An exponentially convergent method for nonlinear operator equations: two-sided approximations and global convergence

I.P. Gavrilyuk[1], I. I. Lazurchak[2], V.L.Makarov and D. Sytnyk [3]

[1] University of Cooperative Education, Berufsakademie Eisenach, Staatliche Studienakademie Thueringen, Am Wartenberg 2, D-99817 Eisenach, Germany, ipg@ba-eisenach.de
[2] State Pedagogical University Drogobych, Ukraine, lazurchak@mail.ru.
[3] Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., Kyiv-4, 01601, Ukraine, makarov@imath.kiev.ua

Abstract:

We propose a new analytical-numerical method for solving nonlinear operator and differential operator equations. The method possesses the exponential convergence rate and can provide two-sided approximations. The user can control the exponential convergence through an embedded control mechanism, i.e. one can arrive the exponential convergence independent of the value of the Lipschitz constant. The numerical examples confirm the theoretical results.

1 Introduction

There are various approaches to construct exponentially convergent approximations to solutions of nonlinear differential equations. To the classical ones belong e.g. the variational methods [26]. The spectral methods (see [5] and the literature therein) are based on the spectral expansions or a special interpolation. One can obtain exponentially convergent methods for a wide class of differential equations with operator coefficients combining an appropriate quadrature formula for the Dunford-Cauchy representation of the operator exponential with a special interpolation of the nonlinearity (see [20] and the literature cited therein). One more alternative which seems to be very promising is the use of the homotopy or perturbation idea (see e.g. [15, 16]) which is closely related to the Adomian decomposition method (ADM) [2-4]. One of the important elements of this approach are Adomian's polynomials [2-4, 34, 48]. There are many publications in the last decade which are devoted to the application of Adomian's decomposition method (ADM) for the both linear and nonlinear operator and differential operator equations [16, 24, 29-31]. The class of problem for which these approximations provide the exponential convergence rate is restricted by the character of the nonlinearity: there are nonlinearities which do not fulfill the known convergence conditions and the approximation method do not contain some convergence control, i.e. the convergence is not global. The crucial role for the known convergence conditions plays the constant $L = n!M\alpha^n$ bounding the $n-$th Fréchet derivative of the nonlinearity (the generalized Lipschitz constant) and, roughly speaking, the known convergence results assert convergence provided that this constant is small enough. In the present paper we propose a control mechanism which guarantees the exponential convergence for a wide class of nonlinearities independent of the constant L (i.e. global). The idea of such approach for eigenvalue problems was recently announced in [32].

Let us remind of the idea of ADM which can be also interpreted as the FD-method proposed in [31] for the Sturm-Liouville problems and is very close to the homotopy perturbation methods.

If we have to solve the operator equation

$$u = -N(u) + F, \qquad (1.1)$$

then we can imbed it into the family of equations

$$u(t) = -tN(u(t)) + F, \quad t \in [0,1] \qquad (1.2)$$

and obtain obviously

$$u(1) = u. \tag{1.3}$$

We look for the solution of (1.1) in the form

$$u(t) = \sum_{j=0}^{\infty} t^j u^{(j)}, \tag{1.4}$$

and represent

$$N(\sum_{j=0}^{\infty} t^j u^{(j)}) = \sum_{j=0}^{\infty} t^j A_j, \tag{1.5}$$

where

$$A_j = \frac{1}{j!} \frac{\partial^j N(\sum_{k=0}^{\infty} t^k u^{(k)})}{\partial t^j} \Bigg|_{t=0}. \tag{1.6}$$

Substituting (1.4) into (1.1) we have

$$\sum_{j=0}^{\infty} t^j u^{(j)} = -tN\left(\sum_{j=0}^{\infty} t^j u^{(j)} \right) + F. \tag{1.7}$$

Applying to this equality successively the operator $\dfrac{1}{(j+1)!}\dfrac{d^{j+1}}{dt^{j+1}}$ and then setting $t = 0$ we obtain the following recurrence formulas

$$u^{(j+1)} = -A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right), \quad j = 0,1,\ldots$$
$$A_0\left(N; u^{(0)}\right) = N\left(u^{(0)}\right), \quad u^{(0)} = F. \tag{1.8}$$

Here $A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right)$ are the Adomian polynomials with the following explicit representation

$$A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right) =$$
$$\sum_{\alpha_1 + \cdots + \alpha_j = j} N^{(\alpha_1)}\left(u^{(0)}\right) \frac{\left(u^{(1)}\right)^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \cdots \frac{\left(u^{(j-1)}\right)^{\alpha_{j-1} - \alpha_j}}{(\alpha_{j-1} - \alpha_j)!} \frac{\left(u^{(j)}\right)^{\alpha_j}}{(\alpha_j)!}, \tag{1.9}$$

where the sequence of indices natural α_i is not increasing, $N^{(i)}(u)$ is the $i-$th (Fréchet) derivative of the operator N.

The solution of (1.1) can be now represented by (provided that the convergence radius of series (1.4) is not less then 1)

$$u = u(1) = \sum_{j=0}^{\infty} u^{(j)} \tag{1.10}$$

and the truncated sum

$$\overset{m}{u} = \sum_{j=0}^{m} u^{(j)} \tag{1.11}$$

represents an approximation to the exact solution.

The following theorem from [24] gives some sufficient conditions for the convergence of (1.4) for all $t \in [0,1]$.

Theorem 1.1 *Let H be a Banach space and $F \in H$. If the operator $N(u): H \to H$ is analytic in a ball $\|u - u_0\| < R$ with the center u_0 and if for all $n \geq 0$ there holds $\|N^{(n)}(u_0)\| \leq n! M \alpha^n$ with some $M > 0$, $\alpha > 0$, then the conditions*

1. $4M\alpha \leq 1$, for $R = \infty$,
2. $5M\alpha \leq 1$, for $R < \infty$.

provide the convergence of (1.4) for all $t \in [0,1]$ and, therefore, the convergence of (1.10).

2 Application to parabolic problems

We consider the problem

$$\frac{\partial u(t)}{\partial t} + Au(t) = f(t, u(t)), \quad t \in (0,1],$$
$$u(0) = u_0, \tag{2.1}$$

where $u(t)$ is an unknown vector valued function with values in a Banach space X, $u_0 \in X$ is a given vector, $f(t,u): (\mathrm{P}_+ \times X) \to X$ is a given function (nonlinear operator) and A is a linear densely defined closed operator with the domain $D(A)$ acting in X. The abstract setting (2.1) covers many applied problems such as nonlinear heat conduction or diffusion in porous media, the flow of electrons and holes in semiconductors, nerve axon equations, chemically reacting systems, equations of the population genetics theory, dynamics of nuclear reactors, Navier-Stokes equations of the viscous flow etc. (see e.g. [23] and the references therein). This fact together with theoretical interest are important reasons to study efficient discrete approximations of problem (2.1).

A simple example of a partial differential equation covered by the abstract setting (2.1) is the nonlinear heat equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x,u) \tag{2.2}$$

with the initial condition $u(0,x) = u_0(x)$, where the operator A is defined by

$$D(A) = \{v \in H^2(0,1): v(0) = 0, v(1) = 0\},$$
$$Av = -\frac{d^2 v}{dx^2} \qquad \text{for all } v \in D(A). \tag{2.3}$$

Given a discretization parameter N we are interesting in approximations possessing an exponential convergence rate with respect to $N \to \infty$ which for a given tolerance ε provide algorithms of optimal or low complexity [12, 13]. Exponentially convergent algorithms were proposed recently for various linear problems.

The homogeneous equation

$$\frac{dT(t)}{dt} + AT(t) = 0, \quad T(0) = I, \tag{2.4}$$

where I is the identity operator and $T(t)$ is an operator valued function defines the semi-group of bounded operators $T(t) = e^{-At}$ generated by A (called also the operator exponential or the solution operator of the homogeneous equation (2.1)). Given the solution operator, the initial vector u_0 and the right-hand side $f(t)$, the solution of the homogeneous initial value problem (2.1) can be represented by

$$u(t) = u_h(t) = T(t)u_0 = e^{-At}u_0. \tag{2.5}$$

Problem (2.1) is equivalent to the nonlinear Volterra integral equation

$$u(t) = u_h(t) + u_{nl}(t), \tag{2.6}$$

where

$$u_h(t) = T(t)u_0, \tag{2.7}$$

$T(t) = e^{-At}$ is the operator exponential (the semi-group) generated by A and the nonlinear term is given by

$$u_{nl}(t) = \int_0^t e^{-A(t-s)} f(s, u(s)) ds. \tag{2.8}$$

The equation (2.6) is of the type (1.1) with $u = u(t) = u(t,x), \quad u_0 = u_0(x)$

$$N(t,u) = N(u) = u_{nl} = \int_0^t e^{-A(t-s)} f(s, u(s)) ds,$$
$$F = u_h = F(t,x) = T(t)u_0. \tag{2.9}$$

Let A be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector

$$\Sigma = \{z = a_0 + re^{i\theta} : r \in [0,\infty), |\theta| < \varphi < \frac{\pi}{2}\} \tag{2.10}$$

and on its boundary Γ_Σ and outside the sector the following estimate for the resolvent holds true

$$\Pi(zI - A)^{-1}\Pi \leq \frac{M}{1 + |z|} \tag{2.11}$$

with some positive constant M (compare with [17, 27, 28, 33, 41]). The angle φ is called the spectral angle of the operator A. A practically important example of strongly positive operators in $X = L_p(\Omega), 0 < p < \infty$ represents a strongly elliptic partial differential operator [8-10, 33, 38] where the parameters a_0, φ of the sector Σ are defined by its coefficients.

A convenient representation of the operator exponential is the one provided by the improper Dunford-Cauchy integral

$$e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-tz} (zI - A)^{-1} dz \tag{2.12}$$

where Γ_I is an integration path enveloping the spectrum of A. After parametrizing Γ we get an improper integral of the type

$$e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-tz} (zI - A)^{-1} dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(t,\xi) d\xi. \tag{2.13}$$

The last integral can be discretized by a quadrature rule (desirable exponentially convergent) involving a short sum of resolvents. Such an algorithm inherits a two-level parallelism with respect to both the computation of resolvents and the treatment of different time values.

Two efficient methods for solving linear homogeneous parabolic problems based on the improper Dunford-Cauchy integrals along a path enveloping the spectrum of A were discussed in [10, 12, 18, 39, 40] where the boundary of a sector containing the spectrum of A or a parabola were used as the integration path. The methods from [10, 18, 22] use Sinc-quadratures [1, 42, 43] and possesse the exponential convergence rates for $t > 0$ and a polynomial convergence rates for $t = 0$ depending on the smoothness of the initial vector u_0 from a Hilbert space. An exponential convergence rate for all $t \geq 0$ was proved in [11, 47] under assumptions that the initial function u_0 belongs to the domain of $D(A^\sigma)$ for some $\sigma > 1$, where the preliminary computation of $A^\sigma u_0$ is needed. Note that all these algorithms can not be directly applied to inhomogeneous problems due to the inefficiency of computation of the operator exponential at $t = 0$. In [19] a hyperbola was

used as the integration path which allows one to get the uniform exponential convergence rate with respect to $t \geq 0$ without preliminary computation of $A^\sigma u_0$. An exponentially convergent algorithm for the case of an operator family $A(t)$ depending on the parameter t was proposed in [21]. This algorithm uses an exponentially convergent algorithm for the operator exponential generating by a constant operator.

We can also use the representation

$$e^{-At}u_0 = \frac{1}{2\pi i}\int_{\Gamma_I} e^{-zt}\left[(zI - A)^{-1} - \frac{1}{z}I\right]u_0 dz \tag{2.14}$$

instead of (2.13) (see [19]), where the integration hyperbola is given by

$$\Gamma_I = \{z(\xi) = a_I \cosh\xi - ib_I \sinh\xi : \xi \in (-\infty,\infty)\} \tag{2.15}$$

(note that the hyperbola

$$\Gamma_0 = \{z(\xi) = a_0 \cosh\xi - ib_0 \sinh\xi : \xi \in (-\infty,\infty), b_0 = a_0 \tan\varphi\} \tag{2.16}$$

is called the spectral hyperbola, which pathes through the vertex $(a_0,0)$ of the spectral angle and possesses asymptotes which are parallel to the rays of the spectral angle Σ).

Parametrizing integral (2.14) by (2.15) we get

$$u(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} F(t,\xi)d\xi \tag{2.17}$$

with

$$F(t,\xi) = F_A(t,\xi)u_0,$$
$$F_A(t,\xi) = e^{-z(\xi)t}(a_I \sinh\xi - ib_I \cosh\xi)\left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I\right]. \tag{2.18}$$

We approximate integral (2.17) by the following Sinc-quadrature

$$u_M(t) = \frac{h}{2\pi i}\sum_{k=-M}^{M} F(t,z(kh)) \tag{2.19}$$

with an appropriate h. The following result from [19] characterizes the error of this approximation.

Theorem 2.1 *Let A be a densely defined strongly positive operator and $u_0 \in D(A^\alpha), \alpha \in (0,1)$, then Sinc-quadrature (2.19) represents an approximate solution of the homogeneous initial value problem, i.e. $u(t) = e^{-At}u_0$, and possesses a uniform with respect to $t \geq 0$ exponential convergence rate with an estimate which is of the order $O(e^{-c\sqrt{M}})$ uniformly in $t \geq 0$ provided that $h = O(1/\sqrt{M})$ and of the order $O\left(\max\left\{e^{-\pi dM/(c_1 \ln M)}, e^{-c_1 a_I tM/2 - c_1\alpha \ln M}\right\}\right)$ for each fixed $t \geq 0$ provided that $h = c_1 \ln M/M$.*

In accordance with (1.8), (1.9) for the computation of Adomian's polynomials one should compute first

$$A_0(N;u_0) = N(t,u_0) = \int_0^t e^{-A(t-\tau)}f(\tau,u_0(\tau))d\tau \tag{2.20}$$

and then one after another

$$N^{\alpha_1}(t,u_0) = \int_0^t e^{-A(t-\tau)}\frac{\partial^{\alpha_1}f(\tau,u_0(\tau))}{\partial u^{\alpha_1}}d\tau. \tag{2.21}$$

Using representation (2.14) of the operator exponential we get, e.g. for $N(t,u_0)$

$$N(t,u_0) = \int_0^t \frac{1}{2\pi i} \int_{\Gamma_I} e^{-z(t-s)}[(zI-A)^{-1} - \frac{1}{z}I]f(s,u_0(s))dzds$$

$$= \frac{1}{2\pi i} \int_{\Gamma_I} \left[(z(\xi)I-A)^{-1} - \frac{1}{z(\xi)}I \right] \int_0^t e^{-z(\xi)(t-s)} f(s,u_0(s))ds z'(\xi)d\xi, \qquad (2.22)$$

$$z(\xi) = a_I \cosh\xi - ib_I \sinh\xi.$$

Replacing here the first integral by quadrature (2.19) we get

$$u_p(t) \approx u_{ap}(t) = \frac{h}{2\pi i} \sum_{k=-M}^{M} z'(kh)\left[(z(kh)I-A)^{-1} - \frac{1}{z(kh)}I \right]f_k(t) \qquad (2.23)$$

with

$$f_k(t) = \int_0^t e^{-z(kh)(t-s)} f(s,u_0(s))ds, \ k = -M,...,M. \qquad (2.24)$$

In order to construct an exponentially convergent quadrature for these integrals we change the variables by

$$\frac{t}{2} - s = \frac{t}{2}\tanh\xi \qquad (2.25)$$

and get instead of (2.24)

$$f_k(t) = \int_{-\infty}^{\infty} F_k(t,\xi)d\xi, \qquad (2.26)$$

where

$$F_k(t,\xi) = \frac{t}{2\cosh^2\xi}\exp[-z(kh)t(1+\tanh\xi)/2]f(t(1-\tanh\xi)/2, u_0(t(1-\tanh\xi)/2)). \qquad (2.27)$$

The following assertion was proven in [19].

Lemma 2.1. *Let* $f(t,u_0(t))$ *for* $t \in [0,\infty]$ *can be analytically extended into the sector* $\Sigma_f = \{\rho e^{i\theta_1} : \rho \in [0,\infty], |\theta_1| < \varphi\}$ *and for all complex* $w \in \Sigma_f$ *we have*

$$\|f(w,u_0(w))\| \leq ce^{-\delta|\Re w|} \qquad (2.28)$$

with $\delta \in (0,\sqrt{2}a_0]$ *, then the integrand* $F_k(t,\xi)$ *can be analytically extended into the strip* $D_{d_1}, 0 < d_1 < \varphi/2$ *and belongs to the Hardy class* $H^1(D_{d_1})$ *[43] with respect to* ξ *, where* a_0, φ *are the spectral characterizations (2.10) of* A.

Let the assumptions of Lemma 2.1 hold, then we can use the following quadrature rule to compute the integrals (2.26) (see [42], p.144)

$$f_k(t) \approx f_{k,M}(t) = h \sum_{p=-M}^{M} \mu_{k,p}(t)f(\omega_p(t)), \qquad (2.29)$$

where

$$\mu_{k,p}(t) = \frac{t}{2}\exp\{-\frac{t}{2}z(kh)[1+\tanh(ph)]\}/\cosh^2(ph),$$

$$\omega_p(t) = \frac{t}{2}[1-\tanh(ph)], h = O(1/\sqrt{M}), \qquad (2.30)$$

$$z(\xi) = a_I \cosh\xi - ib_I \sinh\xi.$$

Substituting (2.29) into (2.23) we get the following algorithm to compute an approach $A_{0,M}(N;u_0)$ to $A_0(N;u_0)$

$$A_{0,M}(N;u_0) = A_{ap,M}(t) = \frac{h}{2\pi i} \sum_{k=-M}^{M} z'(kh)[(z(kh)I - A)^{-1} - \frac{1}{z(kh)}I]$$

$$\times h \sum_{p=-M}^{M} \mu_{k,p}(t)f(\omega_p(t),u_0(\omega_p(t))). \tag{2.31}$$

The next theorem characterizes the error of this algorithm.

Theorem 2.2 *Let A be a densely defined strongly positive operator with the spectral characterization a_0, φ and $f(t,u_0(t)) \in D(A^\alpha), \alpha > 0$ for $t \in [0,\infty]$ can be analytically extended into the sector $\Sigma_f = \{\rho e^{i\theta_1} : \rho \in [0,\infty], |\theta_1| < \varphi\}$ where the estimate*

$$\|A^\alpha f(w,u_0(w))\| \le c_\alpha e^{-\delta_\alpha |\Re w|}, \; w \in \Sigma_f \tag{2.32}$$

with $\delta_\alpha \in (0, \sqrt{2}a_0]$ holds, then algorithm (2.31) converges with the error estimate

$$\|E_M(t)\| = \|A_0(N;u_0)(t) - A_{0,M}(t)\| \le ce^{-c_1\sqrt{M}} \tag{2.33}$$

uniformly in t with positive constants c, c_1 depending on α, φ, a_0 and independent of M.

In this way one can compute all Adomian's polynomials needed in (1.11) with an exponential accuracy.

Let the nonlinear operator $f(s,u)$ for each s is analytic as function of u in some disc $\|u - u_0\| = \rho < r$ with the boundary Γ, then [7, p. 157] it holds

$$| f_{u^n}^{(n)}(s,u_0) | \le \tilde{M}(s,\rho)n!\rho^n \le M(\rho)n!\rho^n, \tag{2.34}$$

provided that $\tilde{M}(s,\rho) \le M(\rho)$, where $\tilde{M}(s,\rho) = \max_{u \in \Gamma} \|f(s,u)\|$. Thus, for $M(\rho)$ small enough we are in the situation of Theorem 1.1, so that the method (1.11) converges exponentially.

An alternative exponentially convergent method based on the interpolation of the nonlinearity on a Gauss-Lobatto grid was proposed in [20].

One can obtain an operator equation of type (1.1) applying to PDE (2.1) the operator $L_t^{-1} = \int_0^t$ analogously to [6, 35] but there are not any theoretical justification of convergence of Adomian's method in this case.

Remark 1 *In the recent paper [29] the following modification of ADM was proposed. One looks the summands of (1.10) in accordance with the recurrence formulas*

$$u^{(j+1)} = -\overline{A_j}\left(u^{(0)},u^{(1)},\ldots,u^{(j)}\right), \quad j = 0,1,\ldots$$

$$u^{(0)} = F, \tag{2.35}$$

where $\overline{A_j}\left(u^{(0)},u^{(1)},\ldots,u^{(j)}\right)$ are the modified Adomian polynomials given by

$$\overline{A_j}\left(u^{(0)},u^{(1)},\ldots,u^{(j)}\right) = N\left(u^{(0)}+\ldots+u^{(j)}\right) - N\left(u^{(0)}+\ldots+u^{(j-1)}\right) \tag{2.36}$$

In the paper [29] for the problem

$$\frac{d^k y(t)}{dt^k} + \beta(t)f(y(t)) = \kappa(t), \quad t \in (0,T), \quad \frac{d^p y(t)}{dt^p} = c_p, \; p = \overline{0,k-1}, \tag{2.37}$$

with the given c_p, $p = \overline{0, k-1}$, with $M = \max\limits_{t \in [0,T]} |B(t)|$ and with the right-hand side $f(y)$ satisfying the Lipschitz condition with a constant L it was shown that the modified Adomian methods converges as a geometrical progression with the quotient α and with the error estimate

$$\left| y(t) - \sum_{j=0}^{m} y_j(t) \right| \leq \frac{\alpha^m}{1-\alpha} \|y_1\|_\infty \tag{2.38}$$

provided that

$$\alpha = \frac{LMT^k}{k!} < 1. \tag{2.39}$$

Numerical experiments have shown that the modified ADM converges faster then the ordinary one. But it was ignored in [29] that the modified ADM in fact coincides with the usual fixed point iteration. Actually, the relations (2.35), (2.36) imply

$$u^{m+1} = -N(u^m) + F, \quad m = 0,1,\ldots$$
$$u^0 = F. \tag{2.40}$$

Now the conclusions of [29] about the advantages of the modified ADM become understandable and are well known long ago (see e.g. [36]).

The natural question arises in the case when the assumptions of Theorem 1 are not fulfilled: what can we do in order to arrive the convergence? One of the aims of this paper is to answer this question and to construct an iteration method which converges whereas the fixed point iteration (2.40) can be divergent. An other aim is to show that the odd and the even iterations provide the two-sided approximations, therefore they can be used e.g. for aposteriori error estimates. We accompany our theoretical considerations by the numerical examples which confirm our conclusions.

3 Two-sided iteration method

Let $\overline{S}_r(a) = \left\{ x \in X : \|x - a\| \leq r \right\}$ be a closed ball in a Banach space X. Then the following assertion about the fixed point iteration for equation (1.1) holds true (see e.g. [26]).

Theorem 3.1 *Let the operator N satisfies the conditions*
1° $\forall u, v \in \overline{S}_r(a)$ *it holds*

$$\|N(u) - N(v)\| \leq q\|u - v\|, \quad q \in (0;1), \tag{3.1}$$

2° *for $F \in \overline{S}_r(a)$ it holds*

$$\|N(F)\| \leq (1-q)r, \tag{3.2}$$

Then the equation

$$u = -N(u) + F, \quad N(0) = 0 \tag{3.3}$$

possesses a unique solution $u_\Sigma \in \overline{S}_r(a)$, which can be obtained by the fixed point iteration

$$u_{n+1} = -N(u_n) + F, \quad n = 0,1,\ldots \tag{3.4}$$

with the error estimate

$$\|u_\Sigma - u_n\| \leq \frac{q^n r}{1-q}. \tag{3.5}$$

Let us clarify the conditions on the operator N under which the iterations (3.4) provide the two-sided approximations to u_Σ.

Let $K \subset X$ be a cone with a partial order $°$, i.e. we write $v ° u$ when $u - v \in K$. Further we make the following assumptions.

$3°$ $F \in K$.

$4°$ The operator N is positive in the sense that
$$N(K) \subset K \tag{3.6}$$

$5°$ There exists the Frechét derivative $N'(v)$ with the property
$$\left\| N'(v) \right\| \le q, \, 0° \, N'(v)u \quad \forall u, v \in \overline{S}_r(F) \cap K \tag{3.7}$$

$6°$

$$0° \, u_1 = -N(u_0) + F = -N(F) + F \tag{3.8}$$

Then the following assertion holds true.

Theorem 3.2 *Let the conditions $2°$ -- $6°$ hold. Then the fixed point iteration (3.4) converges to the unique solution u_Σ of the equation (3.3) and provide the two-sided approximation, i.e.*

$$u_\Sigma ° \ldots ° u_{2k} ° \ldots ° u_2 ° u_0$$
$$u_1 ° u_3 ° \ldots ° u_{2k+1} ° \ldots ° u_\Sigma \tag{3.9}$$

Proof. See [30].

Example 3.1 *Let us consider the Dirichlet boundary value problem*
$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0;1),$$
$$u(0) = u(1) = 0, \tag{3.10}$$

with $f(x) = \pi^2 \sin \pi x + M(\sin \pi x)^3$ and a given constant $M \ge 0$.
The exact solution of (3.10) is
$$u(x) = \sin \pi x. \tag{3.11}$$
Problem (3.10) is equivalent to the following nonlinear Fredholm integral equation
$$u(x) = \int_0^1 G(x, \xi)[-Mu^3(\xi) + f(\xi)]d\xi,$$
$$G(x, \xi) = \begin{cases} x(1-\xi), \, x \le \xi \\ \xi(1-x), \, \xi < x. \end{cases} \tag{3.12}$$
This equation is of the form (3.3) with
$$N(u) = M\int_0^1 G(x, \xi)u^3(\xi)d\xi, \quad F = \int_0^1 G(x, \xi)f(\xi)d\xi.$$
For $M = 1$, $r = 1.2$, $q = 0.54$ all assumption of Theorem 3.2 are fulfilled where

$$X = C[0;1], \quad K = \{v(x) \in C[0;1] : v(x) \ge 0, \, x \in [0;1]\}$$

$$\|F\|_\infty = 1.087\ldots < r, \quad \|N(F) - F\|_\infty = 0.101\ldots < (1-q)r,$$

$$0 \leq N(v)u = 3\int_0^1 G(x,\xi)v^2(\xi)u(\xi)d\xi \leq \frac{3r^3}{8} = \frac{3*1.44}{8}r = qr,$$

$$u_1 = -N(u_0) + F = -\int_0^1 G(x,\xi)f(\xi)d\xi + f(x) \geq 0.$$

The numerical results (obtained with Maple) are presented in Table 1 and on Fig. 1, 2. The table exhibits the advantages of the fixed point iteration to the usual Adomian's method. The figures demonstrate that the even and the odd Adomian's iterations include the exact solution.
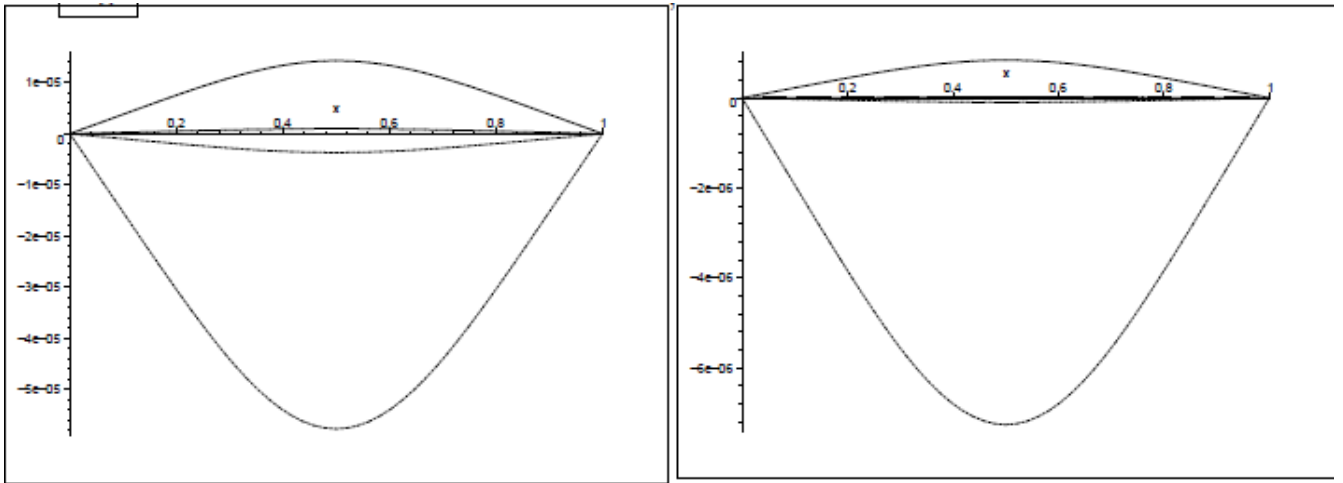

Figure 1: The errors of the usual (left) and the modified (right) Adomian's iterations number 4, 6 (dash) and 5, 7 (continuous) for $M = 0.5$.
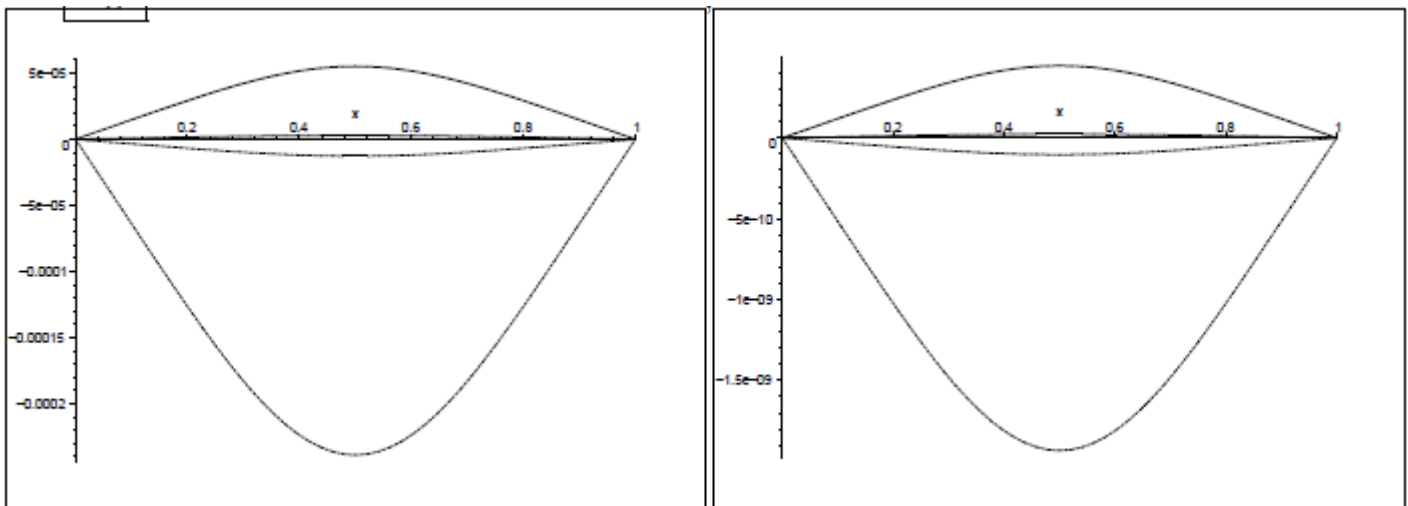

Figure 2: The errors of the modified Adomian's iterations (left) number 4, 6 (dash) and 5, 7 (continuous) as well as (right) 12, 14 (dash) and 13,15 (continuous) for $M = 1$.

n	ε_A	ε_{mA}
0	.4477593957331e-1	.394026825275758e-1
1	.6148944791993e-2	.473705632178769e-2
2	.114506764843e-2	.544572291857201e-3
3	.246398359542e-3	.629246333480784e-4
4	.5768833894e-4	.726644599750120e-5
5	.14273551991e-4	.839176371831571e-6
6	.367192582e-5	.969127578152334e-7
7	.972342751e-6	.111920351333820e-7
8	.26329372e-6	.129251941616996e-8
9	.72571985e-7	.149267628681118e-9
10	.2029413e-7	.172381275559081e-10

Table 1: Error of the usual (ε_A) and the modified (ε_{mA}) Adomian's method for $M = 0.5$

4 An iteration method for nonlinear problems with the controllable exponential convergence

In this section we give the description of an algorithm with the controllable exponential convergence. In order to avoid technical difficulties we justify this algorithm in the case of the following nonlinear model problem

$$u''(x) - N(u(x))u(x) = -f(x), \quad x \in (0,1),$$
$$u(0) = u(1) = 0 \tag{4.1}$$

with a nonlinear function $N(u): P^1 \to P^1$ satisfying the conditions

$$N(u) \geq 0, \quad [uN(u)]' \geq 0, \quad N''(u) \geq 0, \quad \forall u \in P^1. \tag{4.2}$$

Using the Green function for the differential operator defined by

$$D(A) = \{u : u \in W_2^2(0,1) : u(0) = u(1)\},$$
$$Au = -\frac{d^2 u}{dx^2} \quad \forall u \in D(A) \tag{4.3}$$

one can reduce problem (4.1) to the operator equation of kind (1.1). In the case when the fixed point iteration (2.40) is divergent we propose the following method based on the idea of the FD-method [32] which is closed to a homotopy perturbation method.

We introduce a grid

$$\bar{\bar{\omega}} = \left\{ x_i \in [0,1], i = \overline{1,K}: \quad 0 = x_1 < x_2 < \ldots < x_{K-1} < x_K = 1 \right\}$$

partitioning the interval $[0,1]$ into subintervals $[x_{i-1}, x_i]$, $i = \overline{1,K}$ of the length $h_i = x_i - x_{i-1}, |h| = \max_i h_i$ and imbed problem (4.1) into the parametric family of problems

$$
\begin{aligned}
&\frac{\partial^2 u(x,t)}{\partial x^2} - \left\{ N(u(x_{i-1},t)) + t\left[N(u(x,t)) - N(u(x_{i-1},t)) \right] \right\} u(x,t) \\
&= -f(x), x \in (0,1), \\
&[u(x)]_{x=x_i} = 0, \quad \left[\frac{du(x)}{dx} \right]_{x=x_i} = 0, \quad i = \overline{1,K-1}, \\
&u(0,t) = u(1,t) = 0, \quad t \in [0,1].
\end{aligned}
\tag{4.4}
$$

It is clear that for $t = 1$ the solution of problem (4.4) coincides with the solution of problem (4.1), i.e.

$$u(x,1) = u(x),$$

For $t = 0$ we obtain the following base problem

$$
\begin{aligned}
&\frac{d^2 u^{(0)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1}) \right) u^{(0)}(x) = -f(x), \\
&x \in (x_{i-1}, x_i), \quad i = \overline{1,K}, \\
&[u^{(0)}(x)]_{x=x_i} = 0, \quad \left[\frac{du^{(0)}(x)}{dx} \right]_{x=x_i} = 0, \quad i = \overline{1,K-1}, \\
&u^{(0)}(0) = u^{(0)}(1) = 0,
\end{aligned}
\tag{4.5}
$$

where $[v(x)]_{x=\xi} = v(\xi + 0) - v(\xi - 0)$ denotes the jump of the function $v(x)$ at the point $x = \xi$.

The last problem as well as problem (4.4) are representatives of the class of boundary value problems with piecewise constant argument which are in the focus of attention of many researchers for some time (see e.g. [46] and the literature therein).

We look for the solution of problem (4.4) in the form

$$u(x,t) = \sum_{j=0}^{\infty} t^j u^{(j)}(x). \tag{4.6}$$

Substituting (4.6) into (4.4) and comparing the coefficients in front of powers of t we obtain the following recurrence sequence of problems for $u^{(j)}(x)$ (with piecewise constant argument):

$$
\begin{aligned}
&\frac{d^2 u^{(j+1)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1}) \right) u^{(j+1)}(x) \\
&= N'\left(u^{(0)}(x_{i-1}) \right) u^{(j+1)}(x_{i-1}) u^{(0)}(x) + F^{(j+1)}(x), \\
&x \in (x_{i-1}, x_i), \quad i = \overline{1,K},
\end{aligned}
\tag{4.7}
$$

where

$$F^{(j+1)}(x) = \sum_{p=1}^{j} A_{j+1-p}\left(N; u^{(0)}(x_{i-1}), \ldots, u^{(j+1-p)}(x_{i-1})\right) u^{(p)}(x)$$

$$+ \sum_{p=0}^{j} \left[A_{j-p}\left(N; u^{(0)}(x), \ldots, u^{(j-p)}(x)\right) \right.$$

$$- A_{j-p}\left(N; u^{(0)}(x_{i-1}), \ldots, u^{(j-p)}(x_{i-1})\right) \right] u^{(p)}(x)$$

$$+ A_{j+1}\left(N; u^{(0)}(x_{i-1}), \ldots, u^{(j)}(x_{i-1}), 0\right) u^{(0)}(x), \qquad (4.8)$$

$$[u^{(j+1)}(x)]_{x=x_i} = 0, \quad \left[\frac{du^{(j+1)}(x)}{dx} \right]_{x=x_i} = 0, \quad i = \overline{1, K-1},$$

$$u^{(j+1)}(0) = u^{(j+1)}(1) = 0, \quad j = 0, 1, \ldots,$$

$A_j(N; v_0, v_1, \ldots, v_j)$ are Adomian's polynomials for the nonlinear function $N(v)$ given by the explicit formula (1.9). The solution of problem (4.1) is then given by

$$u(x) = \sum_{j=0}^{\infty} u^{(j)}(x) \qquad (4.9)$$

(provided that the convergence radius of (4.6) is not less then 1) and the approximate solution by

$$u(x) \approx \overset{m}{u}(x) = \sum_{j=0}^{m} u^{(j)}(x), \qquad (4.10)$$

where the exponential convergence will be controlled by the parameter $|h|$.

Let us consider the base problem (4.6). This problem is equivalent to the system of nonlinear equations

$$u^{(0)}(x_i) = \int_0^1 G\left(x_i, \xi, \vec{N}(u)\right) f(\xi) d\xi, \quad i = \overline{1, K-1}, \qquad (4.11)$$

where

$$\vec{N}(u) = \left(N\left(u^{(0)}(x_1)\right), \ldots, N\left(u^{(0)}(x_{K-1})\right) \right) \qquad (4.12)$$

and $G\left(x_i, \xi, \vec{N}\right)$ is the Green function of problem (4.5) provided that the vector $\vec{u} = \{u^{(0)}(x_1), u^{(0)}(x_2), \ldots, u^{(0)}(x_{K-1})\}$ is known.

We introduce the operator

$$B(\vec{u}) = \left(\int_0^1 G\left(x_i, \xi, \vec{N}(u)\right) f(\xi) d\xi \right)_{i=1}^{K-1} \qquad (4.13)$$

which is continuous on a closed ball

$$\overline{S} = \left\{ \vec{u} \in \mathrm{P}^{K-1} : \|\vec{u}\|_{0,\infty,\hat{\omega}_N} = \max_{1 \le i \le K-1} |u^{(0)}(x_i)| \le r \right\}$$

with r defined by $\|f\|_{0,\infty,[0,1]}$ and translate the ball $\overline{S}$ into itself. Therefore, by Brower's fixed point theorem (see e.g. [25]) there exists a fixed point of this operator in $\overline{S}$, i.e. the system of equations (4.11) is solvable.

Remark 2 *The fixed point iteration for equation (4.11) is equivalent to the solution of sequence of the following problems*

$$\frac{d^2 u^{(0),n+1}(x)}{dx^2} - N\!\left(u^{(0),n}(x_{i-1})\right)\!u^{(0),n+1}(x) = -f(x),$$
$$x \in (x_{i-1}, x_i), \qquad i = \overline{1, K}$$
$$u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0,$$
$$n = 0,1,\ldots,$$

(4.14)

where $\overrightarrow{u^{(0),0}} = \left(u^{(0),0}(x_i)\right)_{i=\overline{1,K}}$ is an arbitrary vector from the ball $\overline{S}$. For this problem there exists the following exact difference scheme

$$\left(a^n(x_i)u^{(0),n+1}(x_i)_{\bar{x}}\right)_x - d^n(x_i)u^{(0),n+1}(x_i) = -\varphi^n(x_i), \qquad i = \overline{1, K},$$
$$u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0,$$

(4.15)

with

$$a^n(x_i) = \left[\frac{\sinh(\sqrt{\mu_i^n} h_i)}{h_i \sqrt{\mu_i^n}}\right]^{-1}, \qquad \mu_i^n = N\!\left(u^{(0),n}(x_{i-1})\right),$$

(4.16)

$$d^n(x_i) = \frac{\sqrt{\mu_i}}{\hbar_i}\tanh\frac{\sqrt{\mu_i}h_i}{2} + \frac{\sqrt{\mu_{i+1}}}{\hbar_i}\tanh\frac{\sqrt{\mu_{i+1}}h_{i+1}}{2}, \qquad \hbar_i = \frac{h_i + h_{i+1}}{2},$$

(4.17)

$$\varphi^n(x_i) = \frac{1}{\hbar_i}\sum_{\alpha=1}^{2}(-1)^{\alpha}\left[\frac{dW_{\alpha}^i(x_i)}{dx} - W_{\alpha}^i(x_i)(-1)^{\alpha+1}\sqrt{\mu_{i-1+\alpha}^n}\coth\sqrt{\mu_{i-1+\alpha}^n}\,h_{i-1+\alpha}\right],$$

(4.18)

where $W_{\alpha}^i(x), \quad \alpha = 1,2$ are solutions of the following two Cauchy problems

$$\frac{d^2 W_{\alpha}^j(x)}{dx^2} - N\!\left(u^{(0),n}(x_i)\right)\!W_{\alpha}^j(x) = -f(x),$$
$$x_{j-2+\alpha} < x < x_{j-1+\alpha},$$
$$W_{\alpha}^j(x_{j+(-1)^\alpha}) = \frac{dW_{\alpha}^j(x)}{dx}\bigg|_{x=x_{j+(-1)^\alpha}} = 0, \qquad \alpha = 1,2.$$

(4.19)

In order to compute the coefficients of the exact difference scheme for one iteration step one should solve $2(K-1)$ Cauchy problems by an IVP-solver, each on a small interval with the length of the corresponding step-size. Then the difference scheme with a tridiagonal matrix can be solved by the special elimination method (method of chasing, method ``progonki'') which in our case is stable [37].

Remark 3 *An other algorithmic implementation of the fixed point iteration for (4.11) can be done by the multiple shooting method [44] in the following way. Let*

$u^{(0)}(x) = u^{(0)}(x; x_{i-1}, s_{i-1}, s_{i-1}^{(1)})$ *and* $\dfrac{d}{dx}u^{(0)}(x) = \dfrac{d}{dx}u^{(0)}(x; x_{i-1}, s_{i-1}, s_{i-1}^{(1)})$ *be the solution and*

its derivative of the differential equation (4.5) on the subinterval (x_{i-1}, x_i) *subject to the initial*

values s_{i-1} *and* $s_{i-1}^{(1)}$ *(and obtained by some IVP-solver), i.e. we have*

$$u^{(0)}(x_i; x_{i-1}, s_{i-1}, s_{i-1}^{(1)}) = s_i,$$
$$\frac{d}{dx} u^{(0)}(x_i; x_{i-1}, s_{i-1}, s_{i-1}^{(1)}) = s_i^{(1)}, i = 1, ..., K, \qquad (4.20)$$
$$s_0 = 0, \ s_K = 0,.$$

Analogously to the multiple shooting method this system of equations can be written down in the form $s = F(s)$ *where* $s = (s_0^{(1)}, s_1, s_0^{(1)}, ..., s_{K-1}, s_{K-1}^{(1)}, s_K^{(1)})^T$ *and* $F(s)$ *for an arbitrary* s *can be calculated using an IVP-solver. From the discussion above it follows that the fixed point iteration*

$$\overset{m+1}{s} = F(\overset{m}{s}), m = 0, 1, ... \qquad (4.21)$$

converges provided that $\overset{0}{s}$ *was chosen within the corresponding ball.*

Let us rewrite the equations (4.7) in the form

$$\frac{d^2 u^{(j+1)}(x)}{dx^2} - q(x) u^{(j+1)}(x)$$
$$= N'\big(u^{(0)}(x_{i-1})\big)\big[u^{(j+1)}(x_{i-1}) u^{(0)}(x) - u^{(j+1)}(x) u^{(0)}(x_{i-1})\big] \qquad (4.22)$$
$$+ F^{(j+1)}(x), \quad x \in (x_{i-1}, x_i), \quad i = \overline{1, K},$$
$$u^{(j+1)}(0) = u^{(j+1)}(1) = 0, \quad j = 0, 1, ...$$

with

$$q(x) = N\big(u^{(0)}(x_{i-1})\big) + N'\big(u^{(0)}(x_{i-1})\big) u^{(0)}(x_{i-1}), \quad x \in [x_{i-1}, x_i), \quad i = \overline{1, K}. \qquad (4.23)$$

Given $u^{(0)}(x_i), i = 1, ..., K-1$, let $G(x, \xi, q(\cdot))$ be the Green function corresponding to the operator on the left side of (4.22) with the Dirichlet boundary conditions. Then problem (4.22) can be performed to

$$u^{(j+1)}(x) = \sum_{p=1}^{K} \int_{x_{p-1}}^{x_p} G(x_i, \xi, q(\cdot)) \int_{x_{p-1}}^{\xi} \frac{du^{(j+1)}(\eta)}{d\eta} d\eta \, u^{(0)}(\xi) d\xi N'\big(u^{(0)}(x_{k-1})\big) -$$
$$- \sum_{p=1}^{K} \int_{x_{p-1}}^{x_p} G(x_i, \xi, q(\cdot)) \int_{x_{p-1}}^{\xi} \frac{du^{(0)}(\eta)}{d\eta} d\eta \, u^{(j+1)}(\xi) d\xi N'\big(u^{(j+1)}(x_{p-1})\big). \qquad (4.24)$$
$$- \int_{0}^{1} G(x_i, \xi, q(\cdot)) F^{(j+1)}(\xi) d\xi, \quad i = \overline{1, K}.$$

In order to estimate $u^{(j+1)}(x)$ we need to estimate the Green function $G(x, \xi, q(\cdot))$, which can be explicitly represented by the formula (see e.g. [37])

$$G(x_i, \xi, q(\cdot)) = \frac{1}{v_1(1)} \begin{cases} v_1(x) v_2(\xi), & x \le \xi, \\ v_1(\xi) v_2(x), & \xi \le x. \end{cases} \qquad (4.25)$$

Here $v_\alpha(x)$, $\alpha = 1, 2$ are the so called stencil functions which satisfy the equations

$$\frac{d^2}{dx^2} v_\alpha(x) - q(x) v_\alpha(x) = 0, \quad 0 < x < 1,$$
$$\alpha = 1, 2, \qquad (4.26)$$
$$v_1(0) = 0, \quad v_1'(0) = 1, \quad v_2(1) = 0, \quad v_2'(1) = -1$$

as well as the continuity conditions

$$[v_\alpha(x)]_{x=x_i} = 0, \quad [v'_\alpha(x)]_{x=x_i} = 0, \quad \alpha = 1,2 \quad i = \overline{1, K-1}. \tag{4.27}$$

These functions possess the following properties:

1° $v_1(x)$ is a non decreasing, non negative function on $[0,1]$,

2° $v_2(x)$ is non increasing, non negative function on $[0,1]$,

3° $v_1(1) = v_2(0)$,

4° $v'_1(x)v_2(x) - v_1(x)v'_2(x) \equiv v_1(1) = v_2(0)$,

These properties as well as the maximum principal imply the estimates

$$0 \le G(x,\xi,q(\cdot)) \le G(x,\xi,0),$$

$$\left| \frac{\partial G(x,\xi,q(\cdot))}{\partial x} \right| \le 1. \tag{4.28}$$

Using (4.28) as well as the assumptions (4.2) we obtain from (4.24)

$$\|u^{(j+1)}\|_{1,\infty,[0,1]} \le |h| \, \|u^{(0)}\|_{1,\infty,[0,1]} N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right) \|u^{(j+1)}\|_{1,\infty,[0,1]} + \|F^{(j+1)}\|_{1,\infty,[0,1]}. \tag{4.29}$$

For $|h|$ small enough this inequality can be transformed to

$$\|u^{(j+1)}\|_{1,\infty,[0,1]} \le c_1 \|F^{(j+1)}\|_{1,\infty,[0,1]} \tag{4.30}$$

with

$$c_1 = \left[1 - |h| \, \|u^{(0)}\|_{1,\infty,[0,1]} N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right) \right]^{-1}. \tag{4.31}$$

and the norms

$$\|v\|_{0,\infty,[0,1]} = \max_{x\in[0,1]} |v(x)|, \qquad \|v\|_{1,\infty,[0,1]} = \max\{ \max_{x\in[0,1]} |v(x)|, \max_{x\in[0,1]} |v'(x)| \}.$$

Further we will need the following two auxiliary statements.

Lemma 4.1 *Let $N(u)$ be represented by the power series $N(u) = \sum\limits_{i=1}^{\infty} a_i u^{2i}$, $a_i \ge 0$ and*

$u^{(p)}(x) \in C^1[0,1]$, $p = 0,1,\dots$, *then*

$$\left\| A_k\left(N(u); u^{(0)}(x),\dots,u^{(k)}(x)\right) - A_k\left(N(u); u^{(0)}(x_{i-1}),\dots,u^{(k)}(x_{i-1})\right) \right\|_\infty \le$$

$$\le 2h \sum_{i=1}^{\infty} i a_i A_k\left(N(u); \|u^{(0)}\|_{1,\infty,[0,1]}, \|u^{(1)}\|_{1,\infty,[0,1]},\dots,\|u^{(k)}\|_{1,\infty,[0,1]}\right) \tag{4.32}$$

$$= |h| A_k\left(N'(u); \|u^{(0)}\|_{1,\infty,[0,1]}, \|u^{(1)}\|_{1,\infty,[0,1]},\dots,\|u^{(k)}\|_{1,\infty,[0,1]}\right)$$

Proof. Since the Adomian polynomials are linear operators with respect to the first argument (see (1.9)), i.e.

$$A_k\left(N(u); u^{(0)}(x),\dots,u^{(k)}(x)\right) = \sum_{i=1}^{\infty} a_i A_k\left(u^{2i}; u^{(0)}(x),\dots,u^{(k)}(x)\right)$$

it is sufficient to consider the case $N(u) = u^{2i}$ only. For this case we have

$$A_k\left(u^{2i}; u^{(0)}(x), \ldots, u^{(k)}(x)\right) =$$

$$= \sum_{\alpha_1 + \ldots + \alpha_k = k} N^{(\alpha_1)}\left(u^{(0)}(x)\right) \frac{\left[u^{(1)}(x)\right]^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \cdot \ldots \cdot \frac{\left[u^{(k-1)}(x)\right]^{\alpha_{k-1} - \alpha_k}}{(\alpha_{k-1} - \alpha_k)!} \frac{\left[u^{(k)}(x)\right]^{\alpha_k}}{(\alpha_k)!} =$$

$$= \sum_{\alpha_1 + \ldots + \alpha_k = k} 2i(2i-1) \cdot \ldots \cdot (2i - \alpha_1 + 1)\left[u^{(0)}(x)\right]^{2i - \alpha_1} \frac{\left[u^{(1)}(x)\right]^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \cdot \ldots$$

$$\cdot \frac{\left[u^{(k-1)}(x)\right]^{\alpha_{k-1} - \alpha_k}}{(\alpha_{k-1} - \alpha_k)!} \frac{\left[u^{(k)}(x)\right]^{\alpha_k}}{(\alpha_k)!},$$

$$\left\| A_k\left(u^{2i}; u^{(0)}(x), \ldots, u^{(k)}(x)\right) - A_k\left(u^{2i}, u^{(0)}(x_{i-1}), \ldots, u^{(k)}(x_{i-1})\right) \right\| \le$$

$$\le \sum_{\alpha_1 + \ldots + \alpha_k = k} 2i(2i-1) \ldots (2i - \alpha_1 + 1)(2i - \alpha_1 + \alpha_1 - \alpha_2 + \ldots + \alpha_{k-1} - \alpha_k + \alpha_k) \cdot$$

$$\cdot \left\| u^{(0)} \right\|_{1,\infty,[0,1]}^{2i - \alpha_1} \frac{\left\| u^{(1)} \right\|_{1,\infty,[0,1]}^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \cdot \ldots \cdot \frac{\left\| u^{(k-1)} \right\|_{1,\infty,[0,1]}^{\alpha_{k-1} - \alpha_k}}{(\alpha_{k-1} - \alpha_k)!} \frac{\left\| u^{(k)} \right\|_{1,\infty,[0,1]}^{\alpha_k}}{(\alpha_k)!} | h | =$$

$$= 2i | h | A_k(u^{2i}; \left\| u^{(0)} \right\|_{1,\infty,[0,1]}, \left\| u^{(1)} \right\|_{1,\infty,[0,1]}, \ldots, \left\| u^{(k)} \right\|_{1,\infty,[0,1]})$$

$$= | h | A_k([u^{2i}]'; \left\| u^{(0)} \right\|_{1,\infty,[0,1]}, \left\| u^{(1)} \right\|_{1,\infty,[0,1]}, \ldots, \left\| u^{(k)} \right\|_{1,\infty,[0,1]}).$$

The lemma is proven.

Lemma 4.2 *Let $N(u)$ be represented by the power series $N(u) = \sum_{j=1}^{\infty} a_j u^{2j}$ then*

$$A_{j+1}(N(u); V_0, \ldots, V_j, 0)$$
$$= \frac{1}{(j+1)!} \left\{ \frac{d^{j+1}}{dz^{j+1}} \left[N(f(z)) - (f(z) - V_0)N'(V_0) \right] \right\}_{z=0}, \qquad (4.33)$$
$$j = 0, 1, \ldots,$$

with $f(z) = \sum_{j=0}^{\infty} z^j V_j$.

Proof. Proof is obvious.

Returning to (4.30) and taking into account (4.2), (4.3) we obtain

$$\left\| u^{(j+1)} \right\|_{1,\infty}$$

$$\le c_1 \Big\{ \sum_{p=1}^{j} A_{j+1-p}(N(u); \left\| u^{(0)} \right\|_{1,\infty,[0,1]}, \left\| u^{(1)} \right\|_{1,\infty,[0,1]}, \ldots, \left\| u^{(j+1-p)} \right\|_{1,\infty,[0,1]}) \left\| u^{(p)} \right\|_{1,\infty,[0,1]}$$

$$+ h \sum_{p=0}^{j} A_{j-p}(N'(u)u; \left\| u^{(0)} \right\|_{1,\infty,[0,1]}, \ldots, \left\| u^{(j-p)} \right\|_{1,\infty,[0,1]}) \left\| u^{(p)} \right\|_{1,\infty,[0,1]} \qquad (4.34)$$

$$+ \frac{1}{(j+1)!} \left[\frac{d^{j+1}}{dz^{j+1}} \left(N\left(\sum_{s=0}^{\infty} z^s \left\| u^{(s)} \right\|_{1,\infty,[0,1]} \right) - \sum_{s=1}^{\infty} z^s \left\| u^{(s)} \right\|_{1,\infty,[0,1]} N'\left(\left\| u^{(0)} \right\|_{1,\infty,[0,1]} \right) \right) \right]_{z=0} \Big\}$$

Introducing in (4.34) the new variables by

$$|h|^{-j}\|u^{(j)}\|_{\infty} = v_j, \qquad (4.35)$$

then changing v_j to V_j and the inequality sign to the equality one we arrive at the following system of equations

$$
\begin{aligned}
V_{j+1} = c_1 \Bigg\{ &\sum_{p=1}^{j} A_{j+1-p}\big(N(u);V_0,\ldots,V_{j+1-p}\big)V_p \\
&+ \sum_{p=0}^{j} A_{j-p}(N'(u)u;V_0,\ldots,V_{j-p})V_p \\
&+ \frac{1}{(j+1)!}\frac{d^{j+1}}{dz^{j+1}}\left(N\left(\sum_{s=0}^{\infty}z^s V_s\right)\right)_{z=0} - V_{j+1}N'(V_0)\Bigg\}, \\
&j = 0,1,\ldots, \quad V_0 = v_0 = \|u^{(0)}\|_{\infty,[0,1]}
\end{aligned}
\qquad (4.36)
$$

or

$$
\begin{aligned}
V_{j+1} = \frac{c_1}{1+c_1 N'(V_0)} \Bigg\{ &\sum_{p=1}^{j} A_{j+1-p}\big(N(u);V_0,\ldots,V_{j+1-p}\big)V_p \\
&+ \sum_{p=0}^{j} A_{j-p}(N'(u)u;V_0,\ldots,V_{j-p})V_p \\
&+ \frac{1}{(j+1)!}\frac{d^{j+1}}{dz^{j+1}}\left(N\left(\sum_{s=0}^{\infty}z^s V_s\right)\right)_{z=0}\Bigg\}.
\end{aligned}
\qquad (4.37)
$$

The solution of this system is a majorant for the solution of (4.34), i.e. $v_j \leq V_j$, $j = 0,1,\ldots$. Using the method of generating functions we obtain from (4.37)

$$
\begin{aligned}
f(z) - V_0 = \frac{c_1}{1+c_1 N'(V_0)} \big\{ &[f(z)-V_0][N(f(z))-N(V_0)] \\
&+ zf^2(z)N'(f(z)) + N(f(z)) - N(V_0)\big\}
\end{aligned}
\qquad (4.38)
$$

From this equation we can express z as a function of f

$$
z = \frac{1}{f^2 N'(f)}\left\{\left(\frac{1}{\tilde{C}} - N(f) + N(V_0)\right)(f-V_0) - N(f) + N(V_0)\right\},
$$
$$
V_0 \leq f, \quad \tilde{C} = \frac{c_1}{1+c_1 N'(V_0)},
\qquad (4.39)
$$

and then find f_m, for which z arrives its maximum $z_m = R$. The condition

$$|h|V_0\big[N'(V_0)\big]^2 < 1 \qquad (4.40)$$

guarantees the existence of f_m because under assumption (4.40) we have

$$z(V_0) = 0, \quad \lim_{f\to\infty} z(f) = 0$$

$$\frac{d}{df}[z(f)f^2 N'(f)]|_{f=V_0} = \frac{1}{\tilde{c}} - N'(V_0) = \frac{1}{c_1} = 1 - |h|\,\|u^{(0)}\|_{\infty,[0,1]}N'(\|u^{(0)}\|_{\infty,[0,1]}) > 0.$$

The value z_m defines the convergence radius of series (4.38), i.e.

$$R^j V_j = C \frac{1}{(j+1)^{1+\varepsilon}}, \qquad (4.41)$$

with an arbitrarily small positive ε. Returning to the old notations we have

$$\|u^{(j)}\|_{\infty,[0,1]} \leq \frac{C}{(j+1)^{1+\varepsilon}} \left(\frac{h}{R}\right)^j, \quad j = 0,1,\dots, \qquad (4.42)$$

which leads to the following sufficient convergence condition for the series $f(z) = \sum_{j=0}^{\infty} z^j V_j$:

$$\frac{h}{R} \leq 1. \qquad (4.43)$$

Thus, we have proved the following assertion.

Theorem 4.1 *Under the assumptions of Lemma4.1 the method (4.10) for problem (4.38) converges super-exponentially (converges) with the error estimate*

$$\|\overset{m}{u} - u\|_{\infty,[0,1]} \leq \frac{C}{(1+m)^{1+\varepsilon}} \frac{(h/R)^{m+1}}{1-h/R}, \quad (C \sum_{j=m+1}^{\infty} \frac{1}{(j+1)^{1+\varepsilon}}) \qquad (4.44)$$

provided that

$$h < R, \quad (h = R). \qquad (4.45)$$

5 Examples

Example 5.1 *Let us consider the Dirichlet boundary value problem*
$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0;1),$$
$$u(0) = u(1) = 0, \qquad (5.1)$$

with $f(x) = \pi^2 \sin \pi x + M(\sin \pi x)^3$ *and a given constant* $M \geq 0$. *The exact solution of (5.1) is*
$$u(x) = \sin \pi x. \qquad (5.2)$$

Problem (5.1) is equivalent to the following nonlinear Fredholm integral equation
$$u(x) = \int_0^1 G(x,\xi)[-Mu^3(\xi) + f(\xi)]d\xi,$$
$$G(x,\xi) = \begin{cases} x(1-\xi), \ x \leq \xi \\ \xi(1-x), \ \xi < x. \end{cases} \qquad (5.3)$$

This equation is of the form (1.1) with
$$N(u) = M \int_0^1 G(x,\xi)u^3(\xi)d\xi, \quad F = \int_0^1 G(x,\xi)f(\xi)d\xi.$$

For $M = 20$ *the usual fixed point iteration as well as the usual ADM are divergent. Due to symmetry we apply the FD-method with two steps to the modified problem (5.1):*
$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0, 0.5),$$
$$u(0) = 0, \quad u'(0.5) = 0. \qquad (5.4)$$

Problem (4.4) for this example has the form

$$\frac{\partial^2 u(x,t)}{\partial x^2} - M\left\{u(\frac{1}{4},t)^2 + t\left[u(x,t)^2 - u(\frac{1}{4},t)^2\right]\right\}u(x,t) = -f(x),$$

$x \in (0, 0.25),$

$$\frac{\partial^2 u(x,t)}{\partial x^2} - M\left\{\frac{1}{2}[u(\frac{1}{4},t) + u(\frac{1}{2},t)]^2 + t\left[u(x,t)^2 - \frac{1}{2}\left(u\frac{1}{4},t) + u(\frac{1}{2},t)\right)^2\right]\right\}u(x,t) = -f(x),$$

$x \in (0.25, 0.5),$

$$u(0,t) = 0, \quad \frac{\partial u(1/2,t)}{\partial x} = 0, \quad [u(x,t)]_{x=0.25} = 0, \quad \left[\frac{\partial u(x,t)}{\partial x}\right]_{x=0.25} = 0$$

(5.5)

The numerical results obtained with Maple are presented in Table 2 and Table 3, where $\Delta^m(x) = u(x) - \overset{m}{u}(x).$

Table 2: Two subintervals ($K = 2$)

x	0.25	0.5
$\Delta^0(x)$	.0109383495	.051202866
$\Delta^1(x)$	.0020822061	.0068667797

Table 3: One subinterval ($K = 1$)

x	0.25	0.5
$\Delta^0(x)$	.02842879773388	.1384486690937460
$\Delta^1(x)$	.01785603134623	.0287451291955546
$\Delta^2(x)$	.00133217977494	.0010162058541
$\Delta^3(x)$	.00455624274781	.0034293443153
$\Delta^4(x)$	.00003810934901	.00201582469655

Example 5.2 *This example goes back to Troesch (see e.g. [45]) and represents a well-known test problem for numerical software:*

$$u'' = \lambda \sinh(\lambda u), \quad x \in (0,1), \quad \lambda > 0, \quad u(0) = 0, \quad u(1) = 1. \qquad (5.6)$$

Due to the hyperbolic type of the nonlinearity here, moderate increase of λ leads to tremendous variations in the derivative of nonlinear part and therefore of the solution. The exact solution of Troesch's test problem can be represented in the form (see, for example [44])

$$u(x,s) = \frac{2}{\lambda}arcsinh\left(\frac{s \cdot sn(\lambda x, k)}{2 \cdot cn(\lambda x, k)}\right), \quad k^2 = 1 - \frac{s^2}{4}, \qquad (5.7)$$

where $sn(\lambda x, k)$, $cn(\lambda x, k)$ *are the elliptic Jacobi functions and the parameter* s *satisfies the equation*

$$\frac{2}{\lambda} arcsinh\left(\frac{s \cdot sn(\lambda, k)}{2 \cdot cn(\lambda, k)}\right) = 1.$$

For example, for the parameter value $\lambda = 10$ it holds $s = 0.35833778463 \cdot 10^{-3}$. To our best knowledge a numerical method working for the highest value of $\lambda = 62$ was presented in [14]. Furthermore it turns out that modern numerical software (Maple, Mathematica) have difficulties even with numerical solution of the algebraic problem above with respect to s for $\lambda > 62$.

Our goal here is to illustrate the presented approach rather then to improve the result $\lambda = 62$. However this approach in theory allows one to threat problem (5.6) with any λ using appropriate number of subdivisions K.

All computation presented below have been performed using QD libary with quad-double precision (appx. 60 decimal digits of accuracy) and GSL templates for quadrature formulas and matrix calculations. To eliminate the influence of quadrature errors we was using Runge type estimate with $Eps = 10^{-20}$. A final stoping criterion in (4.10) was $\|u^{(j+1)}\| < 10^{-10}$.

Table 4: Troesch test with $K = 4$ and $K = 256$

m	$\lambda = 1, K = 4$ $\|\bar{u} - u\|_m$	$\lambda = 10, K = 256$ $\|\bar{u} - u\|_m$
0	.1116088037901624715702912 86e-2	.39195545356289242597643 6496e-1
1	.6242894211299905397328780 93e-4	.94093762712806697528551 7882e-3
2	.6132747526004530011203605 43e-4	.59143890839451892845408 9104e-4
3	.4911153951759336617396356 86e-7	.15807288278568234842569 1183e-5
4	.1341936136602905761448884 19e-8	.14379639773096183699442 8146e-5
5	.8260364313419361366029057 61e-11	.44037684223009852192676 8794e-7
7		.53184447073354566106051 9677e-9
8		.71076147951413241721402 1573e-11

References

[1]   About Sinc-approximations. In K. Bowers and J. Lund, editors, *Proceedings of Computation and Control III*, pages --, 1993. Birkhäuser.

[2]   Adomian, G. A new approach to nonlinear partial differential equations. *J. Math. Anal. Appl.*, 102:420--434, 1984.

[3]   Adomian, G. A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.*, 135:501--544, 1988.

[4]   Adomian, G. *A review of the decomposition method in applied mathematics*. MA: Kluwer, Boston, 1994.

[5]   Canuto, C. and Hussaini, M.Y. and Quarteroni, A. and Zang, T.A. *Spectral Methods in Fluid Dynamics*. Springer-Verlag, Berlin, Heidelberg, New York et. all., 1988.

[6]   Daftardar-Gejji, V. and Bhalekar, S. Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method. *Applied Mathematics and Computation*, 202:113--120, 2008.

[7]   Ruehs F. *Funktionentheorie*. DVW, Berlin, 1971.

[8]   Fujita, H. and Saito, N. and Suzuki, T. *Operator Theory and Numerical Methods*. Elsevier, Heidelberg, 2001.

[9]   Gavrilyuk, I.P. Strongly P-positive operators and explicit representation of the solutions of initial value problems for second order differential equations in Banach space. *Journ.of Math. Analysis and Appl.*, 236:327--349, 1999.

[10] Gavrilyuk, I.P. and Hackbusch, W. and Khoromskij, B.N. $\mathcal{H}$-matrix Approximation for the Operator Exponential with Applications. *Numer. Math.*, 92:83--111, 2002.

[11]   Gavrilyuk, I.P. and Hackbusch, W. and Khoromskij, B.N. Data- Sparse Approximation to the Operator-Valued Functions of Elliptic Operator. *Math. Comp.*, 73:1297--1324, 2004.

[12]   Gavrilyuk, I.P. and Hackbusch, W. and Khoromskij, B.N. Tensor-Product Approximation to Elliptic and Parabolic Solution Operators in Higher Dimensions (to appear). *Computing*, 2004

[13]   Gavrilyuk, I.P. and Hackbusch, W. and Khoromskij, B.N. Data-sparse approximation of a class of operator-valued functions. *Math. Comp.*, 74:681--708, 2005.

[14]   Gavrilyuk, I.P. and Hermann,M. and Kutniv, M.V. and Makarov, V.L. Difference schemes for nonlinear BVPS using Runge-Kutta IVP-solvers. *Advances in Difference Equations*, 2006:1--29, 2006.

[15]   Gavrilyuk, I.P. and Klimenko, A.V. and Makarov, V.L. and Rossokhata, N.O. Exponentially convergent parallel algorithm for nonlinear eigenvalue problems. *IMA J. Numer. Anal.*, 27:818--838, 2007.

[16]   Gavrilyuk, I.P. and Klimenko, A.V. and Makarov, V.L. and Rossokhata, N.O. FD-method for eigenvalue problems with nonlinear potential. *Ukrainian Mathematical Journal*, 59:14--28, 2007.

[17] Gavrilyuk, I.P. and Makarov, V.L. Representation and approximation of the solution of an initial value problem for a first order differential eqation in Banach space. *Z. Anal. Anwend.(ZAA)*, 15:495--527, 1996.

Gavrilyuk, I.P. and Makarov, V.L. Algorithms without accuracy saturation for evolution equations in Hilbert and Banach spaces. *Math. Comp.*, 74:555--583, 2005.

[18] Gavrilyuk, I.P. and Makarov, V.L. Exponentially convergent parallel discretization methods for the first order evolution equations. *Computational Methods in Applied Mathematics (CMAM)*, 1(4):333-355, 2001.

[19] Gavrilyuk, I.P. and Makarov, V.L. Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces. *SIAM Journal on Numerical Analysis (to appear)*, 43(5):2144--2171, 2005.

[20] Gavrilyuk, I.P. and Makarov, V.L. An exponentially convergent algorithm for nonlinear differential equations in Banach spaces. *Math. Comp.*, 76:1895--1923, 2007 .

[21] Gavrilyuk, I.P. and Makarov, V.L. Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces. *SIAM Journal on Numerical Analysis*, 43(5):2144--2171, 2007 .

[22] Gavrilyuk, I.P. and Makarov, V.L. and Vasylyk, V. A new estimate of the Sinc method for linear parabolic problems including the initial point. *Computational Methods in Applied Mathematics (CMAM)*, 4:1-- 27, 2004.

[23]   Henry, D. *Geometrical Theory of Semilinear Parabolic Equations*. Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[24] Abbaoui K. and Pujol M. J.and Cherruault Y.and Himoun N. and Grimalt P. A new formulation of Adomian method: convergence result. *Kybernetes*, 30(9-10):1183-- 1191, 2001.

[25] Kantorovich, L.V. and Akilov, G.P. *Functional Analysis*. Pergamon Press, Princeton, 1982.

[26] Krasnosel'skii, M.A. and Vainikko, G.M. and Zabreiko, P.P. and Rutitskii, Ja.B. and Stezenko, V.Ja. *Approximate solution of operator equations*. Wolters-Noordhoff, Groningen, 1972.

[27] Krasnosel'skij, M.A. and Sobolevskij, P.E. Fractional powers of operators acting in Banach spaces (in Russian). *Doklady AN SSSR*, 129(3):499--502, 1959.

[28] Krein, S.G. *Linear Differential Operators in Banach Spaces*. Amer. Math. Soc., New York, 1971.

[29] El-Kalla I. L. Error analysis of Adomian series solution to a class of nonlinear differential equations. *Appl. Math. E-Notes*, 7:214--221, 2007.

[30] Lazurchak,I.I. and Makarov, V. L. and Sytnyk,D. Two-sided approximations for nonlinear operator equations. *CMAM*, 8(4):386--392, 2008.

[31] Makarov, V.L. About functional-discrete method of arbitrary accuracy order for solving Sturm-Liouville problem with piecewise smooth coefficients. *DAN SSSR*, 320(1):34-- 39, 1991.

[32] Makarov, V.L. The functional-discrete method for solving the eigenvalue problems for nonlinear differential equations. *Dopovidi NAN Ukr.*, (8):16-- 22, 2008.

[33] Pazy, A. *Semigroups of linear operator and applications to partial differential equations*. Springer Verlag, New York, Berlin, Heidelberg, 1983.

[34] Rach, R. A convenient computational form for the Adomian polynomials. *J. Math. Anal. Appl.*, 102:415--419, 1984.

[35] Ray, S.S. A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends. *Applied Mathematics and Computation*, 202:544--549, 2008.

[36] Rozhdestvenskii, B. L. The Picard method as a solution method for problems of mathematical physics. *Chislennye metody mehaniki sploshnoj sredy (Novosibirsk)*, 5(2):96--103, 1974.

[37] Samarskii, A.A.. *Theory of difference schemes*. M.Dekker, New York, 2001.

[38] Samarskii, A.A. and Gavrilyuk, I.P. and Makarov, V.L. Stability and regularization of three-level difference schemes with unbounded operator coefficients in Banach spaces. *SIAM Journal on Numerical Analysis*, 39(2):708--723, 2001.

[39] Sheen, D. and Sloan, I. H. and Thomée, V. A parallel method for time-discretization of parabolic problems based on contour integral representation and quadrature. *Math.Comp.*, 69:177--195, 2000.

[40] Sheen, D. and Sloan, I. H. and Thomée, V. A parallel method for time-discretization of parabolic equations based on Laplace transformation and quadrature. *IMA Journal of Numerical Analysis*, 23:269--299, 2003.

[41] Solomjak, M.Z. Application of the semi-group theory to investigation of differential equations in Banach spaces (in Russian). *Doklady AN SSSR*, 122(5):766--769, 1958.

[42] Stenger, F. *Numerical methods based on Sinc and analytic functions*. Springer Verlag, New York, Berlin, Heidelberg, 1993.

[43] Stenger, F. Collocating Convolutions. *Math.Comp.*, 64:211--235, 1995.

[44] Stoer, J. and Bulirsch, R. *Introduction to Numerical Analysis*. Springer, Berlin, 2008.

[45] B. A. Troesch. A simple approach to a sensitive two-point boundary value problem. *J. Comput. Phys.*, 21:279--290, 1976.

[46] Akhmet M. U. Integral manifolds of differential equations with piecewise constant argument of generalized type. *Nonlinear Anal.*, 66(2):367--383, 2007.

[47] Vasylyk, V. Uniform exponentially convergent method for the first order evolution equation with unbounded operator coefficient. *Journal of Numerical and Applied Mathematics (ISSN 0868-6912)*, 1:99--104 (in Russian), 2003.

[48] Wazwaz, A. M. A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. Comput.*, 111:53--69, 2000.