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About Some Geometric Characteristic of the Generalized Möbius Listing's surfaces GML_2^n

[1]I. Tavkhelidze, [2]C. Cassisa

[1] I. Vekua Institute of Applied Mathematics, Tbilisi State University, University Street 2, Tbilisi 0186, Georgia -- e-mail: ilia.tavkhelidze@math.sci.tsu.ge
[2] Università di Roma ``La Sapienza", Dipartimento di Matematica, P.le A. Moro, 2, 00185 Roma, Italia -- e-mail: cassisa@mat.uniroma1.it

Abstract

In previous articles [1-7] a wide class of geometric figures – "Generalized Twisting and Rotated" bodies (sometimes called "surface of Revolution" see [11]) - shortly GTR_m^n - was defined through their analytic representation. In particular cases, this analytic representation gives back many classical objects (torus, helicoid, helix, Möbius strip ... etc.). Aim of this article is to consider some geometric properties of a wide subclass of the already defined surfaces, by using their analytical representation. In previous articles [1-7] a set of the Generalized Möbius Listing's bodies - shortly GML_m^n, which are a particular case of the GTR_m^n bodies, have been defined. In the present paper we show some geometric properties of Generalized Twisting and Rotated - surfaces and relationships between the set GML_2^n and the sets of Knots and Links.

Key words and phrases. *Analytic representation, Möbius strip, Möbius-Listing's surfaces, First fundamental form, Second fundamental form, Mean curvature, Gaussian Curvature, Hyperbolic point, Parabolic point, Knots, Link-2.*

Notations

In this article we use following notations:

- X, Y, Z, or x, y, z - is the ordinary notation for coordinates;
- τ, ψ, θ - are space values (local coordinates or parameters in parallelogram):

$$\begin{aligned}
&1.\quad \tau \in [\tau_*, \tau^*],\ where\ \tau_* \le \tau^*\ usually\ are\ non-negative\ constants;\\
&2.\quad \psi \in [0, 2\pi]; \qquad\qquad (1)\\
&3.\quad \theta \in [0, 2\pi h],\ where\ h \in R\ (Real);
\end{aligned}$$

But sometimes, as a special case, we suppose that

$$\tau \in [-\tau^*, \tau^*] \qquad (1^*)$$

- $P_m \equiv A_1 A_2 \dots A_m$ - denotes an "Plane figure with m-symmetry", in particular P_m is a ``regular polygon" and $m$ is the number of its angles or vertices. In the general case the edges of ``regular polygons" are not always straight lines ($A_i A_{i+1}$ may be, for example: edge of epicycloid, or edge of hypocycloid, or part of lemniscate of Bernoulli, and so on) (see e.g. Fig. 1f);
- $PR_m \equiv A_1 A_2 \dots A_m A'_1 A'_2 \dots A'_m$ denotes an orthogonal prism, whose ends $A_1 A_2 \dots A_m$ and $A'_1 A'_2 \dots A'_m$ are "Plane m-symmetric figures" P_m (see e.g. Fig. 1b);

For example:

- PR_0 - is a segment and P_0 is a point;

- PR_1 - is an orthogonal cylinder, whose cross section is a P_1 - plane figure without symmetry;
- $PR_2 \equiv A_1A_2A_1'A_2'$ is a rectangle, if $P_2 \equiv A_1A_2$ is a segment of straight line; but also PR_2 maybe a cylinder with cross section P_2 (ellipse, or lemniscate of Bernoulli and so on);
- PR_∞ - is an orthogonal cylinder, whose cross section is a P_∞-circle.

$$x = p(\tau,\psi), \quad z = q(\tau,\psi) \tag{2}$$

or

$$x = p(\tau,\psi)\cos\psi, \quad z = p(\tau,\psi)\sin\psi \tag{2*}$$

are the analytic representations of a **“Plane figure with m-symmetry”** P_m, usually $p(0,0) = q(0,0) = 0$ and the point $(0,0)$ is the center of symmetry of this polygon (see [1-5]).

For example, when the function $p(\tau,\psi)$ in formula (2^*) has the form

$$p(\tau,\psi) \equiv \tau\sum_{i=0}^{m-1}\varepsilon(\psi-\psi_i), \tag{2’}$$

where the arguments τ and ψ are defined in (1); $\psi_i \in [0,2\pi)$ are some constants for each $i = \overline{1,m-1}$, with $\psi_i \neq \psi_j$ if $i \neq j$, and

$$\varepsilon(\psi-\psi_i) \equiv \begin{cases} 0 & if \quad \psi = \psi_i \\ 1 & if \quad \psi \neq \psi_i, \end{cases}$$

then the corresponding plane figure P_m (some time in this article P_m^*) is:

1. a **“simple star”** with m “wings” or “vertices” when $\tau_* \equiv 0$ (see e.g. Fig. 1.a.) $\overline{i = 0,6}$);
2. a set of m segments of straight lines lying on the radiuses of a circle centered at the origin when $\tau_* > 0$ (see e.g. Fig. 1.e.)).

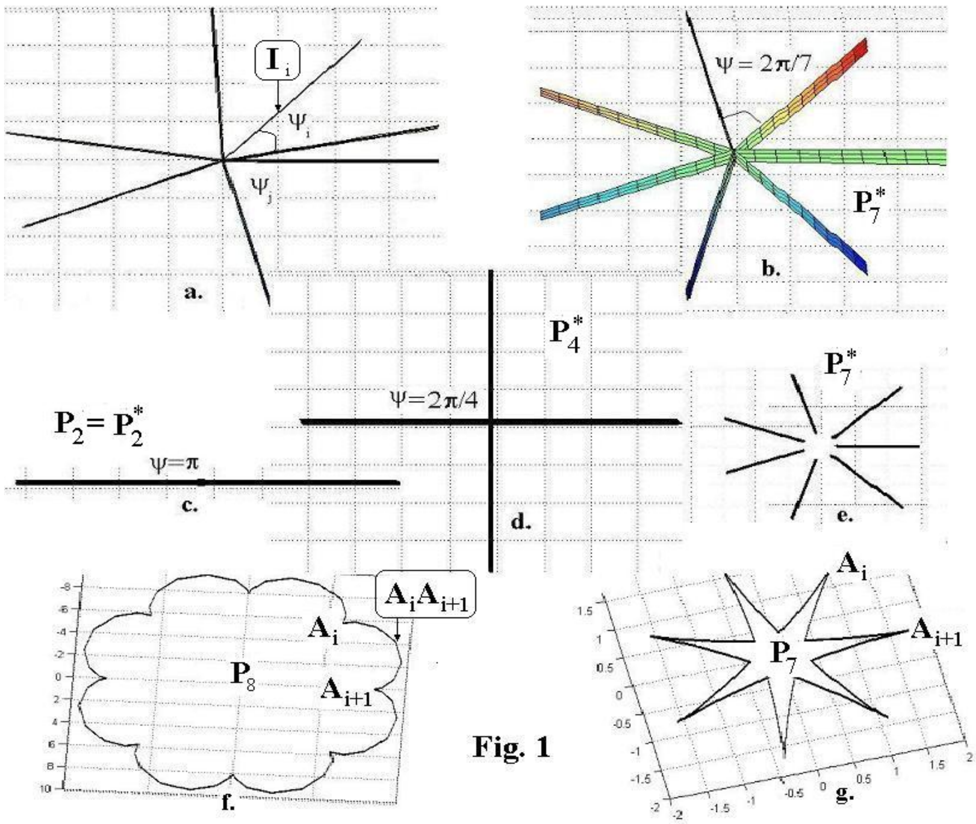


Fig. 1

Conclusion 1. In the case when $\psi_i \equiv \dfrac{2\pi i}{m}, i = \overline{0, m-1}$, then P_m^* is a **"Regular simple star"** (see Figs. 1 b., d., c.);

- if $m = 2$ and $\tau_* \equiv 0$, then (2') is a representation of P_2^* (see e.g. Fig. 1.c.)), which is a segment of straight line $[-\tau^*, \tau^*]$. In this particular case we set

$$p(\tau, \psi) \equiv \tau \tag{2**}$$

where the "argument" τ satisfies (1^*) ;

- $D(p,q)$ or $D(p)$ - diameter of plane figure P_m;
- OO' - axis of symmetry of the prism PR_m;
- L_ρ - Family of lines situated on the plane, whose parametric representations are

$$L_\rho = \begin{cases} X = f_1(\rho, \theta) \\ Y = f_2(\rho, \theta) \end{cases} \quad \rho \in [0, \rho^*), \quad \theta \in [0, 2\pi h], \quad h \in Z \tag{3}$$

or

$$L_\rho = \begin{cases} X(\rho, \theta) = \rho_1(\theta)\cos\theta \\ Y(\rho, \theta) = \rho_2(\theta)\sin\theta \end{cases} \tag{3*}$$

We assume the following hypotheses:

i) For any parameters $\rho_1, \rho_2 \in [0, \rho^*], \rho_1 \neq \rho_2$, the lines L_{ρ_1} and L_{ρ_2} have not intersection.

ii) If L_ρ is a closed curve, then for every fixed $\rho \in [0, \rho^*]$ f_i - are $2p$-periodic functions $f_i(\rho, \theta + 2\pi) = f_i(\rho, \theta), (i = 1,2)$.

- $g(\theta)$ - be an arbitrary sufficiently smooth function

$$g(\theta) : [0, 2h\pi] \to [0, 2h\pi] \tag{4}$$

and if $h = 1$, then for every $\Theta \in [0, 2\pi]$ there exists $\theta \in [0, 2\pi]$, such that $\Theta = g(\theta)$;

- $mod_m(n)$ - natural number $< m$; for every two numbers $m \in N$ (natural) and $n \in Z$ (integer) there exists a unique representation $n = km + j \equiv km + mod_m(n)$, where $k \in Z$ and $j \equiv mod_m(n) \in N \cup \{0\}$;

- $$\mu \equiv \begin{cases} n/m, & when \quad m \in N \quad and \quad n \in Z \\ n & when \quad m = \infty \quad and \quad n \in Z \quad (or\ n \in R \text{ (Real)}) \end{cases} \tag{5}$$

- **Generalized Twisting and Rotated bodies** - shortly GTR_m^n (sometimes called **"Surfaces of revolution"** see[11]) are defined by the parametric representations:

$$\begin{aligned} X(\tau, \psi, \theta) &= f_1\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right], \theta\right) \\ Y(\tau, \psi, \theta) &= f_2\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right], \theta\right) \\ Z(\tau, \psi, \theta) &= Q(\theta) + p(\tau,\psi)\sin(\mu g(\theta)) + q(\tau,\psi)\cos(\mu g(\theta)), \end{aligned} \tag{6}$$

or

$$\begin{aligned} X(\tau,\psi,\theta) &= \left[\rho_1(\theta) + p(\tau)\cos(\psi + \mu g(\theta))\right]\cos(\theta) \\ Y(\tau,\psi,\theta) &= \left[\rho_2(\theta) + p(\tau)\cos(\psi + \mu g(\theta))\right]\sin(\theta) \\ Z(\tau,\psi,\theta) &= Q(\theta) + p(\tau)\sin(\psi + \mu g(\theta)), \end{aligned} \tag{6*}$$

where, respectively:

- the arguments (τ,ψ,θ) are defined in (1);
- the functions f_1 and f_2 or $\rho_1(\theta)$ and $\rho_2(\theta)$ in (3) or (3^*) define the L_R **"Shape of plane basic line"**, more precisely "Shape of orthogonal projection on the plane XOY of the basic line" of corresponding body (see e.g.: circle in – Figs. 2b, 2c, 2g; ellips in – Fig. 2e; spiral in – Figs. 2d, 2f, 2i and square in – Fig. 2h.);
- R is a some fixed real number, which defines the **"Radius"** of the "plane basic line" L_R;
- Functions $p(\tau,\psi)$ and $q(\tau,\psi)$ or $p(\tau)$ in (2) or (2^*) define the **"Shape of the radial cross section"** of corresponding figure. In general case this functions may be depends from arguments ψ (see for example (2')) and θ, i.e. "Shape of the radial cross section" depends from the "place" of this cross section (see e.g. Fig 2i.);
- The function $g(\theta)$ from (4) defines the **"Rule of twisting around basic line"**;
- The number μ in (5) defines the **"Characteristic of twisting"**;
- $Q(\theta)$ is a smooth function which defines the **"Law of vertical stretching of figure"**.

Therefore, this parametric representation defines a GTR_m^n body (some examples are shown in Fig. 2) with the following restrictions:

1) The OO'-axis of symmetry (middle line) of the prism PR_m is transformed into a **"Basic line"** (sometimes called **"Profile curve"**) - $(L_R, Q(\theta))$;

2) Rotation at the end of the prism (2) or (2^*) is semi-regular along the middle line OO', or the twisting of the shape of radial cross section around the basic line is semi-regular (depending from $g(\theta)$) .

- **Generalized Möbius Listing's body** - shortly GML_m^n - is obtained by identifying the opposite ends of the prism PR_m in such a way that:

A) For any integer $n \in Z$ and $i = 1,\ldots,m$ each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge $A_i A_{i+1}$ coincides with the edge

$$A'_{i+n} A'_{i+n+1} \equiv A'_{mod_m(i+n)} A'_{mod_m(i+n+1)}$$

correspondingly;

B) The integer $n \in Z$ denotes the number of rotations of the end of the prism with respect to the axis OO' before the identification. If $n > 0$, the rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise. Some particular examples of GML_m^n and its graphical realizations can be found in [6,7] (see e.g. Fig 2e.).

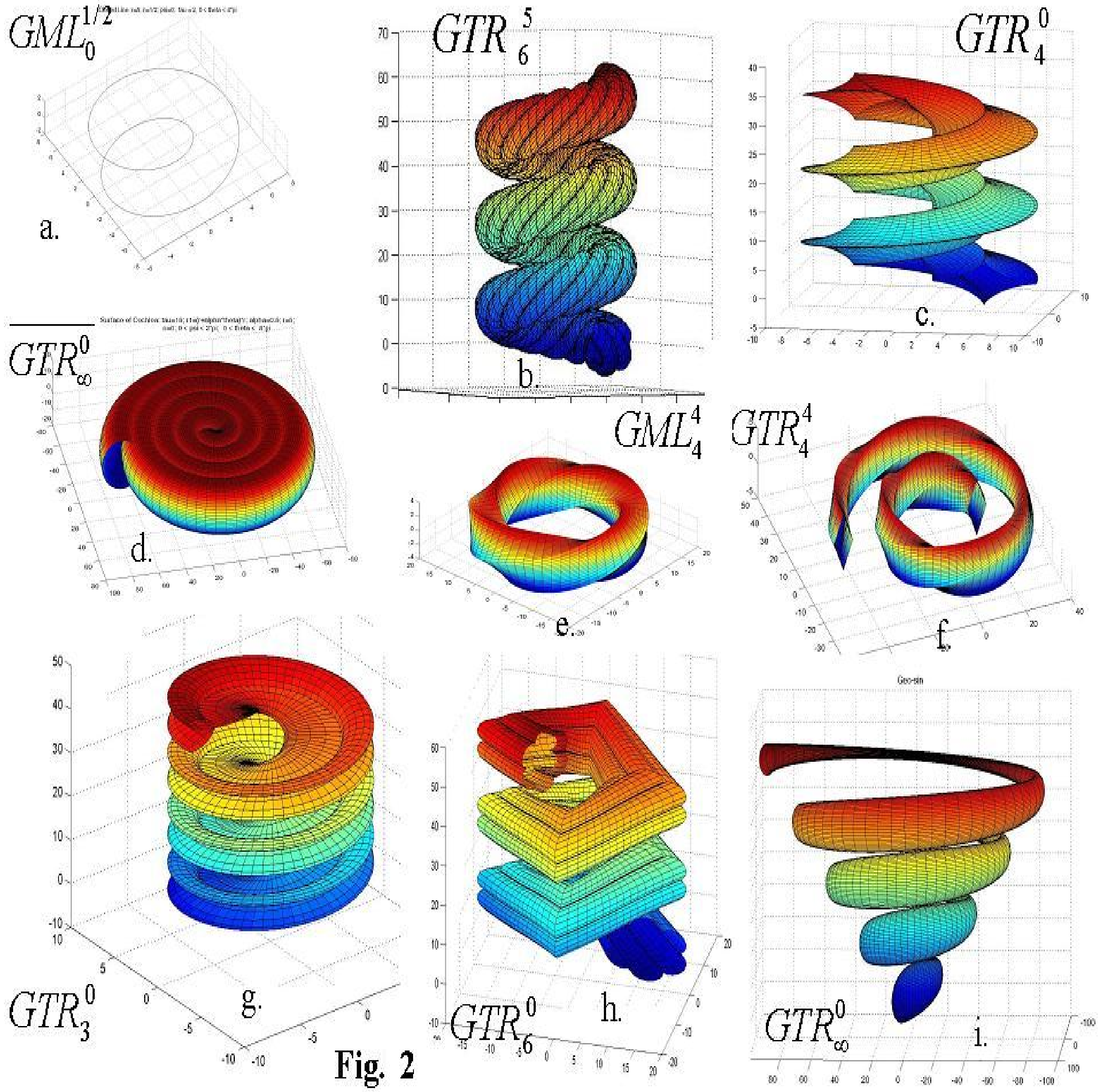


Fig. 2

Conclusion 2. We can assert that:

a.) The GML_m^n body is a particular case of the GTR_m^n body.

b.) The basic line L_R of a GML_m^n body, is always a closed line and the number μ is such that the boundary of this body is a closed surface (see [6]).

c.) The functions f_1, f_2, $\rho_1(\theta)$, $\rho_2(\theta)$, $Q(\theta)$ in parametric representation (6) and (6^*) of a GML_m^n body are always 2π-periodic functions (see e.g. Fig. 3a) or $Q(\theta) \equiv 0$, (some examples are shown in Figs. 3).

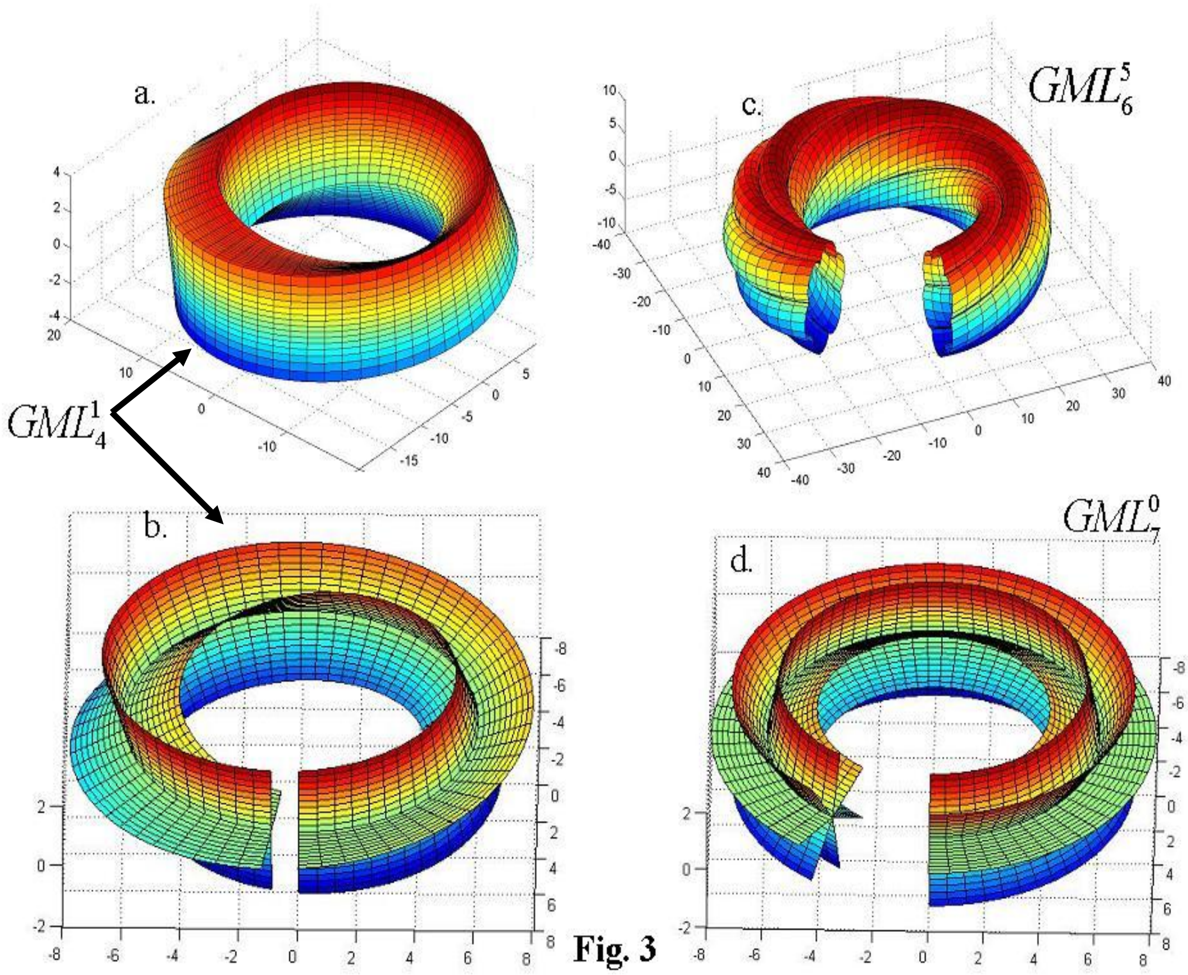


Fig. 3

Some additional information about the classification of GRT_m^n bodies are reported in [3-7].

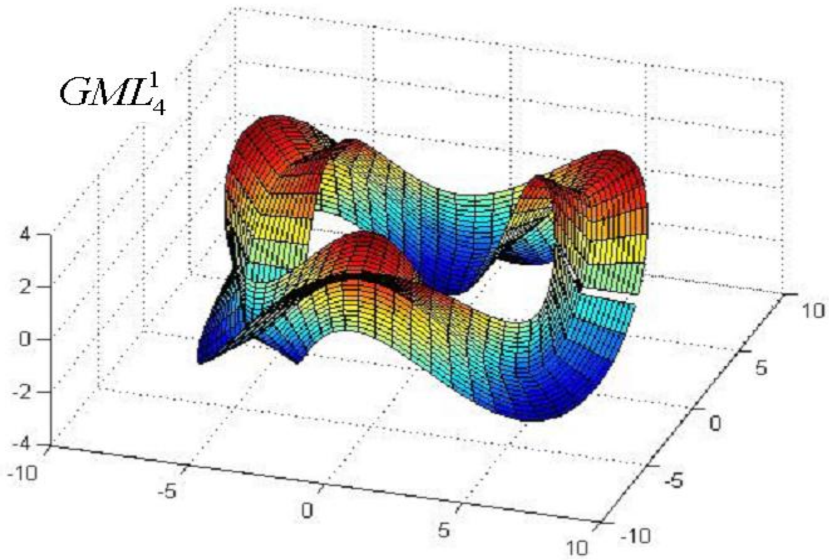


Fig.3a

II. - Some Geometric properties of a "Regular" GML_2^n surfaces.

In this part of our article we study some geometric characteristic of a ``Regular" Generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line. This means that the parametric representations of these surfaces (6) or (6^*) have the following simple form

$$\begin{aligned} X(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta) \\ Y(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta) \\ Z(\tau,\theta) &= \tau\sin\left(\psi + \frac{n\theta}{2}\right), \end{aligned} \tag{7}$$

where, respectively:

- R - radius of basic circle - is constant (see e.g. Figs. 4a., 4b., 4c.);
- In the general case (6) τ, defined in (1), is variable, but now, according to the notation (2**) (Corollary 1.), the argument τ always belongs to the interval $[-\tau^*, \tau^*]$, see (1*) where $\tau^* < R$ is some non-negative constant. So that, when $n=1$ formula (7) is the classical (well known) form of analytic representation of the Möbius strip ([7],[11]). Actually, $2\tau^*$ is the width of the surface GML_2^n;
- the variable θ is defined in (1) and in this case $h \equiv 1$, i.e. $\theta \in [0, 2\pi]$;
- The "rule of twisting around basic line" is **"Regular",** i.e. the function, which is defined by eq. (4) is $g(\theta) \equiv \theta$;
- n - the **"Number or twisting"** of GML_2^n - is an arbitrary integer number, i.e. the number defined by eq. (5) is $\mu \equiv n/2$ (see e.g. *n=1*- **"Möbius strip"** Fig. 4a; *n=2* Fig. 4b; *n=14* Fig. 4c; *n=6* Fig. 4d; *n=0* - Figs. 4e, 4f, 4g, 4h, 4i, 4j.);
- in the present case, ψ is a constant defined in (1) (but when $n = 0$, the number ψ in eq. (7) defines even the type of the corresponding surface, for example: if $\psi = 0$, then the **"Regular"** Generalized Möbius-Listing's surfaces GML_2^0, with basic line a circle, is a Ring ($R > \tau^*$) (see. e.g. Fig. 4e) or Disk ($R = \tau^*$)(see. e.g. Fig. 4g), and if $\psi = \frac{\pi}{2}$, then GML_2^0 is a cylinder (see. e.g. Fig. 4h), in other cases these surfaces are cones or truncated cones (see. e.g. Fig. 4i) (see [3-6])).

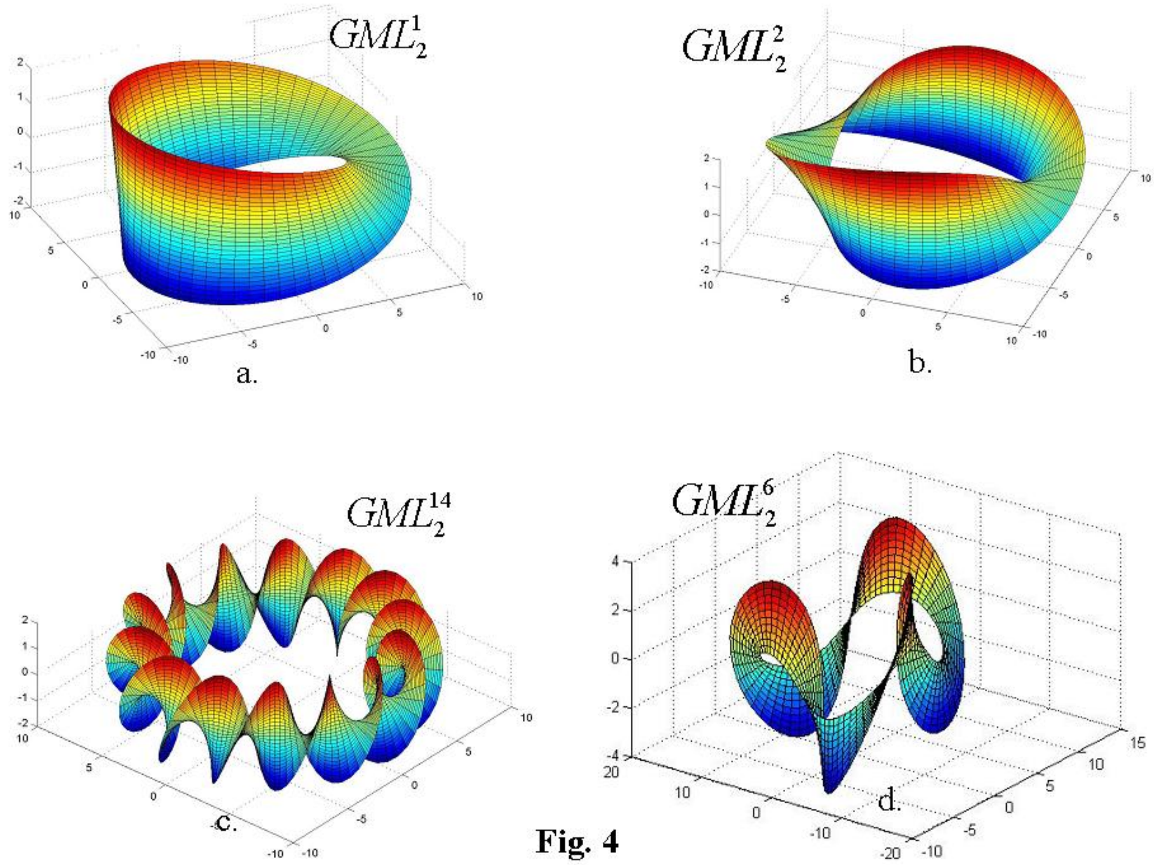
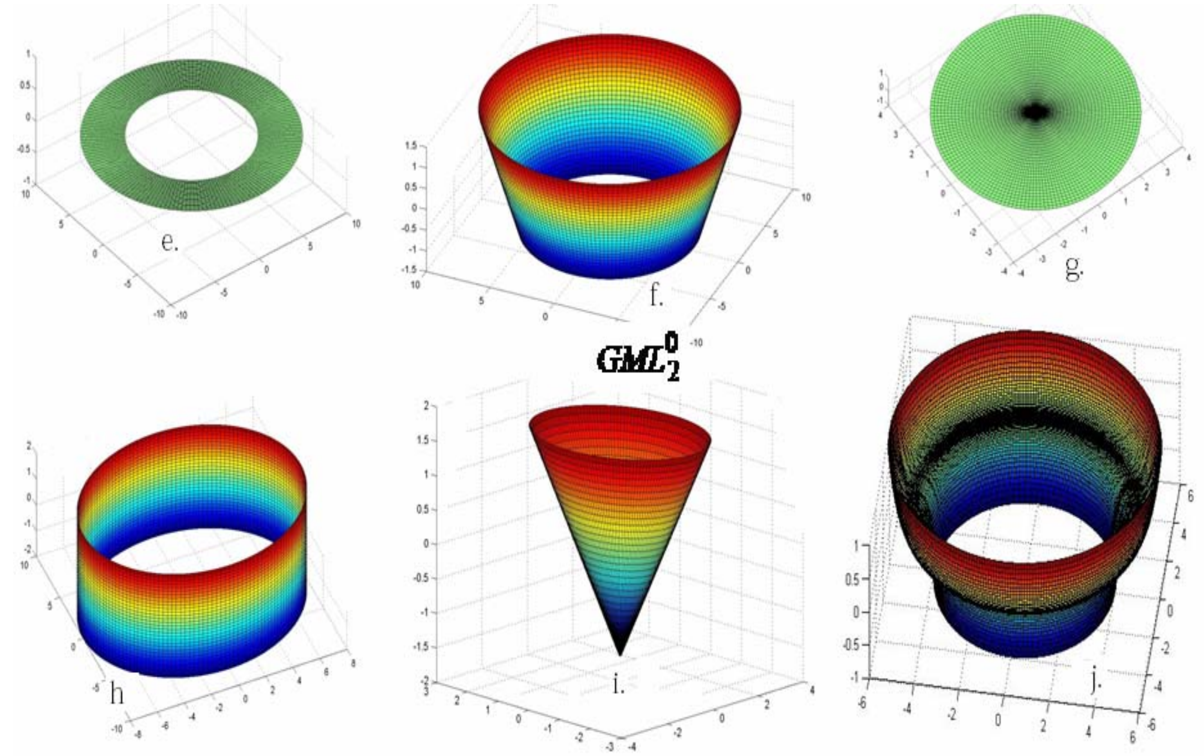


Fig. 4



Remark 1 *Note that:*

a.) For every integer number n eq. (7) defines a one to one correspondence between the strip $[-\tau^*, \tau^*] \times [0, 2\pi)$ and the surface GML_2^n.

b.) If n is an even number, then each function (X, Y, Z) in the representation (7) is a 2π -periodic function of the argument θ.

c.) If n is a odd number, then each function (X,Y,Z) in the representation (7) is a 4π - periodic function satisfying the following properties (Möbius property, see [8])

$$\left(X(\tau,\theta+2\pi);Y(\tau,\theta+2\pi);Z(\tau,\theta+2\pi)\right)=\left(X(-\tau,\theta);Y(-\tau,\theta);Z(-\tau,\theta)\right). \quad \text{(M*)}$$

According to the representation (7), the tangential vectors of the "Regular" Generalized Möbius-Listing's surface GML_2^n, with circle as basic line, are correspondingly

$$\overleftarrow{r}_\tau=\left\{\cos\left(\psi+\frac{n\theta}{2}\right)\cos(\theta);\cos\left(\psi+\frac{n\theta}{2}\right)\sin(\theta);\sin\left(\psi+\frac{n\theta}{2}\right)\right\} \tag{8}$$

and

$$\overleftarrow{r}_\theta=\left\{\begin{array}{c}-\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\cos(\theta);\\ \left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta);\\ \frac{\tau n}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\end{array}\right\}. \tag{9}$$

It is easy to check, that the scalar product of these two vectors (8) and (9) is

$$(\overleftarrow{r}_\tau,\overleftarrow{r}_\theta)=0.$$

Remark 2 *For any integer number n two tangential vectors of a regular GML_2^n, with circle as basic line, are always orthogonal, i.e. the local system of coordinates (τ,θ) in this surface is an orthogonal system.*

Also we may check that

$$\begin{aligned}\frac{\partial(x,y)}{\partial(\tau,\theta)}&=\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos\left(\psi+\frac{n\theta}{2}\right);\\ \frac{\partial(z,x)}{\partial(\tau,\theta)}&=-\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta)-\frac{\tau n}{2}\cos(\theta);\\ \frac{\partial(y,z)}{\partial(\tau,\theta)}&=-\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\left(\psi+\frac{n\theta}{2}\right)\cos(\theta)+\frac{\tau n}{2}\sin(\theta);\end{aligned} \tag{10}$$

and the module of the vector product of these two vectors is

$$|\overleftarrow{r}_\tau\times\overleftarrow{r}_\theta|=\sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}\,. \tag{11}$$

Therefore, the unit normal vector of a ``Regular'' Generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, has the form

$$\overleftarrow{\nu}(\tau,\theta)=\left\{\begin{array}{c}\dfrac{-\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\sin\left(\psi+\dfrac{n\theta}{2}\right)\cos(\theta)+\dfrac{n\tau}{2}\sin(\theta)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}};\\ \dfrac{-\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\sin\left(\psi+\dfrac{n\theta}{2}\right)\sin(\theta)-\dfrac{n\tau}{2}\cos(\theta)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}};\\ \dfrac{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\cos\left(\psi+\dfrac{n\theta}{2}\right)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}}\end{array}\right\}. \qquad (12)$$

Remark 3 *Note that:*

a.) If n is an even number, then the unit normal vector $\overleftarrow{\nu}(\tau,\theta)$ is a 2π-periodic vector and consequently the GML_2^n body is a two-sided surface; i.e. to each point of the GML_2^n corresponds one (external or internal) normal vector of this surface;

b.) If n is a odd number, then the unit normal vector $\overleftarrow{\nu}(\tau,\theta)$ is a 4π-periodic vector function, with the Möbius property

$$\overleftarrow{\nu}(\tau,\theta+2\pi)=-\overleftarrow{\nu}(\tau,\theta) \qquad \text{(M**)}$$

so that the GML_2^n is a one-sided surfaces; i.e. to each point of the GML_2^n surface correspond two normal vectors to this surface and, by the geometric point of view, it is impossible to "distinguish" the external from the internal normal vector to this surface.

The first fundamental form of a regular generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, is given by

$$\begin{aligned}&E(\tau,\theta)=1;\\&F(\tau,\theta)=0;\\&G(\tau,\theta)=\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4},\end{aligned} \qquad (13)$$

so that, it is very easy to see that

Remark 4 *Each point of the corresponding surface (7) is regular, i.e, for each point* (τ,θ) *the corresponding forms satisfy the conditions:* $E(\tau,\theta)>0, G(\tau,\theta)>0$ *and* $EG-F^2>0$.

The second fundamental form of a regular generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, is given by

$$L(\tau,\theta)=0;$$
$$M(\tau,\theta)=\frac{nR}{2\sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}}; \tag{14}$$
$$N(\tau,\theta)=\frac{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{2}}{\sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}}\sin\left(\psi+\frac{n\theta}{2}\right).$$

So that we may rewrite the mean and Gaussian curvatures of a regular GML_2^n, with circle as basic line, in the form

$$H(\tau,\theta)=\frac{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{2}}{\left(\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}\right)^{\frac{3}{2}}}\sin\left(\psi+\frac{n\theta}{2}\right) \tag{15}$$

and

$$K(\tau,\theta)=\frac{-n^2R^2}{4\left(\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\tau^2n^2\right)^2}\quad . \tag{16}$$

Remark 5 *Each point of Regular generalized Möbius-Listing's surfaces* GML_2^n*, with circle as basic line, is:*

a.) Hyperbolic (saddle) point if the number $n\neq 0$, i.e. always $K(\tau,\theta)<0$;

b.) Parabolic point if the number $n\equiv 0$, i.e. always $K(\tau,\theta)\equiv 0$;

III. - About some geometric properties of some subclasses of surfaces GTR_2^n

In this part of the article we consider some cases "of non-regular" Generalized Twisting and Rotated surfaces GTR_2^n and their geometric characteristics.

A.) The "Rule of twisting around basic line" is non-regular and the "basic line" is a circle. In this case the parametric representations (6) or (6*) of a GTR_2^n surface are "similar" to eq. (7), and have the following form:

$$\begin{aligned} X(\tau,\theta)&=\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\cos(\theta)\\ Y(\tau,\theta)&=\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\sin(\theta)\\ Z(\tau,\theta)&=Q(\theta)+\tau\sin\left(\psi+\frac{ng(\theta)}{2}\right),\end{aligned} \tag{17}$$

where, respectively every variables and constants appear in representation (7), but

- $R \equiv const.$ - "Radius of plane basic circle" - or more precisely radius of orthogonal projection of the basic line;
- The argument τ is defined in (1*);
- The variable θ is defined in (1) and h is an arbitrary real number;
- $g(\theta)$ - "Rule of twisting" - is an arbitrary function, defined in (4),
- $Q(\theta)$ - "Law of vertical stretching of figure" - is an arbitrary smooth function, satisfying the conditions:

$$\begin{array}{ll} a. & Q(\theta)\ is\ a\ 2\pi - periodic\ \ function\ \ or\ Q(\theta) \equiv 0\,; \\ b.i. & |\,Q(\theta+2\pi)-Q(\theta)\,| > 2\tau^{*},\ for\ each\ \theta\ or\ Q(\theta) \equiv const. \neq 0 \\ b.ii. & It\ is\ an\ increasing\ (or\ decreasing)\ function.,\ \ i.e.\ Q'(\theta) \geq 0\ (or \leq 0). \end{array} \tag{18}$$

Remark 6 *Note that:*

a.) For every integer number n and for sufficiently smooth functions $g(\theta)$ and $Q(\theta)$, with condition (18), eq. (17) defines a one to one transformation of the strip $[-\tau^{*},\tau^{*}]\times[0,2h\pi)$ into the corresponding surface GTR_2^n; i.e. the condition (18) guarantees that the body GTR_2^n have not self-cross points;

b.) $2\pi hR$ is the **"length"** of the basic line of the corresponding GTR_2^n body;

c.) If the number $h \in (0,1)$, then the corresponding GTR_2^n body is always a non-closed body;

d.) If the number $h \equiv 1$, and:

1. If $Q(\theta) \neq 0$ is a 2π-periodic function, then eq. (17) defines a representation of a non-regular GML_2^n, with space basic line closed.

2. If $Q(\theta) \equiv 0$, then eq. (17) defines a representation of a non-regular GML_2^n, with circle as basic line.

3. If $Q(\theta) \equiv const. \neq 0$ or $Q(\theta)$ is a non-2π-periodic function, with the condition (18), then the corresponding GTR_2^n is a "Helix surface", so that it is a non-closed curve;

e.) If the number $h > 1$, then necessarily $Q(\theta) \equiv const. \neq 0$ or $Q(\theta)$ is a non-2π-periodic function, with condition (18). Under these restrictions, eq. (17) defines a representation of a class of ``Helix surfaces" without self-crossing points (by condition (18)), i.e. the basic line of the corresponding GTR_2^n body is a Helix and its surface has not self-crossing points;

f.) If n is a even number and $g(\theta)$ and $Q(\theta)$ are a 2π-periodic functions, then each function (X,Y,Z) in the representation (17) is a 2π-periodic function of the variable θ.

g.) If n is a odd number and and $g(\theta)$ and $Q(\theta)$ are a 2π-periodic functions or $Q(\theta) \equiv 0$, then each function (X,Y,Z) in the representation (17) is a 4π-periodic function of the argument θ, with the property (M*).

According to the representation (17) the tangential vectors of these surfaces have correspondingly the following forms:

$$\overleftarrow{r}_{\tau} = \left\{ \cos\left(\psi + \frac{ng(\theta)}{2}\right)\cos(\theta);\ \cos\left(\psi + \frac{ng(\theta)}{2}\right)\sin(\theta); \sin\left(\psi + \frac{ng(\theta)}{2}\right) \right\} \tag{19}$$

and

$$\overleftarrow{r}_\theta = \left\{ \begin{array}{c} -\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\sin(\theta)-\frac{\tau ng'(\theta)}{2}\sin\left(\psi+\frac{ng(\theta)}{2}\right)\cos(\theta); \\ \left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\cos(\theta)-\frac{\tau ng'(\theta)}{2}\sin\left(\psi+\frac{ng(\theta)}{2}\right)\sin(\theta); \\ \frac{\tau ng'(\theta)}{2}\cos\left(\psi+\frac{ng(\theta)}{2}\right)+Q'(\theta) \end{array} \right\}. \quad (20)$$

According to the formulas (19) and (20), it is possible to check that the scalar product of the tangential vectors is

$$(\overleftarrow{r}_\tau, \overleftarrow{r}_\theta) = Q'(\theta)\sin\left(\psi+\frac{ng(\theta)}{2}\right) \quad (21)$$

Remark 7 *If* $Q'(\theta)\equiv 0$, *then for any integer number* n *two tangential vectors of the corresponding surface are always orthogonal, i.e.* (τ,θ) *- the local system of coordinates in this surface - is an orthogonal system, i.e.*

a.) $Q(\theta)\equiv 0$ - regular or non-Regular generalized Möbius-Listing's Surfaces GML_2^n, with circle as basic line, see Remark 2;

b.) $Q(\theta)\equiv const.$ - Helix surfaces with constant vertical stretching.

Also, by calculation, is possible to see that

$$\begin{aligned} \frac{\partial(x,y)}{\partial(\tau,\theta)} &= \left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\cos\left(\psi+\frac{ng(\theta)}{2}\right); \\ \frac{\partial(z,x)}{\partial(\tau,\theta)} &= -\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\sin\left(\psi+\frac{ng(\theta)}{2}\right)\sin(\theta) \\ &\quad -\left[\frac{\tau ng'(\theta)}{2}-Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\cos(\theta); \\ \frac{\partial(y,z)}{\partial(\tau,\theta)} &= -\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\sin\left(\psi+\frac{ng(\theta)}{2}\right)\cos(\theta) \\ &\quad +\left[\frac{\tau ng'(\theta)}{2}+Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\sin(\theta) \end{aligned} \quad (22)$$

and correspondingly, the module of the vector product of these two vectors is

$$|\overleftarrow{r}_\tau \times \overleftarrow{r}_\theta| = \sqrt{\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2+\left[\frac{\tau ng'(\theta)}{2}+Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2}. \quad (23)$$

So that, from the structures of the normal vector (22)-(23) of the surfaces (17), it is clear that

Remark 8 *The following properties hold:*

a.) If $Q(\theta)\neq 0$ and $Q(\theta)$ is a non-2π-periodic function, then for any integer number n the corresponding Helix surface is a two-sided surface;

b.) If $Q(\theta)\equiv 0$ or $Q(\theta)$ is a 2π-periodic function, then the corresponding Generalized Möbius-Listing's surface GML_2^n is a two-sided surfaces (the unit normal vector is a 2π-periodic function);

c.) If $Q(\theta)\equiv 0$ or $Q(\theta)$ is a 2π -periodic function, then the unit normal vector is a 4π -periodic function, with the property (M**), so that the corresponding Generalized Möbius-Listing's surface GML_2^n is a one-sided surface;

The first fundamental form of this class of surfaces is given by

$$E(\tau,\theta)=1;$$
$$F(\tau,\theta)=Q'\sin\left(\psi+\frac{ng(\theta)}{2}\right);$$
$$G(\tau,\theta)=\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2+Q'^2(\theta)\sin^2\left(\psi+\frac{ng(\theta)}{2}\right)$$
$$+\left[\frac{(\tau ng'(\theta))}{2}+Q'^2(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2, \tag{24}$$

so that

$$EG-F^2=\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2+\left[\frac{(\tau ng'(\theta))}{2}+Q'^2(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2. \tag{25}$$

Remark 9 *Each point of the corresponding GML_2^n surface (17) is a regular point.*

The second fundamental form of this class of surfaces is given by

$$L(\tau,\theta)=0;$$
$$M(\tau,\theta)=\frac{nRg'(\theta)-Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)}{\sqrt{EG-F^2}}$$
$$N(\tau,\theta)=\frac{1}{\sqrt{EG-F^2}}\left\{\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2\sin\left(\psi+\frac{ng(\theta)}{2}\right)\right.$$
$$+\left[R+\tau\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]\left[\frac{\tau n^2g''(\theta)}{4})+Q''(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]$$
$$\left.+\tau g'(\theta)\left[\frac{\tau g'(\theta)}{2}+Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\sin\left(\psi+\frac{ng(\theta)}{2}\right)\right]\right\} \tag{26}$$

Furthermore, we may rewrite the mean and Gaussian curvatures of this class of surfaces

$$K(\tau,\theta)=\frac{-\left[nRg'(\theta)-Q'(\theta)\cos\left(\psi+\frac{ng(\theta)}{2}\right)\right]^2}{\left[EG-F^2\right]^2}. \tag{27}$$

From the explicit form of Gaussian curvature it is clear that

Remark 10 *The following properties hold:*

a.) Each point of this class of GTR_2^n surfaces, defined by eq. (17), is parabolic or hyperbolic (saddle) point;

b.) If the function $Q(\theta) \equiv const.$, then the surfaces of this class have only saddle points (see also Remark 5.).

B.) The "basic line" is an ellipse and the "Rule of twisting around basic line" is regular.

Even in this case the parametric representations (6) or (6*) of the GTR_2^n surfaces are "similar" to eqs. (7) or (17) and have the following form:

$$\begin{aligned} X(\tau,\theta) &= \left[R_1 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta) \\ Y(\tau,\theta) &= \left[R_2 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta) \\ Z(\tau,\theta) &= Q(\theta) + \tau\sin\left(\psi + \frac{n\theta}{2}\right), \end{aligned} \tag{28}$$

where every variables and constants are respectively given by (7) and (17), but

- R_1 and R_2 - the radiuses of the "basic ellipse", (more precisely of the orthogonal projection on the plane XOY of the elliptic basic line) - are constants;
- The ``Rule of twisting around basic line" is regular, i.e. the functions defined in (4) are given by $g(\theta) \equiv \theta$;
- $Q(\theta)$ - the "Law of vertical stretching of figure" – is given by an arbitrary smooth function, satisfying the condition (18) for each θ, when the function $Q(\psi)$ is a non-2π-periodic function.

Remark 11 *For the GTR_2^n surfaces, with representation (28), all the items of Remark 6 are still valid. The only difference between these two cases is determined by the difference between "plane basic lines" (3).*

According to the representation (28), the tangential vectors of these surfaces have correspondingly the following forms:

$$\overleftarrow{r}_\tau = \left\{\cos\left(\psi + \frac{n\theta}{2}\right)\cos(\theta);\ \cos\left(\psi + \frac{n\theta}{2}\right)\sin(\theta); \sin\left(\psi + \frac{n\theta}{2}\right)\right\} \tag{29}$$

which are identic with (8). Furthermore, we have

$$\overleftarrow{r}_\theta = \left\{\begin{array}{c} -\left[R_1 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta) - \frac{\tau n}{2}\sin\left(\psi + \frac{n\theta}{2}\right)\cos(\theta); \\ \left[R_2 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta) - \frac{\tau n}{2}\sin\left(\psi + \frac{n\theta}{2}\right)\sin(\theta); \\ \frac{\tau n}{2}\cos\left(\psi + \frac{n\theta}{2}\right) + K'(\theta) \end{array}\right\}, \tag{30}$$

which is similar to eq. (9). According to eqs. (29) and (30,) it is possible to check that the scalar product of the tangential vectors is given by

$$(\overleftarrow{r}_\tau, \overleftarrow{r}_\theta) = (R_2 - R_1)\cos\left(\psi + \frac{n\theta}{2}\right)\sin(\theta)\cos(\theta) + Q'(\theta)\sin\left(\psi + \frac{n\theta}{2}\right). \tag{31}$$

Remark 12 *The local system of coordinates* (τ,θ) *on the surface (28) is orthogonal (two tangential vectors of the corresponding surfaces are orthogonal), only if the following conditions hold:* $R_1 \equiv R_2$ *and* $Q'(\theta) \equiv 0$, *i.e.*

a. if $Q(\theta) = 0$ and the plane basic line is a circle, this is the case of a regular generalized Möbius-Listing's Surface GML_2^n with circle as basic line - see Remark 2;

b. if $Q(\theta) = const.$ - Helix surfaces with constant vertical stretching, and with circle as projection in the plane XOY of its basic line - see Remark 7.

Furthermore, by calculation, it is possible to see that, in this

$$
\begin{aligned}
&\frac{\partial(x,y)}{\partial(\tau,\theta)} = \left[R_1\sin^2(\theta) + R_2\cos^2(\theta) + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos\left(\psi + \frac{n\theta}{2}\right),\\
&\frac{\partial(z,x)}{\partial(\tau,\theta)} = -\left[R_1 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin\left(\psi + \frac{n\theta}{2}\right)\sin(\theta) -\\
&-\left[\frac{\tau n}{2} + Q'\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta)\\
&\frac{\partial(y,z)}{\partial(\tau,\theta)} = -\left[R_2 + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin\left(\psi + \frac{n\theta}{2}\right)\cos(\theta) +\\
&+\left[\frac{\tau n}{2} + Q'\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta)
\end{aligned} \tag{32}
$$

and correspondingly, the module of the vector product of these two vectors is

$$
\begin{aligned}
|\overleftarrow{r}_\tau \times \overleftarrow{r}_\theta|^2 = &\left[R_1\sin^2(\theta) + R_2\cos^2(\theta) + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]^2\\
&+\left[(R_1 - R_2)\sin\left(\psi + \frac{n\theta}{2}\right)\sin(\theta)\cos(\theta) + \frac{\tau n}{2} + Q'(\theta)\cos\left(\psi + \frac{n\theta}{2}\right)\right]^2.
\end{aligned} \tag{33}
$$

So that, from the structure of the normal vector (see (32)-(33)) of the surface (28), it is possible to see that

Remark 13

a.) If $Q(\theta) \neq 0$ and $Q(\theta)$ is a non-2π-periodic function, then for any integer number n the corresponding Helix surface is a two-sided surface;

b.) If $Q(\theta) \equiv 0$ or is a 2π-periodic function, then the corresponding Generalized Möbius-Listing's surfaces GML_2^n is a two-sided surfaces, i.e. the unit normal vector is a 2π-periodic function;

c.) If $K(\theta) \equiv 0$ or is a 2π-periodic function, then the corresponding Generalized Möbius-Listing's surfaces GML_2^n is a one-sided surfaces, i.e. the unit normal vector is a 4π-periodic function with properties (M**);

The first fundamental form of this class of surfaces is given by

$$E(\tau,\theta)=1;$$
$$F(\tau,\theta)=(R_2-R_1)\cos\left(\psi+\frac{n\theta}{2}\right)\sin(\theta)\cos(\theta)+Q'(\theta)\sin\left(\psi+\frac{n\theta}{2}\right)$$
$$G(\tau,\theta)=\left[R_1\sin^2(\theta)+R_2\cos^2(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2 \tag{34}$$
$$+\left[(R_1-R_2)\sin(\theta)\cos(\theta)+\frac{n\theta}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\right]^2+\left[\frac{n\theta}{2}\cos\left(\psi+\frac{n\theta}{2}\right)+Q'(\theta)\right]^2$$

and after identical transformation it is possible to write

$$EG-F^2=\left[R_1\sin^2(\theta)+R_2\cos^2(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2$$
$$+\left[(R_1-R_2)\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta)\cos(\theta)+\frac{\tau n}{2}+Q'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2. \tag{35}$$

Now, from the explicit forms (34) and (35), it is easy to deduce that

Remark 14 *Each point of the corresponding surface (28) is regular, i.e, for each point* (τ,θ) *the corresponding forms satisfy the conditions:* $E(\tau,\theta)>0, G(\tau,\theta)>0$ *and* $EG-F^2>0$*;*

The second fundamental form of this class of surfaces is given by

$$L(\tau,\theta)=0;$$
$$M(\tau,\theta)=\frac{1}{\sqrt{EG-F^2}}\left\{\frac{n}{2}\left[R_1\sin^2(\theta)+R_2\cos^2(\theta)\right]+Q'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)\right.$$
$$\left.+(R_2-R_1)\sin 2\left(\psi+\frac{n\theta}{2}\right)\sin 2(\theta)\right\}$$
$$N(\tau,\theta)=\frac{1}{\sqrt{EG-F^2}}\left\{\left[\left[R_1+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\times\right.\right.$$
$$\times\left[R_2+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]+\frac{n^2\tau^2}{2}-Q'\sin\left(\psi+\frac{n\theta}{2}\right)\left]\sin\left(\psi+\frac{n\theta}{2}\right)+\right. \tag{36}$$
$$+\frac{1}{2}(R_2-R_1)\left[n\tau\left[1-2\sin^2\left(\psi+\frac{n\theta}{2}\right)\right]+Q'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)\sin(2\theta)\right]$$
$$\left.+Q''(\theta)\left[R_1\sin^2(\theta)+R_2\cos^2(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos\left(\psi+\frac{n\theta}{2}\right)\right\}.$$

Therefore, we may rewrite the mean and Gaussian curvatures of these class of surfaces in the form

$$K(\tau,\theta)=-\frac{M^2(\tau,\theta)}{\left[EG-F^2\right]^2} \tag{37}$$

From the explicit form of the Gaussian curvature it follows that

Remark 15

Each point of a GTR_2^n surface of this class (28), is a hyperbolic (saddle) or parabolic point;

If $Q'(\theta)\equiv 0$, then each point of a GTR_2^n surface of this class (28), is a hyperbolic (saddle) point; i.e.:

1. if $Q(\theta)\equiv 0$, then the corresponding GML_2^n surfaces have only hyperbolic points;

2. if $Q(\theta)\equiv const.>0$, then the corresponding ``Helix" surfaces (28), have only hyperbolic points;

C.) The "basic line" is some line and the "rule of twisting around basic line" is regular. In this case the parametric representations (6) or (6*) of the corresponding surfaces, have a form "similar" to eqs. (7), (17) or (28):

$$\begin{aligned} X(\tau,\theta) &= \left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)\\ Y(\tau,\theta) &= \left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)\\ Z(\tau,\theta) &= Q(\theta)+\tau\sin\left(\psi+\frac{n\theta}{2}\right), \end{aligned} \tag{38}$$

where, respectively every variables and constants are the same appearing in eqs. (7), (17) and (28), but:

- $R(\theta)=\rho_1(\theta)=\rho_2(\theta)$ - i.e. "Radius" of plane basic line L_R, in eq. (3^*) - is a sufficiently smooth function $R(\theta)>\tau^*>0$, satisfying the conditions:

$$\begin{aligned} &i.\ \ R(\theta)\ is\ a\ 2\pi-periodic\ \ function\quad or\ R(\theta)\equiv const.\\ &or\\ &ii.\ |R(\theta+2\pi)-R(\theta)|>2\tau^* \end{aligned} \tag{39}$$

- $g(\theta)\equiv\theta$ - function defined in (4), i.e. "Rule of twisting around basic line" - is regular;
- $Q(\theta)$ - "Law of vertical stretching of figure" - is an arbitrary smooth function, satisfying the condition (18).

Remark 16 *Note that:*

a.) For every integer number n and for sufficiently smooth functions $Q(\theta)$ and $R(\theta)$, with conditions (18)and (39), eqs. (38) define a one to one transformation between the strip $[-\tau^*,\tau^*]X[0,2\pi h)$ and the corresponding GTR_2^n surface; i.e. the corresponding surfaces have not self-cross points;

b.) If $0<h<1$ and $R(\theta)$ satisfies the condition (39), then the basic line is a non-closed line and consequently the corresponding GTR_2^n surfaces are non-closed;

c.) If $h\equiv 1$ and $Q(\theta)\equiv 0$ or $Q(\theta)$ is a 2π-periodic function and :

1. $R(\theta)\equiv const.>\tau^*>0$ or $R(\theta)$ is a 2π-periodic function, then the basic line of figure is a closed line and eqs. (38) define a corresponding GML_2^n surface;

2. $R(\theta)$ is a strictly decreasing or increasing function with condition(39), then eqs. (38) define a GTR_2^n surface, which is a ``Rolling-Winding" surface; In this case the basic line is a plane spiral L_R if $Q(\theta)\equiv 0$ and a space spiral if $Q(\theta)$ is a 2π-periodic function;

3. In the opposite case for the function $R(\theta)$, eqs. (38) define a GTR_2^n surface, which has self-crossing points, i.e. the relation (38) does not define a one to one correspondence between the points of the strip $[-\tau^*,\tau^*]\times[0,2\pi h)$ and the corresponding GTR_2^n surface;

d.) If $h \geq 1$ and $Q(\theta) \equiv const. \neq 0$ or the function $Q(\theta)$ is non-2π-periodic function satisfying conditions (18) and:

1. $R(\theta) \equiv const. > \tau^* > 0$ or $R(\theta)$ is a 2π-periodic function, then the basic line is a helix line and the corresponding GTR_2^n surface is a ``Helix surface";

2. $R(\theta)$ is a strictly decreasing or increasing function satisfying condition (39), then the corresponding GTR_2^n are **"Cochlea surfaces";**

e.) If n is an even number, $R(\theta)$ and $Q(\theta)$ are a 2π-periodic functions, then each function (X,Y,Z) in eqs. (38) is a 2π-periodic function of θ.

f.) If n is an odd number, $R(\theta)$ and $Q(\theta)$ are 2π-periodic functions, then each function (X,Y,Z) in the representation (38) is a 4π-periodic function of θ.

According to eqs. (38), the tangential vectors of these surfaces have correspondingly the following forms:

$$\overleftarrow{r}_\tau = \left\{\cos\left(\psi+\frac{n\theta}{2}\right)\cos(\theta);\ \cos\left(\psi+\frac{n\theta}{2}\right)\sin(\theta); \sin\left(\psi+\frac{n\theta}{2}\right)\right\}, \tag{40}$$

(in this case is identic to (8), (29)) and

$$\overleftarrow{r}_\theta = \left\{\begin{array}{c} -\left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)+\left[R'(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta); \\ \left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)+\left[R'(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta); \\ \frac{\tau n}{2}\cos\left(\psi+\frac{n\theta}{2}\right)+Q'(\theta)\,. \end{array}\right\}. \tag{41}$$

According to eqs. (40) and (41) it is possible to check that the scalar product of the tangential vectors is given by

$$(\overleftarrow{r}_\tau, \overleftarrow{r}_\theta) = R'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)+Q'(\theta)\sin\left(\psi+\frac{n\theta}{2}.\right). \tag{42}$$

Remark 17 *The local system of coordinates* (τ,θ) *on the surface (38) is orthogonal (two tangential vectors of the corresponding surfaces are orthogonal), only if the following conditions hold:* $R(\theta)\equiv const.$*, and* $Q'(\theta)\equiv 0$ *; i.e.*

a. if $Q(\theta)=0$ and the plane basic line is a circle - case of Regular generalized Möbius-Listing's Surfaces GML_2^n with circle as basic line - see Remarks 2 and 16;

b. if $Q(\theta)=const.$ - Helix surfaces with constant vertical stretching, and with circle as a projection of basic line in the plane XOY - see Remark 7.

Also, by calculation, is possible to see that, in this case

$$
\begin{aligned}
&\frac{\partial(x,y)}{\partial(\tau,\theta)} = \left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos\left(\psi+\frac{n\theta}{2}\right),\\
&\frac{\partial(z,x)}{\partial(\tau,\theta)} = \left\{-\left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)+R'(\theta)\cos(\theta)\right\}\sin\left(\psi+\frac{n\theta}{2}\right)\\
&\qquad -\left[\frac{\tau n}{2}+Q'\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta),\\
&\frac{\partial(y,z)}{\partial(\tau,\theta)} = \left\{-\left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)-R'(\theta)\sin(\theta)\right\}\sin\left(\psi+\frac{n\theta}{2}\right)\\
&\qquad -\left[\frac{\tau n}{2}+Q'\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta),
\end{aligned}
\tag{43}
$$

and correspondingly, the module of the vector product of these two vectors is

$$
|\overleftarrow{r}_\tau \times \overleftarrow{r}_\theta| = \sqrt{\left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\left[Q'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)-R'(\theta)\sin\left(\psi+\frac{n\theta}{2}\right)+\frac{\tau n}{2}\right]^2}\,. \tag{44}
$$

So that, from the structure of the normal vector (see (43)-(44)) of the surface (38), it is possible to see, that

Remark 18

a.) If $R(\theta)$ and $Q(\theta)\neq 0$ are non-2π-periodic functions satisfying the conditions (18) and (39), then for any integer number n the corresponding Helix or Cochlea surfaces are two-sided surfaces;

b.) If $R(\theta)$ and $Q(\theta)\neq 0$ are 2π-periodic functions or $Q(\theta)\equiv 0$ and

1. n is an even number, then the corresponding surface GML_2^n is a two-sided surface (the unit normal vector is a 2π-periodic vector function);
2. n is an odd number, then the corresponding surface GML_2^n is a one-sided surface, i.e. the unit normal vector is a 4π-periodic vector function with property (M**);

The first fundamental form of this class of surfaces is given by

$$
\begin{aligned}
&E(\tau,\theta)=1;\\
&F(\tau,\theta)=R'(\theta)\cos\left(\psi+\frac{n\theta}{2}\right)+Q'(\theta)\sin\left(\psi+\frac{n\theta}{2}\right);\\
&G(\tau,\theta)=\left[R(\theta)+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\left[R'(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\right]^2\\
&\qquad+\left[Q'(\theta)+\frac{\tau n}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2,
\end{aligned}
\tag{45}
$$

so that

$$EG - F^2 = \left[R(\theta) + \tau \cos\left(\psi + \frac{n\theta}{2} \right) \right]^2 + \\ + \left[\frac{\tau n}{2} - R'(\theta) \sin\left(\psi + \frac{n\theta}{2} \right) + Q'(\theta) cos\left(\psi + \frac{n\theta}{2} \right) \right]^2 \tag{46}$$

Remark 19 *Each point of the* GTR_2^n *surfaces of this class (38) is regular, i.e. for each point* (τ, θ) *the corresponding forms satisfy the conditions:* $E(\tau, \theta) > 0, G(\tau, \theta) > 0$ *and* $EG - F^2 > 0$*;*

The second fundamental form of this class of surfaces is given by

$$L(\tau, \theta) = 0;$$
$$M(\tau, \theta) = \frac{1}{\sqrt{EG - F^2}} \times$$
$$\times \left\{ \frac{nR}{2} + \left[R'(\theta) \sin\left(\psi + \frac{n\theta}{2} \right) - K'(\theta) \cos\left(\psi + \frac{n\theta}{2} \right) \right] \cos\left(\psi + \frac{n\theta}{2} \right) \right\}$$
$$N(\tau, \theta) = \frac{1}{\sqrt{EG - F^2}} \times \tag{47}$$
$$\times \left\{ \left[R(\theta) + \tau \cos\left(\psi + \frac{n\theta}{2} \right) \right] \left[R(\theta) + \tau \cos\left(\psi + \frac{n\theta}{2} \right) - R''(\theta) \right] \sin\left(\psi + \frac{n\theta}{2} \right) + \right.$$
$$\left. + 2\left[R'(\theta) - \frac{n\tau}{2} \sin\left(\psi + \frac{n\theta}{2} \right) \right] \left[R'(\theta) \sin\left(\psi + \frac{n\theta}{2} \right) - Q'(\theta) \cos\left(\psi + \frac{n\theta}{2} \right) - \frac{n\tau}{2} \right] \right\}$$

Therefore, we can rewrite the mean and Gaussian curvatures of this class of surfaces in the form

$$K(\tau, \theta) = \frac{ - \left[\frac{nR(\theta)}{2} + \left\{ R'(\theta) \sin\left(\psi + \frac{n\theta}{2} \right) - Q'(\theta) \cos\left(\psi + \frac{n\theta}{2} \right) \right\} \cos\left(\psi + \frac{n\theta}{2} \right) \right]^2 }{ \left[EG - F^2 \right]^2 } . \tag{48}$$

From the explicit form of Gaussian curvature it is clear that

Remark 20

a.) Each point of the surfaces GTR_2^n of this class (38), is a parabolic or hyperbolic (saddle) point.

b.) If $Q(\theta) \equiv const.$ and $R(\theta) \equiv const.$, simultaneously, then the surfaces of this class have only saddle points.

D.) **"Tori"** and **"Helix bodies"** surfaces, with regular "rule of twisting around the basic line". In this the case parametric representations (6) or (6*) the corresponding surfaces, have ``similar'' form to eqs. (7), (17), (28) or (38):

$$X(\psi, \theta) = [R + \tau(\psi) \cos(\psi + \mu\theta)] \cos(\theta),$$
$$Y(\psi, \theta) = [R + \tau(\psi) \cos(\psi + \mu\theta)] \sin(\theta), \tag{49}$$
$$Z(\psi, \theta) = Q(\theta) + \tau(\psi) \sin(\psi + \mu\theta),$$

where, respectively every variables and constants are the same appearing in eqs. (7) and (17), but

- $g(\theta) \equiv \theta$ - function defined in (4), i.e. “rule of twisting around basic line” - is regular;
- The number μ - is defined in (5);
- The variables ψ and θ are defined in (1);
- The “Basic line” or more precisely the orthogonal projection on the plane XOY of the basic line is a circle, with radius R;
- $\tau(\psi)$ - “Shape” of radial cross section of bodies - is 2π -periodic function, satisfying the condition

$$0 < \tau(\psi) < \tau^{*} \tag{50}$$

and the corresponding lines (2) satisfy the “stars” condition;
- $Q(\theta)$ - “Law of vertical stretching of figure” - is an arbitrary smooth function, satisfying the condition (18) for each θ, in the case when the function $Q(\theta)$ is non- 2π -periodic function.

Remark 21

a.) For sufficiently smooth functions $\tau(\psi)$ and $Q(\theta)$, satisfying the conditions (18) and (50), eqs. (49) define a one to one transformation between the strip $[0,2\pi)\times[0,2\pi h)$ and the corresponding surface; i.e. corresponding surfaces have not self-cross points;

b.) If $Q(\theta) \equiv 0$ or $Q(\theta)$ is 2π -periodic function then for every integer number n formula (49) is a analytic representation of GTR_2^n surfaces, which in this case are:

1. Classic regular torus or Dupin's Cyclide, when $\mu = 0$,
2. “Winding torus”, when $\mu \neq 0$,
3. Classic regular torus, for any admissible number μ if the function $\tau(\psi) \equiv const. < R$. In this case, difference between these toruses, with different number μ, is a only different angles between the coordinate lines of the corresponding local coordinate system (ψ,θ);

c.) If $Q(\theta) \neq 0$ and $Q(\theta)$ it is non- 2π -periodic function, with condition (18) then for every real number μ the formula (49) is a analytic representation of ``Helix" bodies surfaces;

According to the representation (38) the tangential vectors of the GTR_2^n surfaces in this case have correspondingly the following forms:

$$\overleftarrow{r}_{\psi} = \begin{Bmatrix} [\tau'(\psi)\cos(\psi+\mu\theta) - \tau(\psi)\sin(\psi+\mu\theta)]\cos(\theta); \\ [\tau'(\psi)\cos(\psi+\mu\theta) - \tau(\psi)\sin(\psi+\mu\theta)]\sin(\theta); \\ \tau'(\psi)\sin(\psi+\mu\theta) + \tau(\psi)\cos(\psi+\mu\theta). \end{Bmatrix} \tag{51}$$

and

$$\overleftarrow{r}_{\theta} = \begin{Bmatrix} -[R+\tau(\psi)\cos(\psi+\mu\theta)]\sin(\theta) - \mu\tau(\psi)\sin(\psi+\mu\theta)\cos(\theta); \\ [R+\tau(\psi)\cos(\psi+\mu\theta)]\cos(\theta) - \mu\tau(\psi)\sin(\psi+\mu\theta)\sin(\theta); \\ \mu\tau(\psi)\cos(\psi+\mu\theta) + Q'(\theta) \end{Bmatrix} \tag{52}$$

According the formulas (51) and (52) it is possible to check, that the scalar product of the tangential vectors is a

$$(\overleftarrow{r}_{\psi}, \overleftarrow{r}_{\theta}) = \mu\tau^2(\psi) + Q'(\theta)(\tau'(\psi)\sin(\psi+\mu\theta) + \tau(\psi)\cos(\psi+\mu\theta)). \tag{53}$$

Remark 22 *The local system of coordinates (ψ,θ) on the surface (49) is orthogonal (two tangential vectors of corresponding surfaces are orthogonal), only when simultaneously: $n = 0$ and $Q'(\theta) \equiv 0$; i.e.*

a. if corresponding surface is a Dupin's Cyclide , $Q(\theta) = 0$ and number of winding $\mu = 0$;

b. if $Q(\theta) = const.$ - Helix surfaces with the constant vertical stretching, with the circular projection of basic line in the plane XOY and without winding of the radial cross section around the basic line, i.e. number $\mu = 0$;

Also, according to the calculation, it is possible to see, that in this case

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(\psi,\theta)} &= [R + \tau(\psi)\cos(\psi + \mu\theta)][\tau'(\psi)\cos(\psi + \mu\theta) - \tau(\psi)\sin(\psi + \mu\theta)],\\
\frac{\partial(z,x)}{\partial(\psi,\theta)} &= -[R + \tau(\psi)\cos(\psi + \mu\theta)][\tau'(\psi)\sin(\psi + \mu\theta) + \tau(\psi)\cos(\psi + \mu\theta)]\sin(\theta)\\
&\quad - [\mu\tau(\psi)\tau'(\psi) + Q'[\tau'(\psi)\cos(\psi + \mu\theta) - \tau(\psi)\sin(\psi + \mu\theta)]]\cos(\theta),\\
\frac{\partial(y,z)}{\partial(\psi,\theta)} &= -[R + \tau(\psi)\cos(\psi + \mu\theta)][\tau'(\psi)\sin(\psi + \mu\theta) + \tau(\psi)\cos(\psi + \mu\theta)]\cos(\theta)\\
&\quad + [\mu\tau(\psi)\tau'(\psi) + Q'[\tau'(\psi)\cos(\psi + \mu\theta) - \tau(\psi)\sin(\psi + \mu\theta)]]\sin(\theta),
\end{aligned} \tag{54}$$

and correspondingly the module of the vector product of these two vectors is

$$\begin{aligned}
|\overleftarrow{r}_\psi \times \overleftarrow{r}_\theta|^2 = [R + \tau(\psi)\cos(\psi + \mu\theta)]^2\left[(\tau'(\psi))^2 + \tau^2(\psi)\right] +\\
[Q'(\theta)(\tau'(\psi)\cos(\psi + \mu\theta) - \tau(\psi)\sin(\psi + \mu\theta)) + \mu\tau(\psi)tau'(\psi)]
\end{aligned} \tag{55}$$

So that, from the structure of the normal vector (see (54)-(55)) of surfaces (49), it is possible to see that

Remark 23

Each of the GTR_2^n surfaces (49) - is a two-sided surface.

The first fundamental form of this class of GTR_2^n surfaces is given by

$$\begin{aligned}
&E(\psi,\theta) = (\tau'(\psi))^2 + \tau^2(\psi);\\
&F(\psi,\theta) = \mu\tau^2(\psi) + Q'(\theta)(\tau'(\psi)\sin(\psi + \mu\theta) + \tau(\psi)\cos(\psi + \mu\theta));\\
&G(\psi,\theta) = [R + \tau(\psi)\cos(\psi + \mu\theta)]^2 +\\
&+ [Q'(\theta)\cos(\psi + \mu\theta) + \mu\tau(\psi)]^2 + Q'(\theta)\sin^2(\psi + \mu\theta).
\end{aligned} \tag{56}$$

and so that

$$\begin{aligned}
EG - F^2 = [R + \tau(\psi)\cos(\psi + \mu\theta)]^2\left[(\tau'(\psi))^2 + \tau^2(\psi)\right]^2\\
+ [\mu\tau(\psi)\tau'(\psi) + Q'(\theta)(\tau'(\psi)\cos(\psi + \mu\theta) - \tau(\psi)\sin(\psi + \mu\theta))]^2
\end{aligned} \tag{57}$$

Remark 24 *Each point of the surfaces GTR_2^n of this class (49) is a regular point, i.e. for each point (ψ,θ) the corresponding forms satisfy the conditions: $E(\psi,\theta) > 0, G(\psi,\theta) > 0$ and $EG - F^2 > 0$;*

The second fundamental form of this class of surfaces is given by

$$L(\psi,\theta)=\frac{1}{\sqrt{EG-F^2}}\left[R+\tau(\psi)\cos(\psi+\mu\theta)\right]\left[\tau^2+2(\tau'(\psi))^2-\tau(\psi)\tau''(\psi)\right];$$

$$M(\psi,\theta)=\frac{1}{\sqrt{EG-F^2}}\{\mu[R+\tau(\psi)\cos(\psi+\mu\theta)][(\tau'(\psi))^2+\tau^2(\psi)]-$$
$$-Q'(\theta)[(\tau'(\psi)\cos(\psi+\mu\theta)-\tau(\psi)\sin(\psi+\mu\theta)]^2+$$
$$+\mu[(\tau'(\psi)\cos(\psi+\mu\theta)-\tau(\psi)\sin(\psi+\mu\theta)]\quad\};$$
(58)

$$N(\tau,\theta)=\frac{1}{\sqrt{EG-F^2}}\{2\mu^2\tau^2(\psi)\tau'(\psi)\sin(\psi+\mu\theta)+$$
$$+\left[R+\tau(\psi)\cos(\psi+\mu\theta)\right]^2[(\tau'(\psi)\sin(\psi+\mu\theta)+\tau(\psi)\cos(\psi+\mu\theta)]+$$
$$+\left[R+\tau(\psi)\cos(\psi+\mu\theta)\right]\times$$
$$\times\left[\mu^2\tau^2(\psi)+Q''(\theta)[(\tau'(\psi)\cos(\psi+\mu\theta)-\tau(\psi)\sin(\psi+\mu\theta)]\right]\}$$

In particular if the "radial cross section" is a circle, i.e. $\tau(\psi)\equiv const.$ and the "law of vertical stretching of figure" is constant, i.e. $Q(\theta)\equiv const.$, then the formulas (56), (57), (58) correspondingly reduce to the simple form

$$E(\psi,\theta)=\tau,$$
$$F(\psi,\theta)=\mu\tau^2,$$
$$G(\psi,\theta)=\left[R+\tau\cos(\psi+\mu\theta)\right]^2+\mu^2\tau^2,$$
$$EG-F^2=\tau^2\left[R+\tau(\psi)\cos(\psi+\mu\theta)\right]^2; \quad (59)$$
$$L(\psi,\theta)=\tau,$$
$$M(\psi,\theta)=\mu\tau,$$
$$N(\psi,\theta)=\left[R+\tau\cos(\psi+\mu\theta)\right]\cos(\psi+\mu\theta)+\mu^2\tau.$$

So that in this case it is easy to rewrite the mean and Gaussian curvatures of the surfaces of this class, which are given by

$$H(\psi,\theta)=\frac{\left[R+\tau\cos(\psi+\mu\theta)\right]+\tau\cos(\psi+\mu\theta)}{\left[R+\tau\cos(\psi+\mu\theta)\right]},$$
$$K(\psi,\theta)=\frac{\cos(\psi+\mu\theta)}{\tau(R+\tau\cos(\psi+\mu\theta))}. \quad (60)$$

From the explicit form of the Gaussian curvature (60) it is clear that

Remark 25 *If $\tau(\psi)\equiv const.$ and $Q(\theta)\equiv const.$, then the surfaces GTR_2^n of the class (49) have elliptic, parabolic and hyperbolic points.*

IV. - Relations between the set of Generalized Möbius-Listing's Surfaces and the sets of Knots and Links

The Knots and Links Classifications is well known (see e.g. [8,9,10]). In this part of our article we consider Knots and Links which appear after cutting the Generalized Möbius - Listing's surfaces GML_2^n along "parallel" lines of their basic lines.

We use the following definitions and notations:

Definition 1 *A closed line (similar to the basic or border's line) which is situated on a* GML_2^n *and is "parallel" to the basic (or border's) line of the* GML_2^n *- i.e. distance between this line and basic or border's lines is constant - is called a* ***"Slit line"*** *or shortly an* ***"s-line"*** *(see e.g. Fig. 5.d.).*

- If the distance between an s-line and the basic line is zero, then this s-line coincides with the basic line (and sometimes is called **"B-line"**)(see e.g. Fig. 5.c.).

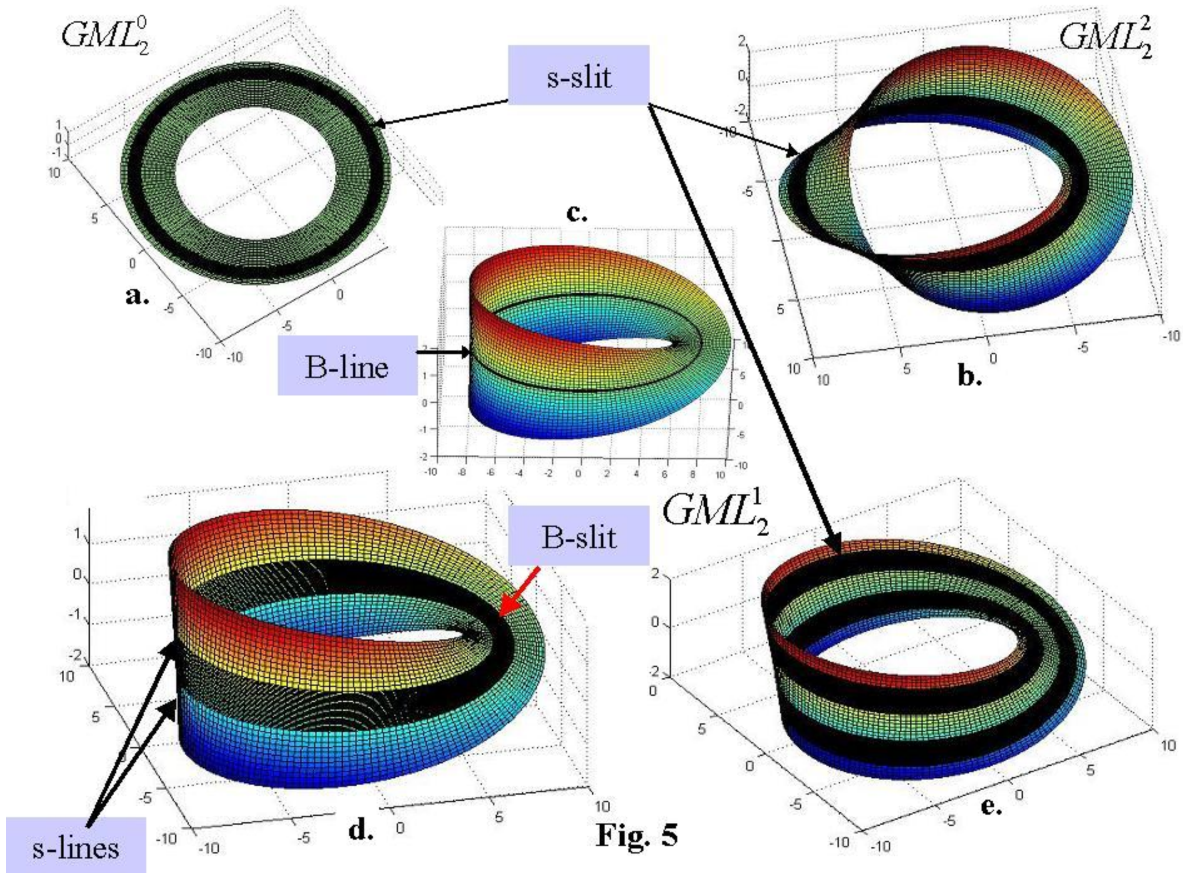


Fig. 5

Definition 2 *A domain situated on the surface* GML_2^n *and such that its border's lines are slit lines, is called a* ***"Slit zone"*** *or shortly an* ***"s-zone"***.

- The distance between the border's lines of an s-zone is the ``**width**" of this s-zone.
- If an s-zone's width equals to zero, then this zone reduces to an s-line.

Definition 3 *If the* ***"B-line"*** *is properly contained inside a* ***"Slit zone"*** *- i.e. his distance to the border's lines is strictly positive - then this* ***"Slit zone"*** *will be called a* ***"B-zone"***.

Definition 4 *The* ***"process of cutting"*** *or shortly the* ***"cutting"*** *is always realized belong some s-lines and produces the vanishing (i.e. elimination) of the corresponding s-zone (which eventually reduces to an s-line).*

- If a GML_2^n surface is cut belong an s-line, then the corresponding vanishing zone will be called an **s-slit** - or "$\xrightarrow{1}$" (see e.g. Figs. 5.a., 5.b., 5.e.).

- If a GML_2^n surface is cut belong its B-line, then the corresponding vanishing zone will be called a **B-slit** - or "$\xrightarrow{B}$" (see e.g. Fig. 5.d.).
- If the vanishing zone - after an **s-slit** (a **B-slit**) - is given by an **"s-zone"** (a **"B-zone"**), then the cutting process will be called an **s-zone-slit** (a **B-zone-slit**).

Without loss of generality and for simplifying the process of proofing, in this article we consider the following restrictions:

- The GML_2^n is a regular generalized Möbius-Listing's Surface (see representation (7) and notation (1*));
- A **B-slit** is symmetric, i.e. the "origin" (domain of its parametric representation) (7) of the corresponding B-zone is of the type

$$T_B \equiv [-\varepsilon, \varepsilon] \times [0, 2\pi) \subset T \equiv [-\tau^*, \tau^*] \times [0, 2\pi) \tag{61}$$

where $\varepsilon \in [0, \tau^*)$ and T is the domain of definition in (7) see also (1*).

- An **s-slit** is parallel to the B-line; i.e.

Case A. If the number of rotations n is even (see e.g. Figs. 5.a., 5.b.), then the ``origin'' of the corresponding s-zone is given by

$$T_s \equiv [\tau_0 - \varepsilon, \tau_0 + \varepsilon] \times [0, 2\pi) \subset T, \tag{62}$$

where ε, τ are some fixed constants and $0 < \tau_0 - \varepsilon < \tau_0 + \varepsilon < \tau^*$.

Case B. If the number of rotations n is odd (see e.g. Fig. 5.e.), then the ``origin'' of the corresponding s-zone is given by the two symmetric strips:

$$T_s^* \equiv [\tau_0 - \varepsilon, \tau_0 + \varepsilon] \times [0, 2\pi) \cup [-\tau_0 - \varepsilon, -\tau_0 + \varepsilon] \times [0, 2\pi) \subset T, \tag{63}$$

where ε, τ are some fixed constants and $0 < \tau_0 - \varepsilon < \tau_0 + \varepsilon < \tau^*$.

- If the vanishing zone is a line, then correspondingly $\varepsilon = 0$.

Theorem 1 *If the GML_2^n surface is cut belong its B-line, then:*

A. if the number of rotations n is even ($n \equiv 2j$), then after a B-zone-slit, an object **"Link-**2**"** appears, whose both components are GML_2^n surfaces, and the topological group of the link-2 is $\{(n)_1^2\}$ (see classification in [9], or [10]), i.e. for each natural $j = 0,1,2,\ldots$

$$GML_2^{2j} \xrightarrow{B} Link-2 \;\; of \;\; GML_2^{2j} \quad Group\,\{(2j)_1^2\}; \tag{64}$$

B. if the number of rotations n is odd ($n \equiv 2j+1$), then after a B-zone-slit, an object **"Knot"** or **"Link-**1**"** of type GML_2^{2n+2} appears and its topological group is $\{(n)_1\}$ (except when $n = 1$, since in this case the topological group is $(0)_1$) (see classification in [8], or [10]), i.e. for each natural $j = 1,2,\ldots$

$$GML_2^{2j+1} \xrightarrow{B} Knot \; of \; GML_2^{4j+4} \quad Group\,\{(2j+1)_1\}; \tag{65}$$

when $j = 0$, i.e. in the case of the classic Möbius strip, then

$$GML_2^1 \xrightarrow{B} Knot \; of \; GML_2^4 \;\; Group \;\; \{(0)_1\}. \tag{66}$$

Proof. The representation (7) is a one to one correspondence between the points of the strip T and the points of the GML_2^n surface, and according to the Remak 1:

Case A. If n is an even number ($n \equiv 2j$), then each of the functions $X(\tau,\theta), Y(\tau,\theta), Z(\tau,\theta)$ is a 2π-periodic function of the argument θ. So that, according to (61) to the B-slit-zone corresponds to the elimination of the T_B in the domain of definition T. But in this case, the domain of definition

$$T \setminus T_B = [-\tau^*, -\varepsilon] \times [0,2\pi) \cup [\varepsilon, \tau^*] \times [0,2\pi) \tag{67}$$

consists of two parts and define two different objects GML_2^n. The one to one correspondence (7) guarantees that the new objects have not self-cross points. Also, by definition of the generalized Möbius-Listing surface, the two strips (new domain of definition) make n rotations around their basic lines before the identification of ends, so that the simplest topological group $\{(n)_1^2\}$ of the link-2 appears. Some simple examples are given in Fig. 6.

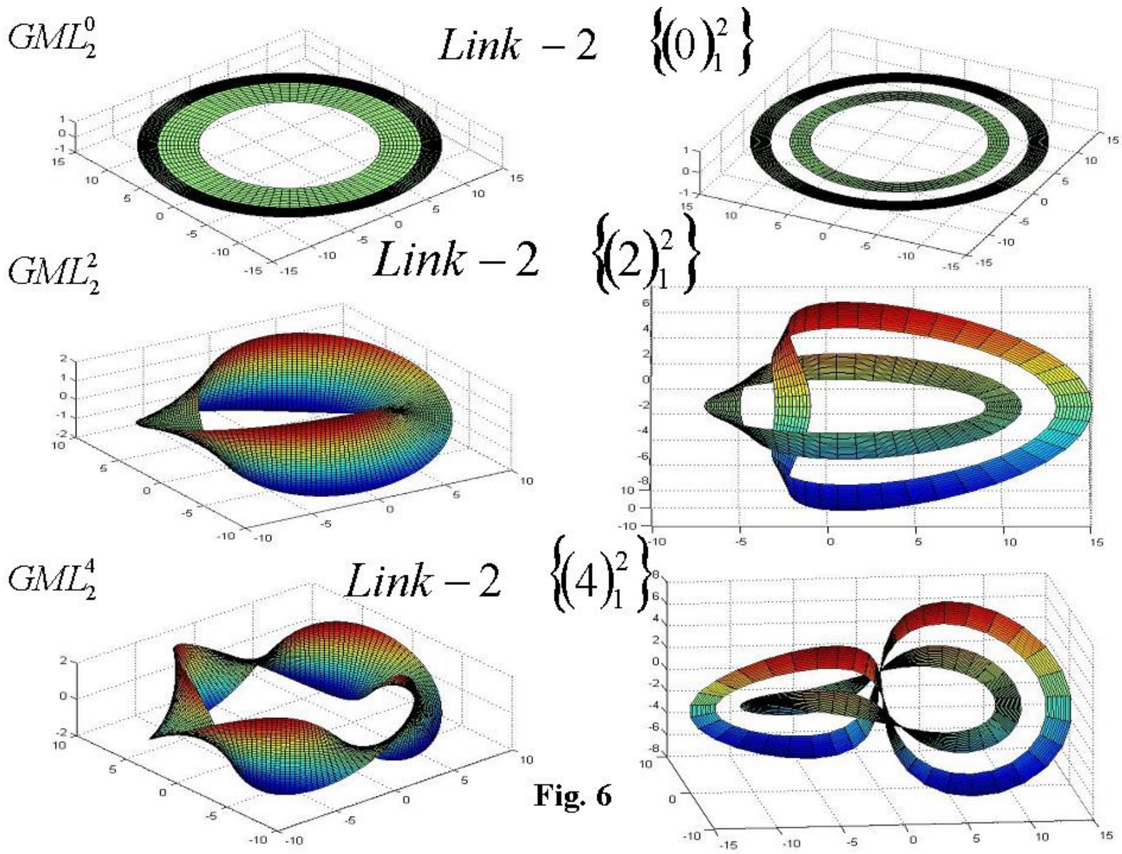


Fig. 6

Case B. If n is a odd number ($n \equiv 2j+1$), then the functions $X(\tau,\theta), Y(\tau,\theta), Z(\tau,\theta)$ are 4π-periodic functions of the argument θ, with property (M*); i.e. for each $\tau \in [-\tau^*, \tau^*]$

$$X(-\tau,0) = X(\tau,2\pi); \quad Y(-\tau,0) = Y(\tau,2\pi); \quad Z(-\tau,0) = Z(\tau,2\pi). \tag{68}$$

In this case, the one to one correspondence (7) defines a single object, in spite of the fact that the domain of definition (67) is disconnected. This new object has a new basic line (in particular, according to eq. (7) and restriction (67)) which is given by

$$\begin{aligned}
X\left(\frac{\tau^{*}-\varepsilon}{2},\theta\right) &= \left[R+\frac{\tau^{*}-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)\\
Y\left(\frac{\tau^{*}-\varepsilon}{2},\theta\right) &= \left[R+\frac{\tau^{*}-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)\\
Z\left(\frac{\tau^{*}-\varepsilon}{2},\theta\right) &= \frac{\tau^{*}-\varepsilon}{2}\sin\left(\psi+\frac{n\theta}{2}\right).
\end{aligned} \tag{69}$$

This is a representation of a really closed line, but now $\theta \in [0,4\pi)$ (see Fig. 2.a. or Remark 2, when $\mu \in Q$ in [6]), and therefore the unit normal vector (12) makes $2n+2$ rotations around the new basic line (69), since it belongs to a GML_2^{2n+2} surface. Also, by definition of the generalized Möbius-Listing surface, the two strips (new domain of definition) make n rotations around the old basic line (circle) before the identification of ends, so that the simplest topological group $\{(n)_1\}$ of the Knot theory appears; but when $n=1$, since the classical group $\{(1)_1\}$ does not exist, then this case corresponds to $\{(0)_1\}$. Some simple examples are shown in Fig. 7.

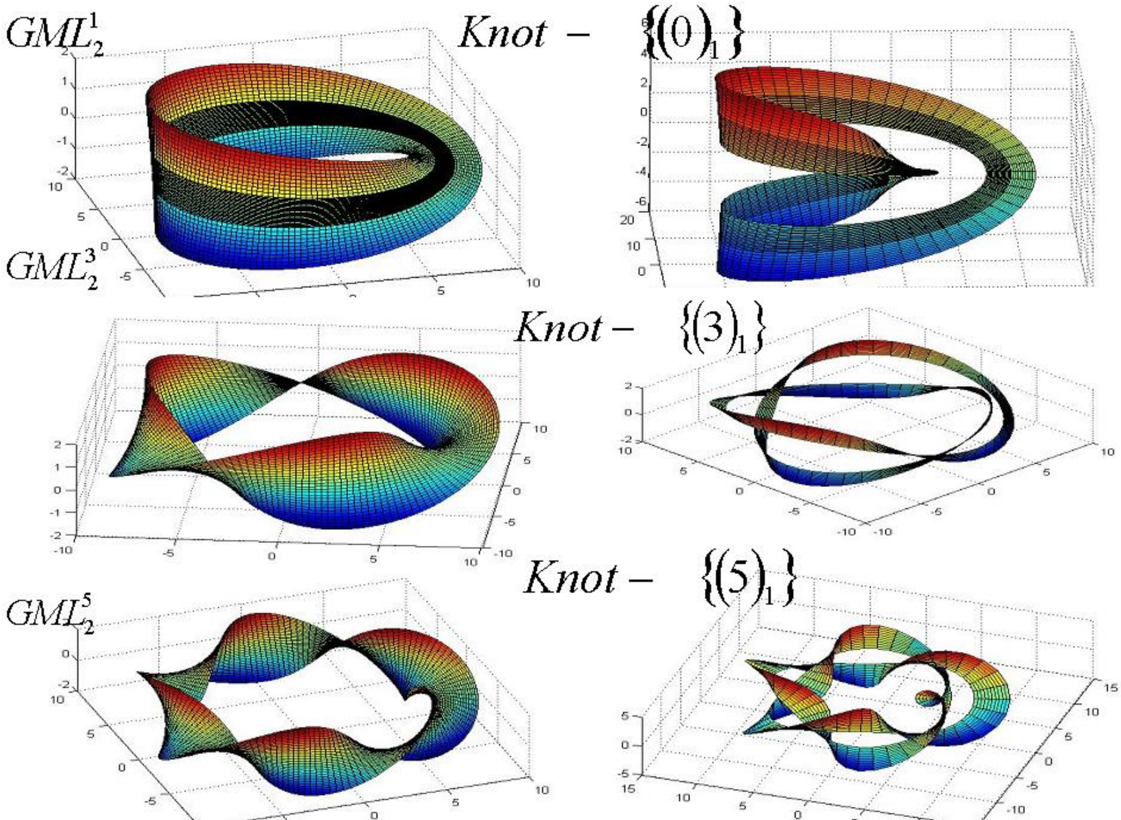


Fig. 7

Theorem 2 *If the GML_2^n surface is cut belong a non trivial s-line (i.e. a line which does not coincide with its basic line), then:*

A. if the number of rotations n is even ($n \equiv 2j$), then after an s-zone-slit, an object ``**Link-** 2 " appears, whose both components are GML_2^n surfaces; furthermore, the topological group of the link-2 is $\{(n)_1^2\}$ (see classification in [9], or [10]), i.e. for each natural $j = 0,1,2,\ldots$

$$GML_2^{2j} \xrightarrow{1} Link-2\; of\; GML_2^{2j} \quad Group\,\{(2j)_1^2\}; \tag{70}$$

B. if the number of rotations n is a odd ($n \equiv 2j+1$), then after an s-zone-slit, an object **``Link-** 2 **"** appears, which is made by two components. One of them is a GML_2^n surface, but the other one is of the type GML_2^{2n+2}, i.e. for each natural $j = 0,2,\ldots$

$$GML_2^{2j+1} \xrightarrow{1} Link - 2\ of GML_2^{2j+1}\ and\ GML_2^{4j+4}. \quad (71)$$

In this case, the topological group of this object is at present unknown.

Proof. The representation (7) with (1*) is a one to one correspondence between the points of the strip T and the points of the GML_2^n surface, and according to the Remak 1:

Case A. If n is an even number ($n \equiv 2j$), then by using the same argument of case A of the previous theorem, but according to the (62), to an s-zone-slit corresponds elimination of the T_s in the domain of definition T. In this case the new domain of definition is given by

$$T \setminus T_s = [-\tau^*, \tau_0 - \varepsilon] \times [0,2\pi) \cup [\tau_0 + \varepsilon, \tau^*] \times [0,2\pi) \quad (72)$$

which also, consist of two parts, but now their widths are different, and define two different objects GML_2^n, with different widths. The one to one correspondence (7) guarantees that the new objects have not self-cross points. Furthermore, by definition of the generalized Möbius-Listing surface, the two strips (new domain of definition) make n rotations around their basic lines before identification of ends, and therefore the simplest topological group $\{(n)_1^2\}$ of the link-2 appears. Some simple examples of differences are shown in Fig. 8. The widths of components of B-zone-slits are equal, but the widths of components of s-zone-slits are different.

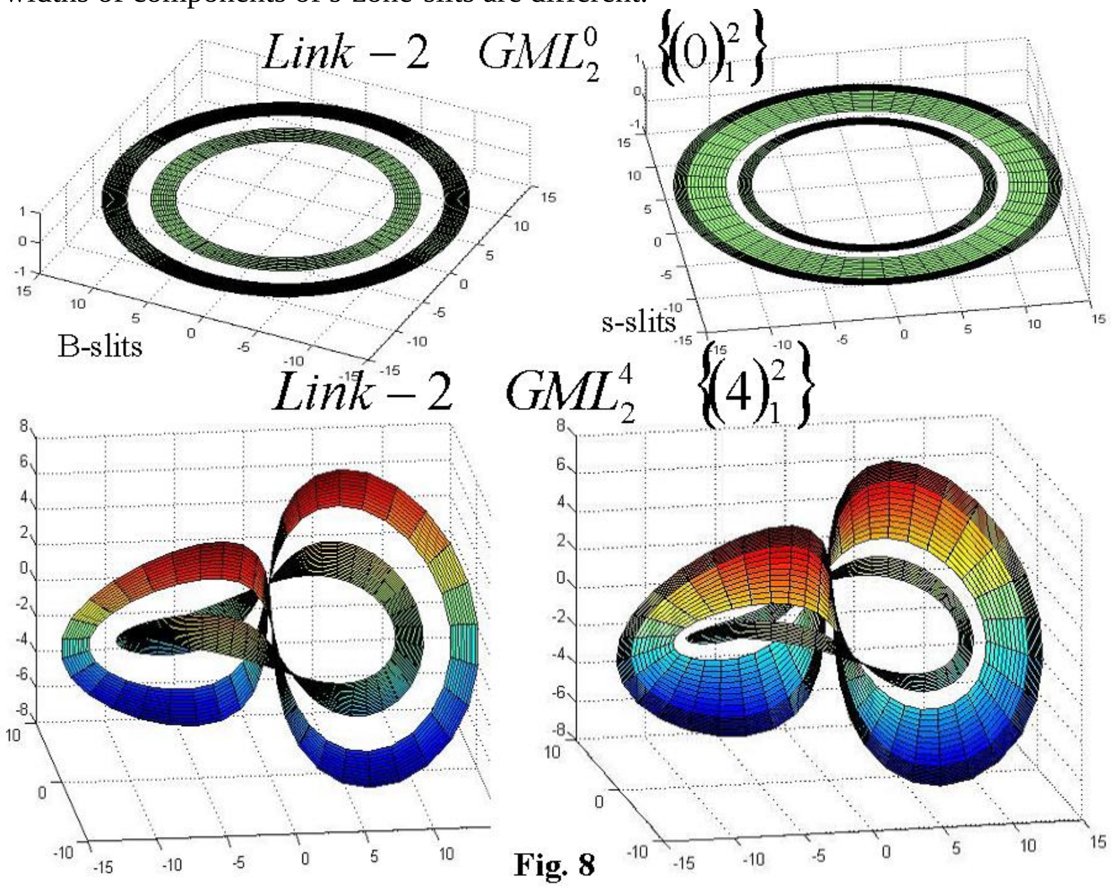


Fig. 8

Case B. If n is an odd number ($n \equiv 2j+1$), then according to (63) after an s-zone-slit the new domain of definition (7) is given by

$$T \setminus T_s^* = [-\tau^*, -\tau_0 - \varepsilon] \times [0,2\pi) \cup [-\tau_0 + \varepsilon, \tau_0 - \varepsilon] \times [0,2\pi) [\tau_0 + \varepsilon, \tau^*] \times [0,2\pi). \quad (73)$$

This is a disconnected domain, which consist to three parties.

The domain defined from the central part of the right hand side of eq. (73), according to the representation formulas (7), defines an object GML_2^n . But its width is smaller with respect to the original surface; i.e. $\tau \in [-\tau_0 + \varepsilon, \tau_0 - \varepsilon]$. Each of the remaining parts of the right hand side of eq. (73), similarly to the case B of the previous theorem, define a geometrical object GML_2^{2n+2}. So that after an s-zone-slit of a GML_2^n, when n is an odd number, appears a link-2, where one of the components is a GML_2^n and the second one is a GML_2^{2n+2}. Unfortunately we do not know, at present, what is the topological group's element of link-2 appearing after an s-zone-slit of a GML_2^n, when n is an odd number. Some simple examples are shown in Fig. 9.

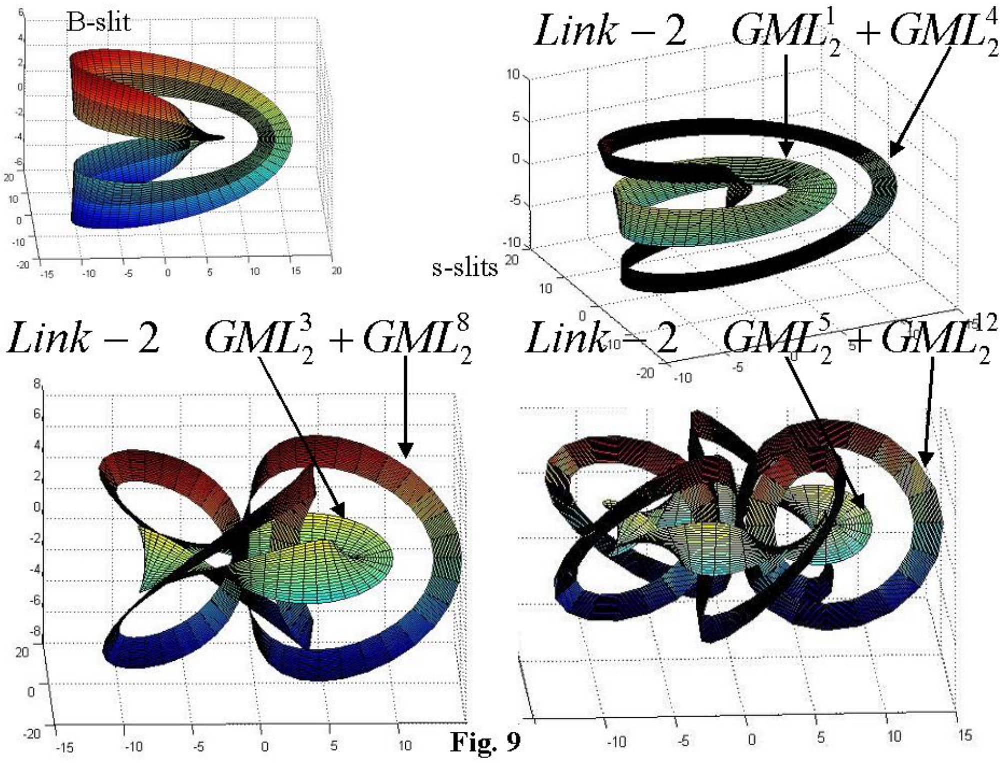


Fig. 9

Lastly, from the above Theorems and Remarks 1,2,3,4,5, we can deduce the following facts

Remark 26 *Both the previous Theorems still hold when the basic line is a closed space line.*

After cutting each regular Möbius-Listing surface GML_2^n, whose basic line is a circle, appear objects (object) with the following properties:

- the tangential vectors of the new objects (object) $\overleftarrow{r_\tau}$ and $\overleftarrow{r_\theta}$ (8),(9) are orthogonal;
- each point of these objects are Hyperbolic (saddle) points if $n \neq 0$;
- each point of these objects are Parabolic points if $n = 0$

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