial x^k}+\mu\delta_{ki}\frac{\partial}{\partial n}\quad (k,i=1,2,3).$$

It is clear, that since the matrix $T(\frac{\partial}{\partial x},n)$ is non-symmetric, respectively the matrix $\Psi(x,z,n,\tau)$ is non-symmetric and its j-th column $\Psi^j(x,z,n,\tau)$ $(j=1,2,3)$ represents the complex stress vector, which corresponds to the complex displacement vector $\Psi^j(x,z,\tau)$.

Theorem V. *For every point* $x\in E_3\setminus\{z\}$

$$\Psi(x,z,n,\tau)|_{\tau=-i\omega}=\Psi(x,z,n,-i\omega)\equiv\Psi(x,z,n,\omega), \tag{3.34}$$

$$\overline{\Psi}(x,z,n,\tau)=\Psi(x,z,n,\overline{\tau}), \tag{3.35}$$

where $\Psi(x,z,n,\omega)$ *is stress tensor, corresponding to the Kupradze's matrix* $\Psi(x,z,\omega)$.

The equalities (3.34) and (3.35) are obtained directly from expressions (3.33) of the elements matrix $\Psi(x,z,n,\tau)$.

Theorem VI. *For every point* $x\in E_3\setminus\{z\}$

$$\lim_{\tau\to 0}\Psi(x,z,n,\tau)=\Psi(x,z,n),$$

where $\Psi(x,z,n)$ *is the stress tensor, corresponding to the matrix* $\Psi(x,z)$ *of Kelvin.*

Indeed, taking into account (3.29) and the following equalities

$$\lim_{\tau\to 0}(1-\tau_k r)e^{\tau_k r}=1\quad (k=1,2),$$

$$\lim_{\tau\to 0}\frac{\tau_2^3 e^{\tau_2 r}-\tau_1^3 e^{\tau_1 r}}{\tau^2}=0,$$

we get

$$4\pi \lim_{\tau\to 0} \Psi_{kj}(x,z,n,\tau) = -\frac{\delta_{kj}}{r^3}\sum_{l=1}^{3} n_l(x)(x^l - z^l)\{1 - \frac{2\mu}{\rho}[\frac{3\rho(\lambda+\mu)}{2\mu(\lambda+2\mu)} - \frac{\rho(\lambda+\mu)}{\mu(\lambda+2\mu)}]\} -$$

$$-\frac{2\mu}{\rho}\sum_{l=1}^{3} n_l(x)(x^l - z^l)\frac{(x^k - z^k)(x^j - z^j)}{r^3}[\frac{15\rho(\lambda+\mu)}{2\mu(\lambda+2\mu)r^2} - \frac{6\rho(\lambda+\mu)}{\mu(\lambda+2\mu)r^2}] +$$

$$+\frac{2\mu}{\rho}\frac{n_k(x)(x^j - z^j) + n_j(x)(x^k - z^k)}{r^3}[\frac{3\rho(\lambda+\mu)}{2\mu(\lambda+2\mu)} - \frac{\rho(\lambda+\mu)}{\mu(\lambda+2\mu)}] -$$

$$-\frac{n_j(x)(x^k - z^j)}{r^3} - \frac{\lambda}{\lambda+2\mu}\frac{n_k(x)(x^j - z^j)}{r^3} = \sum_{l=1}^{3} n_l(x)(x^l - z^l) *$$

$$*[-\frac{\delta_{kj}\mu}{(\lambda+2\mu)r^3} - \frac{3(\lambda+\mu)(x^k - z^k)(x^j - z^j)}{(\lambda+2\mu)r^5}] +$$

$$+\frac{n_k(x)(x^j - z^j) + n_j(x)(x^k - z^k)}{r^3}\frac{\lambda+\mu}{\lambda+2\mu} -$$

$$-\frac{n_j(x)(x^k - z^k)}{r^3} - \frac{\lambda}{\lambda+2\mu}\frac{n_k(x)(x^j - z^j)}{r^3} =$$

$$= [\frac{\mu}{\lambda+2\mu}\delta_{kj} + \frac{3(\lambda+\mu)}{\lambda+2\mu}\frac{(x^k - z^k)(x^j - z^j)}{r^2}]\frac{\partial}{\partial n(x)}\frac{1}{r} +$$

$$+\frac{\mu}{\lambda+2\mu}[n_j(x)\frac{\partial}{\partial x^k}\frac{1}{r} - n_k(x)\frac{\partial}{\partial x^j}\frac{1}{r}] = 4\pi\Psi_{kj}(x,z,n).$$

Q.E.D.

The matrices $\Psi(x,z,\tau)$ and $\Psi(x,z,n,\tau)$ have a certain physical sense, which consist in the following:

1) By virtue of Theorem I, j-th column (row) of the matrix $\Psi(x,z,t)$ (see (3.9)) of fundamental solutions of the operator $A(\frac{\partial}{\partial x},t)$ satisfies the dynamic equation

$$A(\frac{\partial}{\partial x},t)\Psi^j(x,z,t) + Re[e^{\tau t}(\delta_{1j},\delta_{2j},\delta_{3j})]\delta(x-z) = \theta.$$

Thus, with the help of j-th column (row) $\Psi^j = (\Psi_{1j},\Psi_{2j},\Psi_{3j})$ of the matrix $\Psi(x,z,\tau)$ a real displacement of a point $x \in E_3$, under deformation of the homogeneous isotropic elastic medium E_3 is given by the formula

$$\Psi^j(x,z,t) = Re[e^{\tau t}\Psi^j(x,z,\tau)],\ t > \frac{r}{c_2} \tag{3.36}$$

$$(x,z \in E_3,\ x \neq z),$$

when this deformation is caused by the unit point force $E^j(z,t) = Re[e^{\tau t}E^j(z)]$, which is applied to a point $z \in E_3$, is directed along the axis Ox^j $(j = 1,2,3)$, acts at time by the law $f(t) = Re(e^{\tau t}) \equiv e^{\alpha t}\cos\omega t$ (see 3.1)) and begins action at moment $t = 0^+$. In (3.36) c_2 is a propagation speed of a transverse wave in a medium, i.e., r/c_2 is a value of a time interval in which a transverse wave reaches from source z to a point x.

On the basis of the mentioned, evidently, in designation $\Psi_{kj}(x,z,\tau)$ the index k denotes a component of a complex displacement vector of a point x, the index j indicates to the cause of a displacement, namely indicates to the point force, directed along the axis Ox^j. The symbol x denotes a moving point, the symbol z denotes the point in which a point force is applied or conversely, since

$$\Psi_{kj}(x,z,\tau) = \Psi_{kj}(z,x,\tau). \tag{3.37}$$

2) From Theorem IV, point 1) and formula (3.9), it is evident, that by the j-th column of the matrix $\Psi(x,z,n,\tau)$ a real stress, which is caused by the unit point force $E^j(z,t) = Re[e^{\pi} E^j(z)]$ is given by the formula

$$\Psi^j(x,z,n,t) = Re[e^{\pi}\Psi^j(x,z,n,\tau)], \; t > \frac{r}{c_2},$$

$$(x, z \in E_3, \;\; x \neq z).$$

Concerning the designation $\Psi_{kj}(x,z,n,\tau)$, all that was said for $\Psi_{kj}(x,z,\tau)$ remains valid, except the condition (3.37).

4. On Calculation of Elements of Matrices of the Fundamental Solutions $\Psi(x,z,t)$ and the Stress Tensor $\Psi(x,z,n,t)$

Let a concentrated force

$$\Phi^j(z,t) = \Phi^j(z)E^j(z,t) \equiv p(\delta_{1j}, \delta_{2j}, \delta_{3j}) f(t), \quad -\infty < p < \infty, \tag{4.1}$$

be applied to a point $z \in E_3$, which changes by a law $f(t)$ and begins action at the moment $t = 0^+$.

Evidently, a displacement vector $U^j(x,z,t)$ and a stress vector $F^j(x,z,n,t)$, which are caused by the force (4.1) are calculated from correlations

$$\begin{gathered} U^j(x,z,t) = p\Psi^j(x,z,t), \\ F^j(x,z,n,t) = p\Psi^j(x,z,n,t) \quad (j = 1,2,3), \;\; x \neq z, \end{gathered} \tag{4.2}$$

where $\Psi^j(x,z,t)$ is the j-th column (row) of the matrix $\Psi(x,z,t)$, $\Psi^j(x,z,n,t)$ is the j-th column of the matrix $\Psi(x,z,n,t)$, and $|p|$ is the intensity of the force $\Phi^j(z)$, i.e., $|p| = |\Phi^j(z)|$.

In order to calculate the values of the functions $\Psi_{kj}(x,z,t)$ and $\Psi_{kj}(x,z,n,t)$ the magnitudes λ, μ, ρ, r, t involved in them, must be taken in one and the same units' system.

If the magnitudes λ, μ, ρ, r, t are taken in the system SI, then they will have the form (e.g., for a rock)

$$\begin{gathered} \lambda = \lambda_0 10^9 \frac{n}{m^2}, \quad \mu = \mu_0 10^9 \frac{n}{m^2}, \quad \rho = \rho_0 10^3 \frac{kg}{m^3}, \\ r = r_0 10^3 m, \quad t = t_0 sec, \end{gathered} \tag{4.3}$$

where constants λ_0, μ_0, ρ_0 are sought in corresponding tables. r_0 is distance between of points x and z, expressed in km-s.

On the basis of (4.3) it is easy to see, that the dimension of the functions $\Psi_{kj}(x,z,t)$ and $\Psi_{kj}(x,z,n,t)$ in the units' system SI will be

$$[\Psi_{kj}(x,z,t)] = \frac{m}{n}, \quad [\Psi_{kj}(x,z,n,t)] = \frac{1}{m^2} \quad (k = 1,2,3), \tag{4.4}$$

and on the basis of (4.2) and (4.4) the dimension of components of the displacement $U^j(x,z,t)$ and stress $F^j(x,z,n,t)$ in the system SI will be

$$[U_k^j(x,z,t)] = [\Psi_{kj}(x,z,t)][p] = \frac{m}{n}n = m,$$

$$[F_k^j(x,z,n,t)] = [\Psi_{kj}(x,z,n,t)][p] = \frac{1}{m^2}n = \frac{n}{m^2},$$

where [ ] is the sign of a dimension.

Thus, considering vectors $\Psi^j(x,z,t)$ and $\Psi^j(x,z,n,t)$ as displacements and stresses, their components must be multiplied by dimension of the force.

From expression of the function $\Psi_{kj}(x,z,t)$ and $\Psi_{kj}(x,z,n,t)$ (see Section 3) it is seen, that substitution in them the magnitudes (4.3) we shall have to do the division on sufficiently large numbers, what in turn implies growth of an error of calculation.

In order to remove the mentioned difficulty we operate in the following way. If in the expressions of the functions $\Psi_{kj}(x,z,t)$ and $\Psi_{kj}(x,z,n,t)$ we substitute the magnitudes λ,μ,ρ,r,t from representation (4.3), then it is easy to see that in front of expression of the $\Psi_{kj}(x,z,t)$ will be the factor 10^{-12}, and in front of the $\Psi_{kj}(x,z,n,t)$ − 10^{-6}, i.e.,

$$\Psi_{kj}(x,z,t) = 10^{-12}\Psi_{kj}^*(x,z,t),$$
$$\Psi_{kj}(x,z,n,t) = 10^{-6}\Psi_{kj}^*(x,z,n,t), \tag{4.5}$$

where the functions $\Psi_{kj}^*(x,z,t)$ and $\Psi_{kj}^*(x,z,n,t)$ represent, respectively, values of the functions $\Psi_{kj}(x,z,t)$ and $\Psi_{kj}(x,z,n,t)$, when λ,μ,ρ,r,t from (4.3) are substituted by the magnitudes $\lambda_0,\mu_0,\rho_0,r_0,t_0$. On the basis of mentioned we calculate $\Psi_{kj}^*(x,z,t)$ and $\Psi_{kj}^*(x,z,n,t)$, respectively, in the units $\frac{m}{n}$ and $\frac{1}{m^2}$, and the values of the functions $\Psi_{kj}(x,z,t)$ and we find $\Psi_{kj}(x,z,n,t)$ from (4.5).

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