Algebraic Theory of Process Motion

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Abstract

In the proposed work which continues [Z. Z. Khukhunashvili, V. Z. Khukhunashvili, Alternative Analysis Generated by a Differential Equation, E. J. Qualitative Theory of Diff. Equ., No. 2. (2003), pp. 1-31], we study the algebraic properties of processes described by autonomous differential equations. We have found that a wide class of differential equations contains an algebraic object isomorphic to the object consisting of superposed two alternatively acting numerical fields with common neutral elements. Using its own algebraic field, each process constructs its own (differential and integral) calculus with a simultaneous definition of its own frame of reference. It appears that in its own calculus the differential equation of this process takes the linear form, while the arisen system of reference becomes inertial. Along with this, because of the existence of a double algebraic field an alternative antiprocess is assigned to each process. The developed theory makes it possible to describe one process from the standpoint of the other process. It should be said that the inertial systems of one process do not necessarily coincide with the inertial systems of the other process. All the results and conclusions follow exclusively from the algebraic properties of differential equations without using any other postulates and assumptions. These studies enable us to get an idea of the algebraic structure of the Fourier method in the case of nonlinear equations. We succeeded in writing out the exact solution of equations of hydrodynamics in implicit form.

Introduction

In classical mechanics, a classical particle, i.e. a material point is simultaneously an elementary and a fundamental object, while an event, i.e. a process occurring instantly at this point is an elementary physical process. Based on these objects, we form our notion of space and time. The basic laws of motion for a material point are written the form of ordinary differential equations. This is the main pattern of classical mechanics which we conceive as a logically closed complete theory.

However a real particle cannot be interpreted as a material point if we want to preserve its interferential properties, spin, transmutation as well as other observable quantum properties. It is well known that the motion of a real particle is described more precisely and completely if we assign to it, generally speaking, a quantized field, though, in doing so, we enter into a hidden logical contradiction, since, on the one hand, a particle is described by a field and, on the other hand, our notion of the space, where a particle moves, is taken without any modification from the classical theory of a material point.

While moving along its trajectory which is a chain of consecutive events, a classical particle is not only indifferent to what is happening beyond its trajectory, but is also unaware of what may happen to it the next moment: "it only knows what to do when this moment comes" [3]. However by using a field we make the particle become sensitive to the neighboring trajectories [3]. It not only knows what to do at a current instant, but also looks ahead to the future [3]. A mathematical explanation of these facts is that an event is defined at a point of the four-dimensional space and is meaningful only for a material point. As to a field, it is defined and has a meaning only in some given domain of the four-dimensional space.

Based on the above-said, if to probe the outside world one uses a field, then the obtained picture will be finer and deeper than in the case of using the tool of a classical particle. The realization of this program demands the involvement of differential equations because they describe fields and the field laws of motion are already inherent in them. Moreover, in order that processes be described as completely as possible, equations must take into account the interaction of fields, which means that, generally speaking, they must be nonlinear. To reveal the properties of a space and a field and their inter-relationship, we should first and foremost study the algebraic structures contained in differential equations.

In [1], we have investigated the binary operations for quasilinear equations of a sufficiently wide class given on the entire space. It appears that such an equation admits the existence of infinitely many binary laws of addition of solutions. Every binary operation has its alternatively acting counterpart. Along with this, we have studied the group properties of differential equations, their relationship with binary operations and the corresponding geometry of a space. It has also been established in [1] that for each quasilinear equation there always exists the corresponding linear equation and between the relations that exist between their algebro-geometrical structures are homomorphic.

In [2], we have studied ordinary differential equations and described the binary operations that form commutative groups. These operations bring us to an algebraic object consisting of superposed two alternatively acting numerical fields. This double numerical field is used in constructing differential and integral calculi. It appears that standard calculi are always accompanied by the existence of alternative calculi. In the present work we show that each equation has, in general, its own (proper) calculus and between the calculi of equations of the same class, seen as algebraic objects, there exist homomorphic relations. The notion of proper calculus naturally induces the notion of proper time of the process. Equations written in terms of their proper calculi become linear.

Fundamental equations of physical fields reduce to quasilinear systems of partial differential equations. Hence we may proceed to the investigation of algebraic structures arising on the basis of quasilinear systems, which in turn describe certain processes. The algebraic theory of these equations rejects the notion of a classical particle and introduces instead a different object, namely, a process, which in this theory we conceive as a fundamental and, in some sense, elementary object.

1. On the nonlinear expansion of solutions of a quasilinear system

In [1] we have investigated the algebraic structure of a quasilinear system with partial derivatives. In this section we return to this issue to clarify some aspects that have been skipped in [1].

Let us consider a system of differential equations

$$a_n^{\nu k}\left(u^1,\ldots,u^N\right)\frac{\partial u^n}{\partial x^\nu}=F^k\left(u^1,\ldots,u^N\right), \quad (k=1,\ldots,N), \tag{1.1}$$

where summation is performed over the indexes ν and n from 1 to N_0 and N respectively. The elements of the square matrices $a^\nu(u)$ and the vector $F(u)$ are smooth complex-valued functions defined on the whole space Γ^N. Being a vector field, $F(u)$ can have only isolated zeros and infinities. The real independent variables x^ν are the coordinates of the space Γ^{N_0}. For simplicity, it will be assumed that Γ^{N_0} and Γ^N are Euclidean spaces. In the sequel, Γ^{N_0} will be called an external space, and Γ^N an internal space.

(1) Let a solution of equation (1.1) have the form

$$u_\alpha^k=u_\alpha^k\left(z_\alpha\right),$$

where $z_\alpha=(1/l_\alpha)\,\alpha_\nu x^\nu\big(=(1/l_\alpha)\left(\alpha_1x^1+\cdots+\alpha_{N_0}x^{N_0}\right)\big)$, $\alpha\in\Omega$, $k=1,\ldots,N$. Ω is some set from an N_0-dimensional Euclidean space Γ_{N_0} and $l_\alpha=\alpha_\nu l^\nu$,where $l^\nu$are some given real nonzero values. Substituting u_α into (1.1), we obtain the equation

$$a_\alpha\left(u_\alpha\right)\frac{du_\alpha}{dz_\alpha}=F\left(u_\alpha\right), \tag{1.2}$$

where the matrix

$$a_\alpha\left(u_\alpha\right)=\frac{1}{l_\alpha}\alpha_\nu a^\nu\left(u_\alpha\right).$$

In the sequel, $u_\alpha(z_\alpha)$ will be called a plane wave, and (1.2) the equation of plane waves. The vectors u_α and $F(u_\alpha)$ have components u_α^k and $F^k(u_\alpha)$, respectively. Let us introduce the following restriction: $\det a_\alpha(y)$ can have only isolated zeros and infinities when y runs through Γ^N.

(2) A solution of (1.1) is to be sought in the form

$$u^k=\chi^k\left(\ldots,u_\alpha,\ldots\right), \quad (k=1,\ldots,N), \tag{1.3}$$

where u_α is a general solution of equation (1.2) and α runs through the set $\Omega\subset\Gamma_{N_0}$. Substituting (1.3) into (1.1) and taking into account (1.2), we obtain

$$\sum_{\alpha\in\Omega}a_\alpha\left(\chi\right)\frac{\partial\chi}{\partial u_\alpha}a_\alpha^{-1}\left(u_\alpha\right)F\left(u_\alpha\right)=F\left(\chi\right). \tag{1.4}$$

In the sequel, (1.4) will be called the defining equation of a function χ.

(3) Let us consider the characteristic functions $\varphi_\alpha(u_\alpha)$, $\alpha \in \Omega$, of equation (1.2). It has been shown in §2 [1] that a general solution of (1.2) satisfies the equality

$$\varphi_\alpha^k(u_\alpha) = b_\alpha^k z_\alpha + c_\alpha^k,$$
$$(k = 1, \dots, N),$$

where c_α^k are arbitrary integration constants and b_α^k are the components of the nonzero vector b_α. As follows from §2 [1] , there always exists a nonsingular matrix S_α such that when the above equality is multiplied by S_α, the nonzero vector b_α transforms to the vector b with components $b^k = 1$. Hence there always exists a representation of the characteristic functions φ_α that satisfies the equalities

$$\varphi_\alpha^k(u_\alpha) = b^k z_\alpha + c_\alpha^k, \quad b^k = 1, \tag{1.5}$$
$$(k = 1, \dots, N).$$

Let us consider a differential equation

$$\frac{\partial \varphi_\alpha(y)}{\partial y} a_\alpha^{-1}(y) F(y) = b, \tag{1.6}$$

where $\varphi_\alpha^k(y)$ are the unknown functions and y^k are the independent variables. The left-hand side of (1.5) is a particular solution of (1.6), which can be verified by differentiating both sides of (1.5) with respect to z_α and using (1.2).

(4) In (1.3-4) we introduce the new independent variables

$$r_\alpha = \varphi_\alpha(u_\alpha). \tag{1.7}$$

Let us calculate the derivative

$$\frac{\partial \chi}{\partial u_\alpha} = \frac{\partial \chi}{\partial r_\alpha}\frac{\partial r_\alpha}{\partial u_\alpha} = \frac{\partial \chi}{\partial r_\alpha}\frac{\partial \varphi_\alpha(u_\alpha)}{\partial u_\alpha}. \tag{1.8}$$

On the other hand, since $\varphi_\alpha(y)$ is a solution of (1.6), we can write

$$a_\alpha^{-1}(u_\alpha) F(u_\alpha) = \left(\frac{\partial \varphi_\alpha(u_\alpha)}{\partial u_\alpha}\right)^{-1} b. \tag{1.9}$$

After substituting (1.8-9) into (1.4), the defining equation takes the form

$$\sum_{\alpha \in \Omega} a_\alpha(\chi) \frac{\partial \chi}{\partial r_\alpha} b = F(\chi). \tag{1.10}$$

(5) It is obvious that equation (1.10) can be rewritten as

$$\sum_{\alpha \in \Omega} \left(\frac{\partial \varphi_\alpha(\chi)}{\partial \chi} a_\alpha^{-1}(\chi)\right)^{-1} \frac{\partial \varphi_\alpha(\chi)}{\partial r_\alpha} b = F(\chi). \tag{1.11}$$

Let us introduce the notation of the matrix

$$B_\alpha(y) = \frac{\partial \varphi_\alpha(y)}{\partial y} a_\alpha^{-1}(y). \tag{1.12}$$

By (1.12), equation (1.6) takes the form

$$B_\alpha(y)\,F(y)=b. \tag{1.13}$$

Hence we can write the equalities

$$\begin{gathered}B_\alpha(y)\,F(y)=b,\\ B_\alpha^{-1}(y)\,B_\beta(y)\,F(y)=F(y),\\ B_\alpha^{-1}(y)\,b=F(y),\\ B_\alpha(y)\,B_\beta^{-1}(y)\,b=b.\end{gathered} \tag{1.14}$$

(6) Using (1.12), from (1.11) we obtain

$$\sum_{\alpha\in\Omega}B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial r_\alpha}b=F(\chi). \tag{1.15}$$

Let us multiply (1.15) by the matrix $B_\beta(\chi)$. Taking into account (1.14), we have

$$\sum_{\alpha\in\Omega}B_\beta(\chi)\,B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial r_\alpha}b=b. \tag{1.16}$$

Analogously, we can write

$$\sum_{\alpha\in\Omega}B_\gamma(\chi)\,B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial r_\alpha}b=b. \tag{1.17}$$

Subtracting (1.17) from equation (1.16), we obtain

$$\sum_{\alpha\in\Omega}\left(B_\beta(\chi)-B_\gamma(\chi)\right)B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial\chi}\frac{\partial\chi}{\partial r_\alpha}b=0. \tag{1.18}$$

Equation (1.15) can be regarded as a manifold in a *jet*-space with coordinates

$$\left(\chi,\ldots,\frac{\partial\chi}{\partial r_\alpha},\ldots,\frac{\partial\chi}{\partial r_\beta},\ldots\right).$$

It is obvious that on manifold (1.15), the variables $\partial\chi/\partial r_\alpha$ can be assumed to be independent for various values of $\alpha\in\Omega$, while the variable χ is defined from (1.15). As is known [4], such an interpretation is used when studying the group properties of differential equations. In that case, since $\partial\chi/\partial r_\alpha$ is independent, each term in (1.18) must be equal to zero. Thus

$$B_\beta(\chi)\,B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial r_\alpha}b=B_\gamma(\chi)\,B_\alpha^{-1}(\chi)\,\frac{\partial\varphi_\alpha(\chi)}{\partial r_\alpha}b \tag{1.19}$$

for arbitrary $\alpha,\beta,\gamma\in\Omega$. Otherwise, if (1.19) is not fulfilled, we see that, in addition to (1.15), a new independent condition in form (1.18) is imposed on the variables χ and $\frac{\partial\chi}{\partial r_\alpha}$. But this means that (1.16) and (1.17) are independent equations, which contradicts their derivation from (1.15).

Equality (1.19) immediately implies that the result of the multiplication of the vector $B_\alpha^{-1}(\chi)\left[\partial\varphi_\alpha(\chi)/\partial r_\alpha\right]b$ by the matrix $B_\beta(\chi)$ does not depend on the index β. Hence, assuming in the right-hand part of (1.19) that $\gamma = \alpha$, we obtain

$$B_\beta(\chi) B_\alpha^{-1}(\chi) \frac{\partial \varphi_\alpha(\chi)}{\partial r_\alpha} b = \frac{\partial \varphi_\alpha(\chi)}{\partial r_\alpha} b. \tag{1.20}$$

Using (1.20), from (1.16) we eventually find

$$\sum_{\alpha\in\Omega} \frac{\partial \varphi_\alpha^k(\chi)}{\partial r_\alpha^n} b^n = b^k, \tag{1.21}$$

$$(k = 1, \ldots, N).$$

Thus we have shown that the defining equation (1.4) reduces to equation (1.21).

(7) Let us consider a function $\chi \in \Gamma^N$ given in the implicit form

$$\sum_{\alpha\in\Omega} \exp\left[r_\alpha^k - \psi_\alpha^k(\chi)\right] = 1, \tag{1.22}$$

$$(k = 1, \ldots, N),$$

where $\psi_\alpha^k(\chi)$ is an arbitrarily given set of smooth functions defined on the whole space Γ^N. For (1.22) to be solvable with respect to χ it is required that

$$\det\left[\sum_{\alpha\in\Omega} \exp\left(r_\alpha^k - \psi_\alpha^k(\chi)\right) \frac{\partial \psi_\alpha^k(\chi)}{\partial \chi^n}\right] \neq 0. \tag{1.23}$$

Let us differentiate (1.22) with respect to the independent variables r_β^n:

$$\sum_{\alpha\in\Omega} \exp\left[r_\alpha^k - \psi_\alpha^k(\chi)\right] \frac{\partial \psi_\alpha^k(\chi)}{\partial \chi^l} \frac{\partial \chi^l}{\partial r_\beta^n} = \exp\left[r_\beta^n - \psi_\beta^n(\chi)\right] \delta_n^k, \tag{1.24}$$

where δ_n^k is the Kronecker symbol. As different from the index k, in (1.24) summation is performed over the index l from 1 to N. Note that equality (1.24) is the identity for the function χ defined from (1.22). For (1.24) we will consider the following two cases:

a) $n = k$. Then (1.24) can be rewritten in the form

$$\sum_{\alpha\in\Omega} \exp\left[r_\alpha^k - \psi_\alpha^k(\chi)\right] \frac{\partial \psi_\alpha^k(\chi)}{\partial \chi^l} \frac{\partial \chi^l}{\partial r_\beta^k} \exp\left(-\left[r_\beta^k - \psi_\beta^k(\chi)\right]\right) = 1. \tag{1.25}$$

From (1.23) and (1.25) it immediately follows that the cofactor

$$\frac{\partial \chi^l}{\partial r_\beta^k} \exp\left(-\left[r_\beta^k - \psi_\beta^k(\chi)\right]\right) \tag{1.26}$$

does not depend on the parameter β, i.e. the equality

$$\frac{\partial \chi^l}{\partial r_\beta^k} \exp\left(-\left[r_\beta^k - \psi_\beta^k(\chi)\right]\right) = \frac{\partial \chi^l}{\partial r_\gamma^k} \exp\left(-\left[r_\gamma^k - \psi_\gamma^k(\chi)\right]\right) \tag{1.27}$$

holds, where $\beta, \gamma \in \Omega$ are arbitrary.

Let us put cofactor (1.26) of (1.25) into the sum over α. Using property (1.27) and assuming that in each term $\beta = \alpha$, we obtain

$$\sum_{\alpha\in\Omega} \frac{\partial \psi_\alpha^k (\chi)}{\partial r_\alpha^k} = 1, \qquad (k = 1, ..., N)\,. \tag{1.28}$$

b) $n \neq k$. Then identity (1.24) takes the form

$$\sum_{\alpha\in\Omega} \frac{\partial \exp\left(-\psi_\alpha^k (\chi)\right)}{\partial r_\beta^n} \exp r_\alpha^k = 0. \tag{1.29}$$

Differentiating (1.29) with respect to r_γ^m for $m \neq k$, we have

$$\sum_{\alpha\in\Omega} \frac{\partial^2 \exp\left(-\psi_\alpha^k (\chi)\right)}{\partial r_\beta^n \partial r_\gamma^m} \exp r_\alpha^k = 0. \tag{1.30}$$

But since r_α^k are independent variables for various values of α, from (1.29–30) we conclude that

$$\frac{\partial \psi_\alpha^k (\chi)}{\partial r_\beta^n} = 0, \qquad (n \neq k). \tag{1.31}$$

Therefore, using the results a) and b), we come to the identity

$$\frac{\partial \psi_\alpha^k (\chi)}{\partial r_\alpha^n} = \frac{\partial \psi_\alpha^k (\chi)}{\partial r_\alpha^k} \delta_n^k, \qquad (k,\ n = 1, \ldots, N). \tag{1.32}$$

(8) Let us extend the implicit function (1.22) of χ and consider the equality

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[p_\alpha^k \left(r_\alpha^k - \psi_\alpha^k (\chi)\right)\right] = 1, \qquad (k = 1, \ldots, N)\,, \tag{1.33}$$

where q_α, p_α^k are arbitrary values depending only on the parameter $\alpha \in \Omega$. It is required that

$$q_\alpha \neq 0, \quad p_\alpha^k \neq 0$$

for any $\alpha \in \Omega$ and $k = 1, \ldots, N$. The condition under which (1.33) is solvable with respect to χ is written in the form

$$\det\left[\sum_{\alpha\in\Omega} q_\alpha \exp p_\alpha^k \left(r_\alpha^k - \psi_\alpha^k (\chi)\right) \cdot p_\alpha^k \frac{\partial \psi_\alpha^k (\chi)}{\partial \chi^n}\right] \neq 0.$$

Applying the reasoning of (7), we make sure that equalities (1.28) and (1.31– 32) are identically satisfied.

Thus we come to a conclusion that the function χ defined from (1.22) or (1.33) is a solution of the equation

$$\sum_{\alpha\in\Omega}\frac{\partial\psi_\alpha(\chi)}{\partial r_\alpha}=1, \tag{1.34}$$

where $\partial\psi_\alpha(\chi)/\partial r_\alpha$ is the diagonal matrix (1.32).

(9) Let ω be some subset of the set Ω. Consider (1.33) and assume that $q_\alpha = 0$ if $\alpha\in\Omega\setminus\omega$. In this setting, it is obvious that the implicit function

$$\sum_{\alpha\in\omega} q_\alpha \exp p_\alpha^k \left(r_\alpha^k - \psi_\alpha^k(\chi)\right) = 1, \qquad (k=1,\dots,N),$$

is a solution of equation (1.34).

(10) Let us return to the investigation of solutions of equation (1.21). In (1.33), the functions $\psi_\alpha^k(\chi)$ are replaced by the characteristic functions $\varphi_\alpha^k(\chi)$ which are, in turn, solutions of equation (1.6). From (8) it immediately follows that an implicit function χ

$$\sum_{\alpha\in\Omega} q_\alpha \exp p_\alpha^k \left(r_\alpha^k - \varphi_\alpha^k(\chi)\right) = 1, \qquad (k=1,\dots,N), \tag{1.35}$$

is a solution of equation (1.21). But then, in view of (1.5) and (1.7), the solution $u=\chi$ of equation (1.1) is written implicitly as follows:

$$\sum_{\alpha\in\Omega} q_\alpha \exp p_\alpha^k \left[b^k z_\alpha + c_\alpha^k - \varphi_\alpha^k(u)\right] = 1, \qquad (k=1,\dots,N),$$

where c_α^k are arbitrary integration constants of equation (1.2), and $z_\alpha = (1/l_\alpha)\,\alpha_\nu x^\nu$.

Note that (1.35) is a particular solution of equation (1.21). It follows from (1.5) and (1.7) that $r_\alpha^k(u_\alpha)$ contains an arbitrary integration constant c_α^k. It has been mentioned in (3) that φ_α admits the transformation $\varphi_\alpha \to S_\alpha\varphi_\alpha$, where S_α is a nonsingular matrix. Based on these properties, (1.35) can be brought to the form

$$\sum_{\alpha\in\Omega} \exp \left(r_\alpha^k - \varphi_\alpha^k(\chi)\right) = 1, \qquad (k=1,\dots,N).$$

(11) Let the parameters α_ν vary on the intervals $[m_\nu,\ M_\nu]$. Let us partition these intervals. In each cell of this partitioning, we choose points $\tilde\alpha$ (for simplicity, the cell number in $\tilde\alpha$ is omitted). Moreover, in (1.35) it is assumed that

$$q_\alpha = Q(\tilde\alpha)\,\Delta\alpha_1\cdots\Delta\alpha_{N_0},$$

where $\Delta\alpha_\nu$ is the length of a cell of the partitioned interval $[m_\nu,\ M_\nu]$, and $Q(\tilde\alpha) = q_{\tilde\alpha}$. We form the sum

$$\sum_{\tilde\alpha\in\omega_{m,M}} Q(\tilde\alpha)\exp p_{\tilde\alpha}^k\left[r_{\tilde\alpha}^k - \varphi_{\tilde\alpha}^k(\chi)\right]\Delta\alpha_1\cdots\Delta\alpha_{N_0} = 1, \qquad (k=1,\dots,N), \tag{1.36}$$

where $\omega_{m,M} \subset \Omega$. By the result of (9) we conclude that (1.36) is a solution of (1.21) on the set $\omega_{m,M}$. It can be easily verified that the left-hand side of (1.36) is the Riemann integral sum. When a maximal cell length tends to zero, we obtain N_0 repeated integrals. In the arisen integral it is assumed that $m_\nu \longrightarrow -\infty$, $M_\nu \longrightarrow +\infty$, $(\nu = 1, \ldots, N_0)$. Using equations (1.5) and (1.7) we finally obtain

$$\int_{-\infty}^{+\infty} A^k(\alpha) \exp p^k(\alpha) \left[\frac{1}{l_\alpha} \alpha_\nu x^\nu - \varphi_\alpha^k(\chi) \right] d^{N_0}\alpha = 1, \tag{1.37}$$

$$(k = 1, \ldots, N),$$

where $A^k(\alpha) = Q(\alpha) \exp p^k(\alpha) c^k(\alpha)$ figure as arbitrary functions of α. Clearly, equation (1.37) is an algebraic equation with respect to χ. As to $u = \chi$, it is a solution of equation (1.1) represented in intergral terms.

(12) Examples

a) A simple example is the one-dimensional $(N = 1)$ quazilinear equation

$$a^\nu(u) \frac{\partial u}{\partial x^\nu} = F(u). \tag{1.38}$$

Assuming that $p_\alpha = 1$ and $N = 1$,we write the implicit solution (1.35) for equation (1.38)

$$\sum_{\alpha \in \Omega} q_\alpha \exp(r_\alpha - \varphi_\alpha(\chi)) = 1, \tag{1.39}$$

where φ_α is a solution of equation (1.6):

$$\frac{d\varphi_\alpha(y)}{dy} a_\alpha^{-1}(y) F(y) = 1. \tag{1.40}$$

We differentiate (1.39) with respect to r_β and write $\partial\chi/\partial r_\beta$ as

$$\frac{\partial \chi}{\partial r_\beta} = \left(\sum_{\alpha \in \Omega} q_\alpha \exp(r_\alpha - \varphi_\alpha(\chi)) \frac{d\varphi_\alpha(\chi)}{d\chi} \right)^{-1} q_\beta \exp(r_\beta - \varphi_\beta(\chi)). \tag{1.41}$$

Assuming that $N = 1$ and substituting (1.41) into the left-hand part of (1.10), we obtain

$$\sum_{\alpha \in \Omega} a_\alpha(\chi) \left(\sum_{\beta \in \Omega} q_\beta \exp(r_\beta - \varphi_\beta(\chi)) \frac{d\varphi_\beta(\chi)}{d\chi} \right)^{-1} q_\alpha \exp(r_\alpha - \varphi_\alpha(\chi)). \tag{1.42}$$

From (1.40) we define $d\varphi_\alpha(\chi)/d\chi$ and substitute it into (1.42). Simple transformations show that (1.42) is equal to $F(\chi)$. We have thereby proved that (1.39) identically satisfies (1.10).

Analogously, one can verify that (1.39) satisfies equation (1.21). For this, (1.41) is multiplied by $d\varphi_\beta(\chi)/d\chi$ and summed over β.

b) Let a^ν be constant values in (1.38). From (1.40) we find

$$\varphi_\alpha(y) = a_\alpha \mu(y),$$

where

$$\mu(y) = \int \frac{dy}{F(y)}.$$

Consider (1.35) for $N = 1$. Assume $p_\alpha = 1/a_\alpha$. On the other hand, $r_\alpha = \varphi_\alpha(u_\alpha) = z_\alpha + c_\alpha$. Then we obtain

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\frac{1}{a_\alpha}(z_\alpha + c_\alpha) - \mu(\chi)\right] = 1.$$

As a result we have

$$\chi = \mu^{-1}\left(\ln \sum_{\alpha\in\Omega} q_\alpha \exp\frac{z_\alpha + c_\alpha}{a_\alpha}\right),$$

where μ and μ^{-1} are reciprocal functions.

c) Consider the system of linear equations

$$a^\nu \frac{\partial u}{\partial x^\nu} = u, \tag{1.43}$$

where a^ν are constant square matrices of order N.

Let us write the equation of plane waves

$$a_\alpha \frac{du_\alpha}{dz_\alpha} = u_\alpha, \tag{1.44}$$

where $a_\alpha = (1/l_\alpha)\,\alpha_\nu a^\nu$. Assume that a_α is equivalent to a diagonal matrix for any α. Assume, in addition, that

$$\det a_\alpha \neq 0 \tag{1.45}$$

for $\alpha \neq 0$.

Let S_α be the matrices reducing a_α to the diagonal form

$$S_\alpha a_\alpha S_\alpha^{-1} = diag\left(\lambda_\alpha^1, \ldots, \lambda_\alpha^N\right).$$

Then equation (1.44) takes the form

$$\lambda_\alpha^k \frac{dv_\alpha^k}{dz_\alpha} = v_\alpha^k, \tag{1.46}$$

where

$$v_\alpha = S_\alpha u_\alpha.$$

From equation (1.46) we easily find the characteristic functions

$$\varphi_\alpha^k(y) = \lambda_\alpha^k \ln\left(S_{\alpha n}^{\ k} y^n\right), \tag{1.47}$$

where, as different from the index k, summation is performed over n from 1 to N.

In the algebraic equation (1.35) we set $p_\alpha^k = 1/\lambda_\alpha^k$. Taking into account (1.47), we obtain

$$\sum_{\alpha\in\Omega} q_\alpha \frac{S_{\alpha n}^{\ k} u_\alpha^n}{S_{\alpha m}^{\ k} \chi^m} = 1, \tag{1.48}$$

$$(k = 1, \ldots, N).$$

Let us calculate matrix (1.12) when $\varphi_\alpha^k(y)$ is (1.47):

$$B_{\alpha n}^{\ k}(y) = \frac{1}{S_{\alpha m}^{\ k} y^m} S_{\alpha n}^{\ k}. \tag{1.49}$$

It is obvious that the following identities are valid:

$$B_\alpha(y)\,y = b, \tag{1.50}$$

$$B_\alpha^{-1}(y)\,b = y.$$

Recall that the components of the vector b are

$$b^k = 1, \quad (k = 1, \dots, N). \tag{1.51}$$

By virtue of (1.49) the algebraic equation (1.48) can be rewritten as

$$\sum_{\alpha\in\Omega} q_\alpha B_\alpha(\chi)\,u_\alpha = b. \tag{1.52}$$

Multiplying equation (1.52) by the matrix $B_\beta^{-1}(\chi)$ and taking into account (1.50), we obtain

$$\sum_{\alpha\in\Omega} q_\alpha B_\beta^{-1}(\chi)\,B_\alpha(\chi)\,u_\alpha = \chi. \tag{1.53}$$

Analogously, multiplying (1.52) by $B_\gamma^{-1}(\chi)$ and subtracting the obtained result from (1.53), we have

$$\sum_{\alpha\in\Omega} q_\alpha \left(B_\beta^{-1}(\chi) - B_\gamma^{-1}(\chi)\right) B_\alpha(\chi)\,u_\alpha = 0. \tag{1.54}$$

This identity is fulfilled for any $\beta, \gamma \in \Omega$. On the other hand, since u_α are independent values for various α, from (1.54) it follows that

$$B_\beta^{-1}(\chi)\,B_\alpha(\chi)\,u_\alpha = B_\gamma^{-1}(\chi)\,B_\alpha(\chi)\,u_\alpha.$$

Assuming $\gamma = \alpha$ in the right-hand part of this equality, we have

$$B_\beta^{-1}(\chi)\,B_\alpha(\chi)\,u_\alpha = u_\alpha.$$

Using this equality, from (1.53) we eventually obtain the well-known result

$$\chi = \sum_{\alpha\in\Omega} q_\alpha u_\alpha. \tag{1.55}$$

In the case of (1.43), it can be easily verified that (1.55) identically satisfies equation (1.4). Next, it will be shown that (1.55) satisfies equation (1.21). For this, from (1.47) we find

$$\frac{\partial \varphi_\alpha^k(\chi)}{\partial r_\alpha^n} = \lambda_\alpha^k \frac{1}{S_{\alpha m}^{\ k}\chi^m} S_{\alpha l}^{\ k} \frac{\partial \chi^l}{\partial r_\alpha^n}, \tag{1.56}$$

where summation is performed over the indexes $m,\ l$ from 1 to N. From (1.7) and (1.47) we have

$$S_{\alpha n}^{\ k} u_\alpha^n = \exp\left(\frac{1}{\lambda_\alpha^k} r_\alpha^k\right), \tag{1.57}$$

$$u_\alpha^l = \sum_{m=1}^{N} \left(S_\alpha^{-1}\right)_m^l \exp\left(\frac{1}{\lambda_\alpha^m} r_\alpha^m\right).$$

In view of (1.55) and (1.57), we can write

$$\frac{\partial \chi^l}{\partial r_\alpha^n} = q_\alpha \frac{\partial u_\alpha^l}{\partial r_\alpha^n} = q_\alpha \sum_{m=1}^{N} \left(S_\alpha^{-1}\right)_m^l \frac{1}{\lambda_\alpha^m} \exp\left(\frac{1}{\lambda_\alpha^m} r_\alpha^m\right) \delta_n^m,$$

where δ_n^m is the Kronecker symbol. The substitution of the obtained result into (1.56) gives

$$\frac{\partial \varphi_\alpha^k(\chi)}{\partial r_\alpha^n} = q_\alpha \lambda_\alpha^k \frac{1}{S_{\alpha i}^{\ k} \chi^i} S_{\alpha l}^{\ k} \sum_{m=1}^{N} \left(S_\alpha^{-1}\right)_m^l \frac{1}{\lambda_\alpha^m} \exp\left(\frac{1}{\lambda_\alpha^m} r_\alpha^m\right) \delta_n^m.$$

Recall that summation over the indexes i and l is performed from 1 to N. Using (1.57), after some simple transformations we obtain

$$\frac{\partial \varphi_\alpha^k(\chi)}{\partial r_\alpha^n} = \frac{q_\alpha}{S_{\alpha i}^{\ k} \chi^i} S_{\alpha l}^{\ k} u_\alpha^l \delta_n^k. \tag{1.58}$$

This means that equality (1.32) is true. Substituting (1.58) into (1.21) and taking into account (1.48), we obtain the identity

$$b = b.$$

(13) Equation (1.6) can be left-multiplied by a nonsingular matrix so as to make the vector b have the components be equal to

$$\begin{aligned} &b^1 = 1, \\ &b^2 = \cdots = b^N = 0. \end{aligned} \tag{1.59}$$

Now let us introduce the functions

$$\begin{aligned} &\psi_\alpha^1 = \varphi_\alpha^1(y), \\ &\psi_\alpha^2 = \psi_\alpha^2\left(\varphi_\alpha^2(y), ..., \varphi_\alpha^N(y)\right), \\ &\quad ... \\ &\psi_\alpha^N = \psi_\alpha^N\left(\varphi_\alpha^2(y), ..., \varphi_\alpha^N(y)\right), \end{aligned} \tag{1.60}$$

where $\varphi_\alpha(y)$ is a solution of equation (1.6). If in equation (1.6) we replace φ_α by ψ_α, then, in view of (1.59), it can be easily seen that along with φ_α the function ψ_α is also a solution of equation (1.6).

Since equality (1.5) is fulfilled for the solutions of equation (1.9), taking into account (1.59) we obtain from (1.60)

$$\begin{aligned} &\psi_\alpha^1 = z_\alpha + \tilde{c}_\alpha^1, \\ &\psi_\alpha^2 = \tilde{c}_\alpha^2, \\ &\quad ... \\ &\psi_\alpha^N = \tilde{c}_\alpha^N, \end{aligned}$$

where

$$\tilde{c}^1_\alpha = c^1_\alpha,$$
$$\tilde{c}^2_\alpha = \psi^2_\alpha \left(c^2_\alpha, ..., c^N_\alpha\right),$$
$$...$$
$$\tilde{c}^N_\alpha = \psi^N_\alpha \left(c^2_\alpha, ..., c^N_\alpha\right).$$

We thus come to a conclusion that in view of (1.2) transformation (1.60) is equivalent to the transformation of one representation of integration constants to another representation of the these constants.

2. Commutative algebraic operations

(1) We continue the investigation of the algebraic properties of autonomous systems of ordinary differential equations

$$\frac{du^k}{dt} = F^k \left(u^1, \ldots, u^N\right), \quad (k = 1, \ldots, N), \tag{2.1}$$

which has been started in [1–2]. Here $F^k(u)$ is a smooth function given on the Euclidean space Γ^N, t is a real independent variable. The space of solutions of (2.1) is denoted by J^1_N.

In [1], we have established the defining equation (1.3) for binary operations acting in the space of solutions J^1_N. Now we introduce the new variables $r^k_{(1)} = \varphi^k(u_1)$, $r^k_{(2)} = \varphi^k(u_2)$, where $\varphi^k(u)$ are the characteristic functions of equation (2.1). Using equation (2.2) [1] for $\varphi(u)$, the defining equation (1.3) [1] can be rewritten as

$$\left(\frac{\partial\varphi^k(\Phi)}{\partial r^n_{(1)}} + \frac{\partial\varphi^k(\Phi)}{\partial r^n_{(2)}}\right) b^n = b^k, \quad (k = 1, \ldots, N). \tag{2.2}$$

As established in [1], being a function of arbitrary solutions u_1 and u_2, every solution Φ of equation (2.2) defines a binary operation in J^1_N. In particular, a solution represented as an implicit function

$$\exp\left[\varphi^k(u_1) - \varphi^k(\Phi)\right] + \exp\left[\varphi^k(u_2) - \varphi^k(\Phi)\right] = 1, \quad (k = 1, \ldots, N), \tag{2.3}$$

generates a commutative binary operation in J^1_N. Note that a general solution of equation (2.2) has the form

$$\varphi^k(\Phi) = \ln\left(\exp r^k_{(1)} + \exp r^k_{(2)}\right) + Q^k\left(r_{(2)} - r_{(1)}\right),$$

where Q^k are arbitrary functions of N arguments. It is not difficalt to rewrite this solution in the form

$$r^k_{(3)} - r^k_{(1)} = Q^k_1\left(r^1_{(2)} - r^1_{(1)}, ..., r^N_{(2)} - r^N_{(1)}\right),$$

where Q_1 is an arbitrary function of its variables, and $r^k_3 = \varphi^k(\Phi)$.

Without going into details, we explain that in order to find an M-nary operation [5] in the space of solutions J_N^1, equation (2.2) is extended to the equation

$$\sum_{a=1}^{M} \frac{\partial \varphi^k(\Phi)}{\partial r_{(a)}^n} b^n = b^k, \tag{2.4}$$

$$(k = 1, \ldots, N),$$

where $r_{(a)} = \varphi(u_a)$.

Using the results of the preceding paragraph, among other solutions of (2.4) we can write a particular solution

$$\Phi = \varphi^{-1}\left(\ln \sum_{a=1}^{M} \exp \varphi(u_a)\right), \tag{2.5}$$

where φ and φ^{-1} are reciprocal functions. This result proves that some M-nary operations can be reduced to binary operations.

Assuming now $M = 1$ in (2.4), we obtain

$$\frac{\partial \varphi^k(\Phi)}{\partial r^n} b^n = b^k. \tag{2.6}$$

A solution of this equation generates a unary operation in J_N^1. As shown in §3 [1], a unary operation can be interpreted as a mapping of the space J_N^1 into.

Thus we come to a conclusion that equation (2.4) defines a set of combinations of algebraic operations, including unary and M-nary ones. This gives us the right to pose questions whether M-nary operations are reducible or irreducible.

However at this stage our main objective consists in investigating questions connected with commutative binary operations.

In [1–2], we have studied with sufficient completeness the commutative binary operation $u_1 \underset{\varphi}{\dot{+}} u_2$ which is defined from (2.3):

$$u_1 \underset{\varphi}{\dot{+}} u_2 = \varphi^{-1}\left(\ln\left[\exp \varphi(u_1) + \exp \varphi(u_2)\right]\right). \tag{2.7}$$

This operation forms a commutative group in the space J_N^1 of solutions of equation (2.1), where e and h are neutral elements:

$$u \underset{\varphi}{\dot{+}} e = u, \tag{2.8}$$

$$u \underset{\varphi}{\dot{+}} h = h.$$

According to [1], if the components e^k and h^k of the elements e and h are finite values, then e and h are stationary points of equation (2.1).

As shown in [1–2], in addition to (2.7), there exists an alternative sum

$$u_1 \underset{\varphi}{\ddot{+}} u_2 = \varphi^{-1}\left(\ln\left[\exp(-\varphi(u_1)) + \exp(-\varphi(u_2))\right]^{-1}\right), \tag{2.9}$$

that also forms a commutative group. As different from (2.8), there hold

$$u \underset{\varphi}{\ddot{+}} e = e, \tag{2.10}$$

$$u \underset{\varphi}{\ddot{+}} h = u.$$

By virtue of (2.8) and (2.10) these two commutative groups form one algebraic object, i.e. a double commutative group.

In [1–2], we have shown that the space J_N^1 of solutions of equation (2.1) is a discrete fiber space [6]. At that, the base space W_N^1 is understood as the space of solutions of the system

$$\frac{dw^k}{dt} = w^k, \quad (k = 1, \ldots, N), \tag{2.11}$$

which is a collection of N one-dimensional independent equations. The projector is $\exp \varphi : J_N^1 \to W_N^1$.

According to [2], each equation from (2.11) generates the double commutative group (1.2–5). Then, by virtue of (2.7) and (2.9), we can write

$$u_1 \underset{\varphi}{\dot{+}} u_2 = \varphi^{-1}\left(\ln\left[w_1 \dot{+} w_2\right]\right), \tag{2.12}$$

$$u_1 \underset{\varphi}{\ddot{+}} u_2 = \varphi^{-1}\left(\ln\left[w_1 \ddot{+} w_2\right]\right).$$

If each discrete fiber of the space J_N^1 is taken as one element, then, by (1.17–18) [1], from (2.12) we have an isomorphism of the double commutative groups of equations (2.1) and (2.11).

Remark. In a general solution of (2.2) let us choose an arbitrary function Q_1 so that the equality be fulfilled:

$$r^k_{(3)} - r^k_{(1)} = \frac{1}{2}\left(r^k_{(2)} - r^k_{(1)}\right).$$

The obtained function defines a binary operation in the space of solutions of (2.1)

$$u_1 * u_2 = \varphi^{-1}\left(\frac{1}{2}\left[\varphi(u_1) + \varphi(u_2)\right]\right).$$

In particular, in the case of equation (2.11) we obtain

$$(w_1 * w_2)^k = \sqrt{w_1^k \cdot w_2^k}, \quad (k = 1, \ldots, N).$$

It can be easily verified that the operation $u_1 * u_2$ is commutative and nonassociative. If along with the summation operations (2.7) and (2.9), the operation $u_1 * u_2$ is assumed to be a product, then readily conclude that the distributive law does not hold. Hence, instead of $u_1 * u_2$ we introduce an operation of multiplication in the form:

$$u_1 \odot u_2 = \varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right).$$

As in [2], the operation $u_1 \odot u_2$ together with (2.7) and (2.9) forms a double algebraic field. It should be mentioned that, if u_1 and u_2 are solutions of (2.1), then $v = u_1 \odot u_2$ is a solution of

$$\frac{dv}{dt} = 2F(v).$$

(2) Let the spaces of solutions of equations (1.1) and (1.2) be denoted respectively by $J_N^{N_0}$ and $J_{N\alpha}^1$. The superscript of J is the number of independent variables, while the subscript is the number of the unknown functions contained in the respective equation. Obviously, $J_{N\alpha}^1 \subset J_N^{N_0}$ holds for any $\alpha \in \Gamma_{N_0}$, where Γ_{N_0} is the N_0-dimensional Euclidean space we have introduced

in (1) of Section *1*. As we have seen in [1], concurrently there arises a trivial fiber space $P(\Gamma_{N_0}, J_\alpha, \pi)$ with base space Γ_{N_0}, fibers J_α and projector $\pi : P \to \Gamma_{N_0}$ [6]. A solution χ of equation (1.21) should be interpreted as a mapping of the fiber space P in the space $J_N^{N_0}$

$$P \xrightarrow{\chi} J,$$

which we call a χ-mapping. In [1], we have studied with sufficient completeness the algebraic properties of equation (1.1) by means of functions χ. This however does not exhaust the algebraic content of equation (1.1). After all, $J^1_{N\alpha}$ is the space of solutions of ordinary equations (1.2). By (1), in $J^1_{N\alpha}$ there exist M-nary operations acting in the same space. Then the defining equation, which simultaneously gives both a χ-mapping and M -nary operations, has the form

$$\sum_{\alpha\in\Omega}\sum_{a=1}^{M}\frac{\partial\varphi_\alpha^k(\chi)}{\partial r^n_{(a)\alpha}}b^n = b^k, \qquad (k = 1,\dots,N). \tag{2.13}$$

Taking into account (1.7), (1.40) and (2.5), it is not difficult to write a particular solution of equation (2.13) in the implicit form.

(3) For simplicity, in the sequel the neutral elements of the double commutative group (1.2) are assumed to be independent of α, i.e. we have

$$e_\alpha = e, \qquad h_\alpha = h \tag{2.14}$$

for any $\alpha \in \Gamma_{N_0}$.

In [2] it has been shown that on the neutral elements the characteristic functions tend to infinity. From (6.6) [2] we conclude that

$$\varphi_\alpha(e) = -\infty, \qquad \varphi_\alpha(h) = +\infty. \tag{2.15}$$

Taking into account (2.14) and substituting $\varphi \to \varphi_\alpha$ into (2.7–10), we obtain a double commutative group with neutral elements e, h that acts in the space $J^1_{N\alpha}$.

(4) Let us consider the implicit function (1.35) and assume $p^k_\alpha = 1$ for any $\alpha \in \Omega$, $k = 1,\dots,N$. Then with (1.7) taken into account we have

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\varphi^k_\alpha(u_\alpha) - \varphi^k_\alpha(\dot\chi)\right] = 1, \qquad (k = 1,\dots,N). \tag{2.16}$$

Let us discuss some properties of the χ-mapping defined from (2.16).

a) From (1.6) it immediately follows that the characteristic functions φ_α explicitly depend on $\alpha \in \Gamma_{N_0}$. Consider in equality (2.16) two terms with indexes α and β. It is obvious that (2.16) remains unchanged if we exchange there α and β and, accordingly, u_α and u_β. This means that the function $\dot\chi$ defined by (2.16) has the form

$$\dot\chi = \dot\chi(\dots;\alpha,u_\alpha;\dots) \tag{2.17}$$

and is a symmetric function of the blocks (α, u_α), where α runs through the set Ω. In the sequel, instead of (2.17) we will use the short notation

$$\dot{\chi} = \dot{\chi}\left(\ldots, u_\alpha, \ldots\right).$$

b) Let the solution $u_\beta = e$ be given for some $\beta \in \Omega$. By (2.15), the term with index β in sum (2.16) vanishes and we obtain

$$\sum_{\alpha \in \Omega \backslash \beta} q_\alpha \exp\left[\varphi_\alpha^k\left(u_\alpha\right) - \varphi_\alpha^k\left(\dot{\chi}_\Omega\right)\right] = 1.$$

As has been shown in [1], if all $u_\alpha = e$ for $\alpha \in \Omega \setminus \gamma$, then $\dot{\chi}$ is a solution of (1.2) when $\alpha = \gamma$. In particular, if $u_\alpha = e$ for all $\alpha \in \Omega$, then we have

$$\dot{\chi}_\Omega\left(\ldots, e, \ldots, e, \ldots\right) = e. \tag{2.18}$$

If however for some $\beta \in \Omega$ the solution $u_\beta = h$, then, by virtue of (2.15), equality (2.16) is satisfied as soon as $\dot{\chi}_\Omega = h$, i.e.

$$\dot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots, h, \ldots, u_\gamma, \ldots\right) = h. \tag{2.19}$$

(5) Let us now assume in (1.35) that $p_\alpha^k = -1$. By analogy with (2.16) we get

$$\sum_{\alpha \in \Omega} q_\alpha \exp\left[\varphi_\alpha^k\left(\ddot{\chi}\right) - \varphi_\alpha^k\left(u_\alpha\right)\right] = 1, \tag{2.20}$$

$$(k = 1, \ldots, N).$$

The function $\ddot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots\right)$ has the same properties as $\dot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots\right)$, but, as different from $\dot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots\right)$, the neutral elements e and h in $\ddot{\chi}_\Omega$ possess the opposite properties. More precisely, instead of (2.18) we have

$$\ddot{\chi}_\Omega\left(\ldots, h, \ldots, h, \ldots\right) = h, \tag{2.21}$$

while (2.19) is replaced by the equality

$$\ddot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots, e, \ldots, u_\gamma, \ldots\right) = e. \tag{2.22}$$

By virtue of properties (2.18–19) and (2.21–22), in the sequel $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ will be called alternative mappings of the trivial fiber bundle $P\left(\Gamma_{N_0}, J_{N\alpha}^1, \pi\right)$ into the space $J_N^{N_0}$. $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ can also be called alternative expansions of solutions of equations (1.1) into plane waves u_α.

(6) Examples

a) Let us consider equation (1.38) where the coefficients a^ν are constant values. Using the results of (12) of Section *1* , from the equalities $\varphi_\alpha\left(u_\alpha\right) = z_\alpha + c_\alpha$, $\varphi_\alpha\left(u_\alpha\right) = a_\alpha \mu\left(u_\alpha\right)$ we obtain

$$\dot{\chi}_\Omega = \mu^{-1}\left(\ln \sum_{\alpha \in \Omega} q_\alpha \exp \mu\left(u_\alpha\right)\right).$$

Performing analogous calculations in order to find an alternative expansion, for $N = 1$ we derive from (2.20)

$$\ddot{\chi}_\Omega = \mu^{-1}\left(\ln\left[\sum_{\alpha \in \Omega} q_\alpha \exp\left(-\mu\left(u_\alpha\right)\right)\right]^{-1}\right).$$

b) Let us return to equation (1.43). As has been repeatedly shown in [1–2], in linear equations the neutral elements have the form

$$e_0,\ h_0,$$

where $e_0^k = 0$, $h_0^k = \infty$, $(k = 1, ..., N)$. The fulfillment of equalities (2.18-19) immediately follows from (1.55).

Now we will find an alternative sum for (1.55). For this, in (1.35) we assume that $q_\alpha = 1$, $p_\alpha^k = -\left(1/\lambda_\alpha^k\right)$. Taking into account (1.43), we obtain from (1.35)

$$\sum_{\alpha\in\Omega} q_\alpha B_\alpha\left(u_\alpha\right)\ddot{\chi}_\Omega = b,$$

where B_α is matrix (1.49) and the vector b has form (1.51). From the above equality we readily find an alternative expansion

$$\ddot{\chi}_\Omega = \left[\sum_{\alpha\in\Omega} q_\alpha B_\alpha\left(u_\alpha\right)\right]^{-1} b. \tag{2.23}$$

It is not difficult to verify that for the neutral elements e_0, h_0, expansion (2.23) with satisfies conditions (2.21–22).

Using (1.50), we rewrite (1.55) as follows:

$$\dot{\chi}_\Omega = \sum_{\alpha\in\Omega} B_\alpha^{-1}\left(u_\alpha\right) b.$$

It is obvious that $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ are alternative expansions of the solutions of equations (1.43).

(7) Let $\omega \subset \Omega$. Consider equality (2.16) on ω, i.e.

$$\sum_{\alpha\in\omega} q_\alpha \exp\left[\varphi_\alpha\left(u_\alpha\right) - \varphi_\alpha\left(\dot{\chi}\right)\right] = 1. \tag{2.24}$$

Using property (2.15), we add terms of the form $q_\beta \exp\left[\varphi_\beta\left(u_\beta\right) - \varphi_\beta\left(\dot{\chi}\right)\right]$ to equality (2.24) for

$$u_\beta = e, \quad \beta \in \Omega \setminus \omega. \tag{2.25}$$

It is obvious that (2.24) remains unchanged. Then, using condition (2.25), we can write

$$\dot{\chi}_\omega\left(..., u_\alpha, ...\right) = \dot{\chi}_\Omega\left(..., u_\alpha, ...\right). \tag{2.26}$$

Analogously, we can write

$$\ddot{\chi}_\omega\left(..., u_\alpha, ...\right) = \ddot{\chi}_\Omega\left(..., u_\alpha, ...\right) \tag{2.27}$$

for

$$u_\beta = h, \quad \beta \in \Omega \setminus \omega.$$

(8) As is well known, in the classical theory of linear PDEs one can always choose unique u_α and a set Ω in representation (1.55) such that a given solution $u\left(x\right)$ will coincide with sum (1.55). Therefore, in this stage of investigation, it can be assumed without proving that for any given solution of (1.1) there always exist a set Ω and a collection u_α, $\alpha \in \Omega$, such that the solution can be uniquely represented as

$$u = \dot{\chi}_\Omega\left(..., u_\alpha, ...\right). \tag{2.28}$$

Since $J_N^{N_0}$ is a discrete fiber space, solution (2.28) must be the well-defined leaf of this space

[6].

Analogously, by an appropriate choice of $\tilde{\Omega}$ and u_α, the same solution $u \in J_N^{N_0}$ can be uniquely represented in an alternative form

$$u = \ddot{\chi}_{\tilde{\Omega}}\left(..., u_\alpha, ...\right). \tag{2.29}$$

We have thereby come to a conclusion that the existence of $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ generates a double representation of solutions of equations (1.1).

(9) Let us consider equality (1.35). Using (1.7), for $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ we find the respective algebraic equations

$$\dot{\sum_{\alpha\in\Omega}} q_\alpha \exp p_\alpha^k \left[\varphi_\alpha^k\left(u_\alpha\right) - \varphi_\alpha^k\left(\dot{\chi}\right)\right] = 1, \tag{2.30}$$
$$(k = 1, \ldots, N)$$

and

$$\ddot{\sum_{\alpha\in\Omega}} \frac{1}{q_\alpha} \exp p_\alpha^k \left[\varphi_\alpha^k\left(u_\alpha\right) - \varphi_\alpha^k\left(\ddot{\chi}\right)\right] = 1, \tag{2.31}$$
$$(k = 1, \ldots, N).$$

We wish to recall that $\dot{\sum}$ is the summation by the standard rule $\left(a\dot{+}b = a + b\right)$, and $\ddot{\sum}$ is an alternative summation $\left(a\ddot{+}b = \left(a^{-1} + b^{-1}\right)^{-1}\right)$. Note that (2.31) is obtained from (2.30) by the replacement $q_\alpha \to 1/q_\alpha$, $p_\alpha^k \to -p_\alpha^k$.

If equalities (2.14-15) are fulfilled and it is assumed that p_α^k are real and change their sign for different $\alpha \in \Omega$, then equalities (2.18–19) and (2.21–22) do not hold. In that case, we should neglect restriction (2.14) and require that

$$p_\alpha^k \varphi^k\left(e_\alpha\right) = -\infty, \tag{2.32}$$
$$p_\alpha^k \varphi^k\left(h_\alpha\right) = +\infty.$$

This means that the representation of neutral elements varies depending on p_α^k. Not to complicate our discussion, we do not consider this problem here.

3. An induced double algebraic field

(1) While studying the algebraic properties of equation (1.2) in [1], we came to a conclusion that the space of solutions $J_{N\alpha}^1$ is a discrete fiber manifold $J_{N\alpha}^1\left(W_{N\alpha}^1, \exp\varphi_\alpha\right)$ consisting of the base space $W_{N\alpha}^1$, fiber $J_{N\alpha}^1$, projector $\exp\varphi_\alpha : J_{N\alpha}^1 \to W_{N\alpha}^1$ and the discrete group D_α which acts in discrete fibers. Recall that $W_{N\alpha}^1$ is the space of solutions of N one-dimensional equations which, by virtue of (3) of Section *1* , can be written in the form

$$\frac{dw_\alpha^k}{dz_\alpha} = w_\alpha^k, \tag{3.1}$$
$$(k = 1, \ldots, N).$$

Thus we see that a fiber $J_{N\alpha}^1$ of the trivial fiber space $P\left(\Gamma_{N_0},\ J_{N\alpha}^1,\ \pi\right)$ is in turn a properly discrete fiber manifold $J_{N\alpha}^1\ \left(W_{N\alpha}^1, \exp\varphi_\alpha\right)$.

From (2.16) and (2.20) it immediately follows that $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ can be regarded as functions of the variables $w_\alpha = \exp\varphi_\alpha(u_\alpha)$, where $\exp\varphi_\alpha$ is the projector in the discrete fiber manifold $J^1_{N\alpha}(W^1_{N\alpha}, \exp\varphi_\alpha)$. Therefore $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$, are in fact explicit functions of the elements of base manifolds $W^1_{N\alpha}$. Thus we can write

$$\dot{\chi}_\Omega(..., u_\alpha, ...) = \dot{\varkappa}_\Omega(..., w_\alpha, ...), \tag{3.2}$$

$$\ddot{\chi}_\Omega(..., u_\alpha, ...) = \ddot{\varkappa}_\Omega(..., w_\alpha, ...),$$

where u_α and w_α are related by the equality

$$w_\alpha = \exp\varphi_\alpha(u_\alpha). \tag{3.3}$$

Thus, if $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ map the trivial fiber space $P(\Gamma_{N_0},\ J^1_{N\alpha},\ \pi)$ into the space $J^{N_0}_N$, then $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$ map $P(\Gamma_{N_0},\ W^1_{N\alpha},\ \pi)$ into $J^{N_0}_N$. For simplicity, in the sequel we will mainly use the $\varkappa$-representation.

(2) From (2.15) and (3.3) we have

$$e_0 = \exp\varphi_\alpha(e), \quad h_0 = \exp\varphi_\alpha(h), \tag{3.4}$$

where $e^k_0 = 0$, $h^k_0 = \infty$, $(k = 1, \ldots, N)$. Then, by (3.2–3), equalities (2.18-19) take the form

$$\dot{\varkappa}_\Omega(..., e_0, ..., e_0, ..., e_0, ...) = e, \tag{3.5}$$

$$\dot{\varkappa}_\Omega(..., w_\alpha, ..., h_0, ..., w_\alpha, ...) = h.$$

Analogously, (2.21–22) take the form

$$\ddot{\varkappa}_\Omega(..., h_0, ..., h_0, ..., h_0, ...) = h, \tag{3.6}$$

$$\ddot{\varkappa}_\Omega(..., w_\alpha, ..., e_0, ..., w_\alpha, ...) = e.$$

(3) Let $u_1, u_2 \in J^{N_0}_N$. By virtue of the reasoning of (7) and (8) of Section *2*, these solutions can be written as

$$u_1 = \dot{\chi}_\Omega(..., u_{\alpha1}, ...), \tag{3.7}$$

$$u_2 = \dot{\chi}_\Omega(..., u_{\alpha2}, ...).$$

In the space $J^{N_0}_N$ we introduce the binary operation [1]

$$u_1 \dot{\oplus} u_2 = \dot{\chi}_\Omega(..., u_{\alpha1}, ...)\, \dot{\oplus} \dot{\chi}_\Omega(..., u_{\alpha2}, ...) = \dot{\chi}_\Omega\left(..., u_{\alpha1} \underset{\varphi_\alpha}{\dot{+}} u_{\alpha2}, ...\right). \tag{3.8}$$

Since the binary operation $u_{\alpha1} \underset{\varphi_\alpha}{\dot{+}} u_{\alpha2}$ is commutative and associative, the binary operation (3.8), too, possesses the same properties.

By (2.12) and (3.2-3), the binary operation $u_1 \dot{\oplus} u_2$ is written in terms of the $\varkappa$-representation as

$$u_1 \dot{\oplus} u_2 = \dot{\varkappa}_\Omega(..., w_{\alpha1}, ...)\, \dot{\oplus} \dot{\varkappa}_\Omega(..., w_{\alpha2}, ...) = \dot{\varkappa}_\Omega\left(..., w_{\alpha1} \dot{+} w_{\alpha2}, ...\right), \tag{3.9}$$

where $a \dot{+} b = a + b$. By (3.5), we conclude that the equalities below are true:

$$u \dot{\oplus} e = u, \quad u \dot{\oplus} h = h. \tag{3.10}$$

It obviously follows that the elements

$$u = \dot{\varkappa}_\Omega(..., w_\alpha, ...), \qquad (3.11)$$

$$_{(-)}u = \dot{\varkappa}_\Omega(..., -w_\alpha, ...)$$

of the space $J_N^{N_0}$ are the opposite elements with respect to summation (3.9)

$$u\dot{\oplus}\left(_{(-)}u\right) = e. \qquad (3.12)$$

Thus we have come to a conclusion that in the space $J_N^{N_0}$ the binary operation (3.9) forms a commutative group with neutral elements e and h.

Using the assumptions of (7) and (8) of Section *2*, the elements $u_1,\ u_2 \in J_N^{N_0}$ are represented by means of $\ddot{\varkappa}_\Omega$ as

$$u_1 = \ddot{\varkappa}_\Omega(..., \tilde{w}_{\alpha 1}, ...), \qquad (3.13)$$

$$u_2 = \ddot{\varkappa}_\Omega(..., \tilde{w}_{\alpha 2}, ...).$$

where $\tilde{w}_{\alpha 1}$ and $\tilde{w}_{\alpha 2}$ are solutions of equations (3.1). In the space $J_N^{N_0}$ we introduce an alternative binary operation

$$u_1\ddot{\oplus}u_2 = \ddot{\varkappa}_\Omega(..., \tilde{w}_{\alpha 1}, ...)\,\ddot{\oplus}\ddot{\varkappa}_\Omega(..., \tilde{w}_{\alpha 2}, ...) = \ddot{\varkappa}_\Omega\left(..., \tilde{w}_{\alpha 1}\ddot{+}\tilde{w}_{\alpha 2}, ...\right). \qquad (3.14)$$

Since the operation $a\ddot{+}b = \left(a^{-1} + b^{-1}\right)^{-1}$ is commutative and associative, operation (3.14), too, is such. Using (3.6) we find that

$$u\ddot{\oplus}e = e, \qquad (3.15)$$

$$u\ddot{\oplus}h = u.$$

The opposite elements $u,_{(-)}u \in J_N^{N_0}$ are written in terms of the $\ddot{\varkappa}$- representation as

$$u = \ddot{\varkappa}_\Omega(..., \tilde{w}_\alpha, ...),$$

$$_{(-)}u = \ddot{\varkappa}_\Omega(..., -\tilde{w}_\alpha, ...).$$

It is obvious that

$$u\ddot{\oplus}\left(_{(-)}u\right) = h \qquad (3.16)$$

is true.

Thus, since the commutative groups $u_1\dot{\oplus}u_2$ and $u_1\ddot{\oplus}u_2$ are linked by neutral elements e and h, we conclude that in the space $J_N^{N_0}$ of solutions of equations (1.1) they form a double commutative group.

(4) Let us consider the trivial fiber space $P\left(\Gamma_{N_0},\ W_{N\alpha}^1,\ \pi\right)$ and the discrete fiber space $J_N^{N_0}$. As has already been said, $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$ are mappings $P \to J_N^{N_0}$. From (2.30–31) and (3.3) it immediately follows that the structure of the functions $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$ is defined only by characteristic functions φ_α and a set $\Omega \subset \Gamma_{N_0}$. As to $w_\alpha \in W_{N\alpha}^1$, they act as simple arguments of the functions $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$.

By analogy with §2 [2], for every $\alpha \in \Gamma_{N_0}$ we assume that the fiber $W_{N\alpha}^1$ is filled with various smooth N-vector functions $g_\alpha(x)$, where $x \in \Gamma^{N_0}$. The resulting manifold is denoted by $B_{N\alpha}$. It is obvious that $W_{N\alpha}^1 \subset B_{N\alpha}$. Then the trivial fiber space $P = P\left(\Gamma_{N_0},\ W_{N\alpha}^1,\ \pi\right)$ extends to a trivial fiber manifold $P_B = P\left(\Gamma_{N_0},\ B_{N\alpha},\ \pi\right)$. Let us realize the mapping of $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$ of the manifold P_B. In that case, the space $J_N^{N_0}$ of solutions of equations (1.1) also extends to some manifold $B_N^{N_0}$. We would like to emphasize again that we extend only the objects which are subjected to the action of the characteristic functions φ_α and, in doing so, we do not change the functional structure of φ_α. Thus we are able to accomplish a more precise choice of algebraic structures hidden in the depth of differential equations. Since for $W^1_{N\alpha} \to B_{N\alpha}$ the functional structure of $\dot{\varkappa}_\Omega$ and $\ddot{\varkappa}_\Omega$ does not change, the manifold $B^{N_0}_N$ inherits the discrete fiber space $J^{N_0}_N$ unchanged. This means that in each fiber there acts a discrete group D transforming one element of the fiber to other elements of the same fiber [6]. Let $f(x)$ be some N -dimensional vector with smooth components $f^k(x)$. Using a group D, we can form – by means of $f(x)$ – the corresponding discrete fiber in the manifold $B^{N_0}_N$. Thus, by analogy with the construction of the manifold $B_{N\alpha}$, we completely fill $B^{N_0}_N$.

Without loss of generality, assume that for any $f \in B^{N_0}_N$ there always exist collections g_α, $\tilde{g}_\alpha \in B_{N\alpha}$, such that we can write

$$f = \dot{\varkappa}_\Omega(..., g_\alpha, ...)\,, \tag{3.17}$$

$$f = \ddot{\varkappa}_\Omega(..., \tilde{g}_\alpha, ...)\,.$$

(5) Using the results of (3), it seems logical to introduce algebraic operations in the manifold $B^{N_0}_N$. In the latter manifold we introduce the addition rule

$$\dot{\varkappa}_\Omega(..., g_{\alpha 1}, ...)\,\dot{\oplus}\dot{\varkappa}_\Omega(..., g_{\alpha 2}, ...) = \dot{\varkappa}_\Omega\left(..., g_{\alpha 1}\dot{+}g_{\alpha 2}, ...\right), \tag{3.18}$$

and an alternative addition

$$\begin{aligned}&\ddot{\varkappa}_\Omega(..., g_{\alpha 1}, ...)\,\ddot{\oplus}\ddot{\varkappa}_\Omega(..., g_{\alpha 2}, ...)\\ &= \ddot{\varkappa}_\Omega\left(..., g_{\alpha 1}\ddot{+}g_{\alpha 2}, ...\right),\end{aligned} \tag{3.19}$$

where $g_{\alpha 1}$, $g_{\alpha 2} \in B_{N\alpha}$. It is obvious that operations (3.18–19) are commutative and associative.

Equations (3.5–6) hold if w_α is replaced by arbitrary elements $g_\alpha \in B_{N\alpha}$. Then it is not difficult to show that the following equalities are valid:

$$\dot{\varkappa}_\Omega\dot{\oplus}e = \dot{\varkappa}_\Omega,$$

$$\dot{\varkappa}_\Omega\dot{\oplus}h = h,$$

$$\ddot{\varkappa}_\Omega\ddot{\oplus}e = e,$$

$$\ddot{\varkappa}_\Omega\ddot{\oplus}h = \ddot{\varkappa}_\Omega,$$

where $\dot{\varkappa}_\Omega = \dot{\varkappa}_\Omega(..., g_\alpha, ...)\,,\ \ddot{\varkappa}_\Omega = \ddot{\varkappa}_\Omega(..., g_\alpha, ...) \in B^{N_0}_N$.

If $f \in B^{N_0}_N$ is written in form (3.17), then its opposite element is

$$\dot{\varkappa}_\Omega(..., -g_\alpha, ...)\,,$$

$$\ddot{\varkappa}_\Omega(..., -\tilde{g}_\alpha, ...)\,.$$

Thus operations (3.18–9) form a double commutative group in the manifold $B^{N_0}_N$.

(6) Consider a solution of equations (1.1) written in the form

$$u = \dot{\varkappa}_\Omega(..., w_\alpha, ...)\,, \tag{3.20}$$

where $w_\alpha \in W^1_{N\alpha}$. From (3.1) it immediately follows that if $w_\alpha \in W^1_{N\alpha}$, then $c^k_\alpha w^k_\alpha$ $(k = 1, \ldots, N)$ is also a solution of equations (3.1). In that case, along with (3.20), we can write a new solution of equations (1.1):

$$u_c = \dot{\varkappa}_\Omega(..., c_\alpha w_\alpha, ...)\,. \tag{3.21}$$

This means that a differential equation prescribes the existence of an algebraic operation transforming (3.20) to (3.21). We conjecture that this operation is realized by a certain mechanism that constructs solutions with the aid of integration constants.

Let us write an element c of the manifold $B_N^{N_0}$ in the form

$$c = \dot{\varkappa}_\Omega \left(..., c_\alpha, ...\right), \tag{3.22}$$

where c_α^k are arbitrary variables not depending on $x \in \Gamma^{N_0}$. By the same reasoning as is used in [2] to introduce multiplication operations, in the manifold $B_N^{N_0}$ we introduce an algebraic operation by using elements (3.20) and (3.22):

$$c \dot{\otimes} u = \dot{\varkappa}_\Omega \left(..., c_\alpha, ...\right) \dot{\otimes} \dot{\varkappa}_\Omega \left(..., w_\alpha, ...\right) = \dot{\varkappa}_\Omega \left(..., c_\alpha w_\alpha, ...\right), \tag{3.23}$$

where $c_\alpha w_\alpha$ is an N-vector with the coordinates

$$\left(c_\alpha w_\alpha\right)^k = c_\alpha^k w_\alpha^k, \quad (k = 1, \ldots, N). \tag{3.24}$$

Along with (3.21) and (3.22), we construct other elements of the manifold $J_N^{N_0}$:

$$\tilde{u} = \ddot{\varkappa}_\Omega \left(..., w_\alpha, ...\right),$$

$$\tilde{c} = \ddot{\varkappa}_\Omega \left(..., c_\alpha, ...\right).$$

Clearly, $\tilde{u}$ is a solution of equation (1.1). By analogy with (3.23), let us introduce a binary operation in $B_N^{N_0}$:

$$\tilde{c} \ddot{\otimes} \tilde{u} = \ddot{\varkappa}_\Omega \left(..., c_\alpha, ...\right) \ddot{\otimes} \ddot{\varkappa}_\Omega \left(..., w_\alpha, ...\right) = \ddot{\varkappa}_\Omega \left(..., c_\alpha w_\alpha, ...\right), \tag{3.25}$$

where $c_\alpha w_\alpha$ is (3.24).

From (3.23) and (3.25) it immediately follows that $c\dot{\otimes}u$, $\tilde{c}\ddot{\otimes}\tilde{u}$ along with u and $\tilde{u}$ are solutions of (1.1).

Thus, in addition to the binary operations (3.18–19) in the manifold $B_N^{N_0}$, equation (1.1) brings us to the existence of yet another type of a binary operation that due to the presence of two maps $\dot{\varkappa}$ and $\ddot{\varkappa}$ is written in form (3.23) and (3.25). Therefore for arbitrary elements

$$f_a = \dot{\varkappa}_\Omega \left(..., g_{\alpha a}, ...\right),$$

$$\tilde{f}_a = \ddot{\varkappa}_\Omega \left(..., g_{\alpha a}, ...\right), \quad (a = 1, 2) \tag{3.26}$$

of the manifold $B_N^{N_0}$ we have

$$f_1 \dot{\otimes} f_2 = \dot{\varkappa}_\Omega \left(..., g_{\alpha 1}, ...\right) \dot{\otimes} \dot{\varkappa}_\Omega \left(..., g_{\alpha 2}, ...\right) = \dot{\varkappa}_\Omega \left(..., g_{\alpha 1} g_{\alpha 2}, ...\right), \tag{3.27}$$

$$\tilde{f}_1 \ddot{\otimes} \tilde{f}_2 = \ddot{\varkappa}_\Omega \left(..., g_{\alpha 1}, ...\right) \ddot{\otimes} \ddot{\varkappa}_\Omega \left(..., g_{\alpha 2}, ...\right) = \ddot{\varkappa}_\Omega \left(..., g_{\alpha 1} g_{\alpha 2}, ...\right),$$

where

$$\left(g_{\alpha 1} g_{\alpha 2}\right)^k = g_{\alpha 1}^k g_{\alpha 2}^k, \quad (k = 1, ..., N). \tag{3.28}$$

Note that $g_{\alpha 1} g_{\alpha 2}$ is the operation of multiplication in a fiber $B_{N\alpha}$ of the trivial fiber manifold $P_B = P\left(\Gamma_{N_0}, B_{N\alpha}, \ \pi\right)$.

From (3.27–28) it follows that the binary operations $f_1 \dot{\otimes} f_2$ and $f_1 \ddot{\otimes} f_2$ are commutative and associative.

It is not difficult to prove the distributive properties of the obtained binary operations:

$$f_1\dot{\otimes}\left(f_2\dot{\oplus}f_3\right)=\left(f_1\dot{\otimes}f_2\right)\dot{\oplus}\left(f_1\dot{\otimes}f_3\right), \tag{3.29}$$

$$f_1\ddot{\otimes}\left(f_2\ddot{\oplus}f_3\right)=\left(f_1\ddot{\otimes}f_2\right)\ddot{\oplus}\left(f_1\ddot{\otimes}f_3\right).$$

By (3.5) and (3.27) we find

$$f\dot{\otimes}e=e, \tag{3.30}$$

$$f\dot{\otimes}h=h$$

provided that $f\neq e$, $f\neq h$, where f is written in form (3.26). Analogously, by (3.6) and (3.27) we have

$$\tilde{f}\ddot{\otimes}e=e, \tag{3.31}$$

$$\tilde{f}\ddot{\otimes}h=h.$$

(7) In the manifold $B_N^{N_0}$ we form the elements

$$\dot{E}_\Omega=\dot{\varkappa}_\Omega\left(...,1,...,1,...\right) \tag{3.32}$$

and

$$\ddot{E}_\Omega=\ddot{\varkappa}_\Omega\left(...,1,...,1,...\right). \tag{3.33}$$

From (3.27) it follows that

$$\dot{E}_\Omega\dot{\otimes}f=f, \tag{3.34}$$

$$\ddot{E}_\Omega\ddot{\otimes}\tilde{f}=\tilde{f}.$$

Thus we conclude that $\dot{E}_\Omega$ and $\ddot{E}_\Omega$ play the role of unit elements in the operations $\dot{\otimes}$ and $\ddot{\otimes}$ from (3.27), respectively.

(8) Let g_α and m_α be elements of a fiber $B_{N\alpha}$ from the trivial fiber space $P_B=P\left(\Gamma_{N_0},\ B_{N\alpha},\ \pi\right)$. Assume that g_α and m_α are the reciprocal elements with respect to the operation of multiplication in the manifold $B_{N\alpha}$:

$$g_\alpha m_\alpha=1, \tag{3.35}$$

where $g_\alpha m_\alpha$ are calculated according to rule (3.28). Then the elements

$$f=\dot{\varkappa}_\Omega\left(...,g_\alpha,...\right),$$

$$m=\dot{\varkappa}_\Omega\left(...,m_\alpha,...\right)$$

of the manifold $B_N^{N_0}$ are also the reciprocal elements in $B_N^{N_0}$:

$$f\dot{\otimes}m=\dot{E}_\Omega. \tag{3.36}$$

Analogously, if

$$\tilde{f}=\ddot{\varkappa}_\Omega\left(...,g_\alpha,...\right),$$

$$\tilde{m}=\ddot{\varkappa}_\Omega\left(...,m_\alpha,...\right),$$

then we have

$$\tilde{f}\ddot{\otimes}\tilde{m}=\ddot{E}_\Omega. \tag{3.37}$$

Applying the above reasoning, we can introduce the operation of $\varkappa$ -division in the manifold

$B_N^{N_0}$. Let $f_1,\ f_2 \in B_N^{N_0}$. Denote the $\dot{\varkappa}$-relation between $\dot{\varkappa}$-division f_1 and f_2 by

$$f_1 \underset{(\dot{\varkappa})}{/} f_2 = \dot{\varkappa}_\Omega \left(..., \frac{g_{\alpha 1}}{g_{\alpha 2}}, ... \right), \tag{3.38}$$

where the components of the N-vector $g_{\alpha 1}/g_{\alpha 2}$ are

$$\left(\frac{g_{\alpha 1}}{g_{\alpha 2}} \right)^k = \frac{g_{\alpha 1}^k}{g_{\alpha 2}^k}, \qquad (k = 1, ..., N). \tag{3.39}$$

Analogously, we can introduce the $\ddot{\varkappa}$-relation

$$\tilde{f}_1 \underset{(\ddot{\varkappa})}{/} \tilde{f}_2 = \ddot{\varkappa}_\Omega \left(..., \frac{g_{\alpha 1}}{g_{\alpha 2}}, ... \right).$$

Thus we conclude that the $\dot{\varkappa}_\Omega$-mapping transforms the algebraic field generated by (3.1) and acting in the manifold $B_{N\alpha}$ to the algebraic field of the manifold $B_N^{N_0}$, while the $\ddot{\varkappa}_\Omega$-mapping transforms an alternative algebraic field, acting in $B_{N\alpha}$, to an alternative field of the manifold $B_N^{N_0}$. The neutral elements $e,\ h \in B_N^{N_0}$ combine these fields into a double field, i.e. into one whole object.

(9) Let us return to $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ which realize the mapping of the trivial fiber space $P(\Gamma_{N_0},\ J_{N\alpha},\ \pi)$ into $J_N^{N_0}$. In (4) we have introduced the trivial fiber manifold $P(\Gamma_{N_0},\ B_{N\alpha},\ \pi)$. After realizing the mapping $\varphi_\alpha^{-1}(\ln) : B_{N\alpha} \to Q_{N\alpha}$, the manifold $P(\Gamma_{N_0},\ B_{N\alpha},\ \pi)$ transforms to the manifold $P(\Gamma_{N_0},\ Q_{N\alpha},\ \pi)$. It is obvious that $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ map $P(\Omega,\ Q_{N\alpha},\ \pi)$ into the manifold $B_N^{N_0}$.

Using (3.2–3) and (2.12), we obtain, along with (3.8),

$$\tilde{u}_1 \ddot{\oplus} \tilde{u}_2 = \ddot{\chi}_\Omega(..., u_{\alpha 1}, ...) \ddot{\oplus} \ddot{\chi}_\Omega(..., u_{\alpha 2}, ...) = \ddot{\chi}_\Omega\left(..., u_{\alpha 1} \underset{\varphi_\alpha}{\ddot{+}} u_{\alpha 2}, ... \right), \tag{3.40}$$

where $\tilde{u}_1,\ \tilde{u}_2 \in J_N^{N_0}$. Analogously, (3.27) implies the equalities

$$f_1 \dot{\otimes} f_2 = \dot{\chi}_\Omega(..., v_{\alpha 1}, ...) \dot{\otimes} \dot{\chi}_\Omega(..., v_{\alpha 2}, ...) = \dot{\chi}_\Omega\left(..., v_{\alpha 1} \underset{\varphi_\alpha}{\odot} v_{\alpha 2}, ... \right), \tag{3.41}$$

and

$$\tilde{f}_1 \ddot{\otimes} \tilde{f}_2 = \ddot{\chi}_\Omega(..., v_{\alpha 1}, ...) \ddot{\otimes} \ddot{\chi}_\Omega(..., v_{\alpha 2}, ...) = \ddot{\chi}_\Omega\left(..., v_{\alpha 1} \underset{\varphi_\alpha}{\odot} v_{\alpha 2}, ... \right), \tag{3.42}$$

where $v_{\alpha 1},\ v_{\alpha 2} \in Q_{N\alpha}$. In these equalities $v_{\alpha 1} \underset{\varphi_\alpha}{\odot} v_{\alpha 2}$ is the product in the manifold $Q_{N\alpha}$ studied in [2]. The unit element of this product is $E_\alpha \in Q_{N\alpha}$ which is in turn a solution of the equation

$$\varphi_\alpha(E_\alpha) = 0. \tag{3.43}$$

In that case, the unit elements (3.32–33) in the χ-representation take the form

$$\dot{E}_\Omega = \dot{\chi}_\Omega(..., E_\alpha, ...), \qquad \ddot{E}_\Omega = \ddot{\chi}_\Omega(..., E_\alpha, ...). \tag{3.44}$$

By (2.18–19), (2.21–22), it is easy to verify the validity of equations (3.10), (3.15), (3.30-31) and (3.34) in the χ-representation.

Thus we have shown that in $B_N^{N_0}$ the double field is realized both in the $\varkappa$- and in the χ-representation.

(10) Let us consider the one-dimensional quasilinear equation with constant coefficients of the derivatives

$$a^{\nu}\frac{\partial u}{\partial x^{\nu}} = u\left(1-u\right) \tag{3.45}$$

We write the equation of plane waves in the form

$$a_{\alpha}\frac{du_{\alpha}}{dz_{\alpha}} = u_{\alpha}\left(1-u_{\alpha}\right), \tag{3.46}$$

where $a_{\alpha} = \left(1/l_{\alpha}\right)\alpha_{\nu}a^{\nu}$. From this equation we easily find the characteristic functions

$$\varphi_{\alpha}\left(u_{\alpha}\right) = a_{\alpha}\ln\frac{u_{\alpha}}{1-u_{\alpha}}. \tag{3.47}$$

Result (2.7) immediately follows from (2.3). By virtue of Section *1*, we conclude that, along with (2.3), an implicit function

$$\sum_{\alpha=1}^{2}\mathring{q}_{\alpha}\exp\mathring{p}_{\alpha}^{k}\left[\varphi^{k}\left(u_{\alpha}\right)-\varphi^{k}\left(\Phi\right)\right] = 1, \tag{3.48}$$

$$(k = 1,\ldots,N)$$

is also a solution of equation (2.2). This equation obviously implies (2.9) provided that $\mathring{q}_{\alpha} = 1$, $\mathring{p}_{\alpha}^{k} = -1$.

To obtain binary operations for equation (3.46), we assume in (3.48) that $\mathring{q}_{\alpha a} = 1$, $\mathring{p}_{\alpha a} = 1/a_{\alpha}$, and in the alternative case $\mathring{q}_{\alpha a} = 1$, $\mathring{p}_{\alpha a} = -1/a_{\alpha}$. Omitting calculations, we have

$$u_{\alpha 1}\underset{\varphi_{\alpha}}{\dot{+}}u_{\alpha 2} = \frac{u_{\alpha 1}+u_{\alpha 2}-2u_{\alpha 1}u_{\alpha 2}}{1-u_{\alpha 1}u_{\alpha 2}} \tag{3.49}$$

and

$$u_{\alpha 1}\underset{\varphi_{\alpha}}{\ddot{+}}u_{\alpha 2} = \frac{u_{\alpha 1}u_{\alpha 2}}{u_{\alpha 1}+u_{\alpha 2}-u_{\alpha 1}u_{\alpha 2}}. \tag{3.50}$$

In these commutative operations,

$$e = 0$$

$$h = 1$$

are neutral elements.

Thus we have obtained the double commutative group acting in the space $J_{1\alpha}^{1}$ of solutions of equation (3.46). By virtue of (4), $J_{1\alpha}^{1}$ extends to the manifold $Q_{1\alpha}$. Algebraic operations (3.49–50) transform to $Q_{1\alpha}$.

Using equality (6.12) from [2], let us introduce the operation of multiplication in $Q_{1\alpha}$. Then in the case of (3.47) we have

$$g_{\alpha 1}\underset{\varphi_{\alpha}}{\odot}g_{\alpha 2} = \frac{g_{\alpha 1}g_{\alpha 2}}{1-q_{\alpha 1}-q_{\alpha 2}+q_{\alpha 1}q_{\alpha 2}}, \tag{3.51}$$

where $g_{\alpha 1},\ g_{\alpha 2} \in Q_{1\alpha}$. A solution of the equation $\varphi_{\alpha}\left(E_{\alpha}\right) = 0$ [2] is a unit element E_{α} of operation (3.51). Then (3.47) immediately implies

$$E_\alpha = \frac{1}{2}. \tag{3.52}$$

A simple verification shows that the following equations are valid:

$$\begin{gathered} e \underset{\varphi_\alpha}{\dot{+}} g_\alpha = g_\alpha, \\ h \underset{\varphi_\alpha}{\dot{+}} g_\alpha = h, \\ e \underset{\varphi_\alpha}{\ddot{+}} g_\alpha = e, \\ h \underset{\varphi_\alpha}{\ddot{+}} g_\alpha = q_\alpha, \\ E_\alpha \underset{\varphi_\alpha}{\odot} g_\alpha = g_\alpha \underset{\varphi_\alpha}{\odot} E_\alpha = g_\alpha, \\ e \underset{\varphi_\alpha}{\odot} g_\alpha = e, \\ h \underset{\varphi_\alpha}{\odot} g_\alpha = h, \end{gathered} \tag{3.53}$$

where $g_\alpha \in Q_{1\alpha}$. It is obvious that operations (3.49–51) acting in the manifold $Q_{1\alpha}$, generate a double algebraic field.

Now let us find $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$. Using (3.47) and assuming $p_\alpha = 1/a_\alpha$ in (2.30–31), we obtain

$$\dot{\chi}_\Omega = \frac{\sum\limits_{\alpha\in\Omega} q_\alpha \frac{u_\alpha}{1-u_\alpha}}{1 + \sum\limits_{\alpha\in\Omega} q_\alpha \frac{u_\alpha}{1-u_\alpha}}, \tag{3.54}$$

$$\ddot{\chi}_\Omega = \frac{1}{1 + \sum\limits_{\alpha\in\Omega} q_\alpha \frac{1-u_\alpha}{u_\alpha}}.$$

Taking into account (3.46), it is easy to prove that $\dot{\chi}_\Omega$, $\ddot{\chi}_\Omega$ are solutions of (3.45). It is likewise easy to show the fulfillment of (2.18-19) and (2.21-22).

From (3.44), (3.52) and (3.54) we find

$$\dot{E}_\Omega = \frac{\sum\limits_{\alpha\in\Omega} q_\alpha \frac{E_\alpha}{1-E_\alpha}}{1 + \sum\limits_{\alpha\in\Omega} q_\alpha \frac{E_\alpha}{1-E_\alpha}}, \tag{3.55}$$

$$\ddot{E}_\Omega = \frac{1}{1 + \sum\limits_{\alpha\in\Omega} q_\alpha \frac{1-E_\alpha}{E_\alpha}}.$$

In these equalities it is required that:

$$\sum_{\alpha\in\Omega} q_\alpha = 1. \tag{3.56}$$

Taking into account (3.52) and (3.56), from (3.55) we find that

$$\dot{E}_\Omega = \frac{1}{2},$$

$$\ddot{E}_{\Omega} = \frac{1}{2}.$$

It obviously follows that the linear operations (3.8), (3.40–42) and (3.49–51) in the χ-representation generate a double field in the manifold $B_1^{N_0}$ provided that the binary operations (3.49–51) are realized in $Q_{1\alpha}$.

Taking into account (3.3), we realize the mapping

$$w_{\alpha} = \exp \frac{1}{a_{\alpha}} \varphi_{\alpha} \left(u_{\alpha} \right),$$

where φ_{α} is (3.47), i.e.

$$w_{\alpha} = \frac{u_{\alpha}}{1 - u_{\alpha}}. \tag{3.57}$$

But since u_{α} is a solution of (3.46), we have

$$\frac{dw_{\alpha}}{dz_{\alpha}} = \frac{1}{a_{\alpha}} w_{\alpha}. \tag{3.58}$$

In the manifold $B_{1\alpha}$ which is an extension of the space $W_{1\alpha}^1$ of solutions of equation (3.58), binary operations have the from [2]

$$g_{\alpha 1} \dot{+} g_{\alpha 2} = g_{\alpha 1} + g_{\alpha 2},$$

$$g_{\alpha 1} \ddot{+} g_{\alpha 2} = \frac{1}{\frac{1}{g_{\alpha 1}} + \frac{1}{g_{\alpha 2}}}, \tag{3.59}$$

$$g_{\alpha 1} \odot g_{\alpha 2} = g_{\alpha 1} g_{\alpha 2}.$$

In this algebra $e_0 = 0, \quad h_0 = \infty$ are identity elements, and $E_{\alpha} = 1$ is a unit element. It is obvious that the double fields (3.49–53) and (3.59) are isomorphic to each other.

It is interesting to note that when resistors in electric circuits are connected in series, the total resistance obeys the first addition rule in (3.58), while in the case of connection in parallel it obeys the second addition rule in (3.58).

Proceeding from (3.2) and taking into account (3.56), we find a solution of equation (3.45) in the $\varkappa$-representation:

$$\dot{\varkappa}_{\Omega} = \frac{\sum\limits_{\alpha \in \Omega} q_{\alpha} w_{\alpha}}{1 + \sum\limits_{\alpha \in \Omega} q_{\alpha} w_{\alpha}},$$

$$\ddot{\varkappa}_{\Omega} = \frac{1}{1 + \sum\limits_{\alpha \in \Omega} q_{\alpha} \frac{1}{w_{\alpha}}}.$$

Since $E_{\alpha} = 1$ in (3.59), taking into account (3.56), we obtain

$$\dot{E}_{\Omega} = \frac{1}{2},$$

$$\ddot{E}_{\Omega} = \frac{1}{2}.$$

Then from (3.18-19), (3.27) and (3.59) we can easily find a double field acting in the manifold $B_1^{N_0}$ in the $\varkappa$-representation.

(11) Let the solution E_α of equation (3.43) do not depend on the parameter α:

$$E_\alpha^k = E^k, \quad (k = 1, ..., N). \tag{3.60}$$

Recall that the characteristic function $\varphi_\alpha(y)$ is a solution of equation (1.6) and its dependence on α is due to the matrix $a_\alpha = \alpha_\nu a^\nu(y)$ contained in it. In fact, (2.14) and (3.60) are restrictions, imposed on matrix $a^\nu(y)$, which brings about the narrowing of the class of considered equations (1.1). However, in this stage of our investigation these restrictions are not a hindrance to us in comprehending the algebraic gist of differential equations.

Let us return to equations (2.30-31). Proceeding from the example in (10), we assume:

$$E_\alpha^k = E^k, \ \dot{\chi}_\Omega = \ddot{\chi}_\Omega = E^k \quad (k = 1, ..., N), \tag{3.61}$$

for every $\alpha \in \Omega$. Then by analogy with (3.56), from (2.30-31) we obtain

$$\sum_{\alpha \in \Omega} q_\alpha = 1. \tag{3.62}$$

To simplify our further investigation, we will assume that the conditions (2.14) and (3.60-62) hold for all the considered equations (1.1).

(12) For illustration let us consider the following examples:

a) Suppose there is an equation ($N = 1$):

$$\mathring{a}^\nu \frac{\partial u}{\partial x^\nu} = \sin u, \tag{3.63}$$

where $\mathring{a}^\nu$ are given constant magnitudes. The equation of plane waves (1.2) for (3.63) has the form

$$\mathring{a}_\alpha \frac{du_\alpha}{dz_\alpha} = \sin u_\alpha, \tag{3.64}$$

where $\mathring{a}_\alpha = (\alpha_\nu \mathring{a}^\nu)/l_\alpha$. The characteristic function of this equation is

$$\varphi_\alpha = \mathring{a}_\alpha \ln tg \frac{u_\alpha}{2}. \tag{3.65}$$

The general solution of equation (3.64) is

$$u_\alpha = 2arctg \exp\left[\frac{1}{\mathring{a}_\alpha} z_\alpha + c_\alpha\right] + 2\pi n, \tag{3.66}$$

where c_α are arbitrary constants of integration. When in (3.66) n runs through the integers we obtain the collection of elements of a discrete fiber of the space $J_{1\alpha}^1$. $J_{1\alpha}^1$ in turn is a fiber of the trivial fiber bundle $P(\Gamma_{N_0}, \ J_{1\alpha}^1, \ \pi)$.

Assuming in (2.30-31) that $p_\alpha = 1/\mathring{a}_\alpha$, by analogy with the example in (10), we find the solution of equation (3.63)

$$\dot{\chi}_\Omega = 2arctg\left(\dot{\sum_{\alpha \in \Omega}} q_\alpha tg \frac{u_\alpha}{2}\right) + 2\pi n, \tag{3.67}$$

$$\ddot{\chi}_{\Omega}=2arctg\left(\ddot{\sum_{\alpha\in\Omega}}\frac{1}{q_{\alpha}}tg\frac{u_{\alpha}}{2}\right)+2\pi n.$$

For the double field of equation (3.63), the neutral elements are

$$e=2\pi n, \tag{3.68}$$

$$h=\pi+2\pi n.$$

As noted in [1-2], the neutral elements e and h are stationary points of the given equation.

Let us go back to mappings (3.67) and find the identity elements (3.44) of multiplication (3.41-42). Since the characteristic functions of (3.63) is (3.65), we obtain from (3.43) the following:

$$E_{\alpha}=\frac{\pi}{2}+2\pi n. \tag{3.69}$$

Using (3.69), from (3.44) and (3.67), we find that

$$\dot{E}_{\Omega}=2arctg\left(\dot{\sum_{\alpha\in\Omega}}q_{\alpha}\right)+2\pi n,$$

$$\ddot{E}_{\Omega}=2arctg\left(\ddot{\sum_{\alpha\in\Omega}}\frac{1}{q_{\alpha}}\right)+2\pi n.$$

Since (3.62) is fulfilled, we finally derive

$$\dot{E}_{\Omega}=\ddot{E}_{\Omega}=E_{\alpha}=\frac{\pi}{2}+2\pi n.$$

b) Let us now study the linear system (1.43). Obviously, if $\varphi_{\alpha}^{k}(y)$ is a solution of (1.6), then $\varphi_{\alpha}^{k}(y)+c_{\alpha}^{k}$ is also a solution, where c_{α}^{k} are arbitrary constants. Accordingly, instead of (1.47) we introduce the characteristic functions

$$\varphi_{\alpha}^{k}(y)=\lambda_{\alpha}^{k}\left[\ln\left(S_{\alpha n}^{k}y^{n}\right)-\ln\left(S_{\alpha n}^{k}b^{n}\right)\right], \tag{3.70}$$

where b^{k} is (1.51). One can easily see that with (3.70) the results obtained in (12) of Section *1* remain unchanged.

From (3.43) and (3.70) it immediately follows that:

$$E_{\alpha}^{k}=b^{k}=1, \qquad (k=1,...,N). \tag{3.71}$$

Taking into account (3.62), from (6) of Section *2*, we immediately obtain:

$$\dot{E}_{\Omega}=\ddot{E}_{\Omega}=E_{\alpha}=b.$$

(13) As has been shown in [2], in the double field of ordinary differential equations there is one operation of multiplication. In the case of equation (1.1), from (3.27) it follows that the operation of multiplication in the trivial fiber manifold $P\left(\Gamma_{N_0},\ B_{N\alpha},\ \pi\right)$ gets split – because of the double mapping $\dot{\varkappa}_{\Omega}$ and $\ddot{\varkappa}_{\Omega}$ - into two products acting in $B_{N}^{N_0}$. One algebraic field is generated by means of $\dot{\varkappa}_{\Omega}$, and the second alternative field by $\ddot{\varkappa}_{\Omega}$. However both fields are interrelated by the common neutral elements e and h which in these fields manifest the opposite properties.

4. Differential calculus

Having defined the double field generated by the quasilinear system (1.1), we proceed to constructing the differential calculus.

(1) In [2], in studying the differential calculus in the representation of nonlinear ordinary differential equations, there concurrently arose internal times which we call proper times. Analogously, in the theory expounded here the presence of independent variables X^{ν} gives rise to values X^{ν} which play an important role in the formation of derivatives and differentials.

Using (7.4) [2] in the algebra of the manifold $J_{\alpha}\left(W^{1}_{N\alpha},\ \exp\varphi_{\alpha}\right)$, we can write

$$X^{\nu k}_{\alpha} = \varphi^{-1k}_{\alpha}\left(\ln x^{\nu}, ..., \ln x^{\nu}\right), \quad (k = 1, ..., N) \tag{4.1}$$

For the fixed index ν, values X^{ν}_{α} form elements of a discrete fiber of the fiber manifold $Q_{N\alpha}$. From (4.1) it follows that when x runs through the space Γ^{N_0}, the collection X_{α} forms an individual space $\Gamma^{N_0}_{\varphi_{\alpha}}$ which is simultaneously a discrete fiber manifold having the same structure as the manifold $J_{N\alpha}\left(B_{N\alpha},\ \exp\varphi_{\alpha}\right)$, when the N-vector $(x^{\nu}, ..., x^{\nu})$ is an element of $B_{N\alpha}$. In other words, $\Gamma^{N_0}_{\varphi_{\alpha}}$ is N_0 copies of a manifold of the form $J_{N\alpha}\left(x^{\nu},\ \exp\varphi_{\alpha}\right)$.

From the manifold $P\left(\Gamma_{N_0},\ Q_{N\alpha},\ \pi\right)$ we form a trivial fiber submanifold $P\left(\Gamma_{N_0},\ X^{\nu}_{\alpha},\ \pi\right)$, where $X^{\nu}_{\alpha} \in J_{N\alpha}\left(B_{N\alpha},\ \exp\varphi_{\alpha}\right)$. Let us realize the $\dot{\chi}$- and $\ddot{\chi}$-mappings as follows:

$$X^{\nu k} = \dot{\chi}^{k}_{\Omega}\left(..., X^{\nu}_{\alpha}, ...\right), \qquad \tilde{X}^{\nu k} = \ddot{\chi}^{k}_{\Omega}\left(..., X^{\nu}_{\alpha}, ...\right). \tag{4.2}$$

Keeping in mind the fact that we have introduced the N-vector $(x^{\nu}, ..., x^{\nu})$ which is an element of the manifold $B_{N\alpha}$, we rewrite (4.2) in the $\varkappa$-representation as

$$X^{\nu k} = \dot{\varkappa}^{k}_{\Omega}\left(..., x^{\nu}, ...\right), \qquad \tilde{X}^{\nu k} = \ddot{\varkappa}^{k}_{\Omega}\left(..., x^{\nu}, ...\right). \tag{4.3}$$

It is natural to call $X^{\nu k}$ and $\tilde{X}^{\nu k}$ the coordinates of the proper external spaces $\Gamma^{N_0}_{\dot{\varkappa}}$ and $\Gamma^{N_0}_{\ddot{\varkappa}}$ of the manifold $B^{N_0}_{N}$.

(2) Let a function $f(x) \in B^{N_0}_{N}$ in the $\dot{\varkappa}$ -representation have form (3.17). It is assumed that $g_{\alpha}(x) \in B_{N\alpha}$ are smooth functions in the neighborhood of a given point $x_0 \in \Gamma^{N_0}$ for any $\alpha \in \Omega$, where $\Omega \subset \Gamma_{N_0}$. Using the results of (5) of Section *3*, let us form an increment of the function $f(x)$:

$$\Delta_{\dot{\varkappa}} f(x) = f(x) \dot{\ominus} f(x_0) = \dot{\varkappa}_{\Omega}\left(..., g_{\alpha}(x) - g_{\alpha}(x_0), ...\right). \tag{4.4}$$

On the other hand, by the standard calculus, we can write

$$\Delta g^{k}_{\alpha}(x) = g^{k}_{\alpha}(x) - g^{k}_{\alpha}(x_0) = \left[\frac{\partial g^{k}_{\alpha}(x_0)}{\partial x^{\nu}} + \omega^{k}_{\nu}(\Delta x)\right]\Delta x^{\nu}, \tag{4.5}$$

where $\Delta x^{\nu} = x^{\nu} - x^{\nu}_{0}$, and $\omega_{\nu}(\Delta x)$ are infinitesimal values of the same order as $\rho = \sqrt{\left(\Delta x^{1}\right)^{2} + \cdots + \left(\Delta x^{N_0}\right)^{2}}$. The summation in (4.5) is performed over the index ν from 1 to N_0.

Using equalities (3.18), (3.27) and (3.29), we can rewrite (4.4) in the form

$$\Delta_{\dot{\varkappa}} f(x) = \left[\frac{\partial_{\dot{\varkappa}} f(x_0)}{\partial_{\dot{\varkappa}} X^{\nu}} \dot{\oplus} \omega_{\dot{\varkappa}\nu}\right] \dot{\otimes} \Delta_{\dot{\varkappa}} X^{\nu}, \tag{4.6}$$

where the $\dot{\oplus}$-summation (3.18) is performed over the identical indexes ν from 1 to N_0.

In (4.6) we introduce the notation

$$\frac{\partial_{\dot{\varkappa}} f(x_0)}{\partial_{\dot{\varkappa}} X^\nu} = \dot{\varkappa}_\Omega \left(..., \frac{\partial g_\alpha (x_0)}{\partial x^\nu}, ... \right) \tag{4.7}$$

and

$$\Delta_{\dot{\varkappa}} X^\nu = \dot{\varkappa}_\Omega (..., \Delta x^\nu, ...), \tag{4.8}$$

$$\omega_{\dot{\varkappa}\nu} = \dot{\varkappa}_\Omega (..., \omega_\nu (\Delta x), ...).$$

For $\rho \to 0$ we have $\omega_\nu (\Delta x) \to 0$. Hence, in view of (4.8) and (3.5), we obtain

$$\omega_{\dot{\varkappa}\nu} \to \dot{\varkappa}_\Omega (..., e_0, ...) = e \tag{4.9}$$

as $\rho \to 0$.

We assume that all $\Delta x^\nu = 0$ except for one $\Delta x^\sigma \neq 0$. Then $\Delta_{\dot{\varkappa}} X^\nu = e$ if $\nu \neq \sigma$. By (3.30), equality (4.6) implies

$$\Delta_{\dot{\varkappa}} f(x) = \left[\frac{\partial_{\dot{\varkappa}} f(x_0)}{\partial_{\dot{\varkappa}} X^\sigma} \dot{\oplus} \omega_{\dot{\varkappa}\sigma} \right] \dot{\otimes} \Delta_{\dot{\varkappa}} X^\sigma.$$

Dividing this equality by $\Delta_{\dot{\varkappa}} X^\sigma$ in the framework of the $\dot{\otimes}$-operation and taking into account (3.34), (3.36) and (3.38), we find

$$(\dot{\varkappa}) \frac{\Delta_{\dot{\varkappa}} f(x)}{\Delta_{\dot{\varkappa}} X^\sigma} = \frac{\partial_{\dot{\varkappa}} f(x_0)}{\partial_{\dot{\varkappa}} X^\sigma} \dot{\oplus} \omega_{\dot{\varkappa}\sigma}.$$

Using (4.9), for $\Delta x^\sigma \to 0$, we obtain

$$\lim_{\Delta x^\sigma \to 0} (\dot{\varkappa}) \frac{\Delta_{\dot{\varkappa}} f(x)}{\Delta_{\dot{\varkappa}} X^\sigma} = \frac{\partial_{\dot{\varkappa}} f(x_0)}{\partial_{\dot{\varkappa}} X^\sigma}. \tag{4.10}$$

We call (4.10) a $\dot{\varkappa}$-derivative of a function $f(x) \in B_N^{N_0}$ along X^σ.

(3) Let us establish some properties of a $\dot{\varkappa}$-derivative.

a) Assume that $c \in B_N^{N_0}$ is a constant function written in form (3.22). Then (4.7) implies

$$\frac{\partial_{\dot{\varkappa}} c}{\partial_{\dot{\varkappa}} X^\nu} = \dot{\varkappa}_\Omega \left(..., \frac{\partial c_\alpha}{\partial x^\nu}, ... \right) = \dot{\varkappa}_\Omega (..., e_0, ...) = e.$$

The other arithmetic properties of $\dot{\varkappa}$-derivatives are proved analogously:

b)

$$\frac{\partial_{\dot{\varkappa}}}{\partial_{\dot{\varkappa}} X^\nu} (f_1 \dot{\oplus} f_2) = \frac{\partial_{\dot{\varkappa}} f_1}{\partial_{\dot{\varkappa}} X^\nu} \dot{\oplus} \frac{\partial_{\dot{\varkappa}} f_2}{\partial_{\dot{\varkappa}} X^\nu},$$

c)

$$\frac{\partial_{\dot{\varkappa}}}{\partial_{\dot{\varkappa}} X^\nu} (f_1 \dot{\otimes} f_2) = \frac{\partial_{\dot{\varkappa}} f_1}{\partial_{\dot{\varkappa}} X^\nu} \dot{\otimes} f_2 \dot{\oplus} f_1 \dot{\otimes} \frac{\partial_{\dot{\varkappa}} f_2}{\partial_{\dot{\varkappa}} X^\nu},$$

d)

$$\frac{\partial_{\dot{\varkappa}}}{\partial_{\dot{\varkappa}} X^\nu} \left((\dot{\varkappa}) \frac{f_1}{f_2} \right) = (\dot{\varkappa}) \frac{\left[\frac{\partial_{\dot{\varkappa}} f_1}{\partial_{\dot{\varkappa}} X^\nu} \dot{\otimes} f_2 \dot{\ominus} f_1 \dot{\otimes} \frac{\partial_{\dot{\varkappa}} f_2}{\partial_{\dot{\varkappa}} X^\nu} \right]}{f_2 \dot{\otimes} f_2},$$

where $f_1,\ f_2 \in B_N^{N_0}$.

(4) Let us consider a solution of equations (1.1) which is written in form (3.20), where w_α

is a general solution of (3.1). Let us define a $\dot{\varkappa}$-derivative of (3.20). By (4.7) we find

$$\frac{\partial_{\dot{\varkappa}} u}{\partial_{\dot{\varkappa}} X^{\nu}} = \dot{\varkappa}_{\Omega}\left(..., \frac{\partial w_{\alpha}}{\partial x^{\nu}}, ...\right). \tag{4.11}$$

It is obvious that

$$\frac{\partial w_{\alpha}}{\partial x^{\nu}} = \frac{1}{l_{\alpha}}\alpha_{\nu}\frac{dw_{\alpha}}{dz_{\alpha}}.$$

From (3.1) we find

$$\frac{\partial w_{\alpha}}{\partial x^{\nu}} = \frac{1}{l_{\alpha}}\alpha_{\nu}w_{\alpha}.$$

Then (4.11) takes the form

$$\frac{\partial_{\dot{\varkappa}} u}{\partial_{\dot{\varkappa}} X^{\nu}} = \dot{\varkappa}_{\Omega}\left(..., \frac{1}{l_{\alpha}}\alpha_{\nu}w_{\alpha}, ...\right). \tag{4.12}$$

Recall that $l_{\alpha} = \alpha_{\nu}l^{\nu}$. We introduce

$$L^{\nu} = \dot{\varkappa}_{\Omega}\left(..., l^{\nu}, ...\right). \tag{4.13}$$

In the right-hand part of this equality we introduce an N-vector $(l^{\nu}, ..., l^{\nu}) \in B_{N\alpha}$. We multiply equalities (4.12) and (4.13) according to rule (3.27) and then perform the $\dot{\oplus}$-summation over the index ν from 1 to N_0. This results in

$$L^{\nu}\dot{\otimes}\frac{\partial_{\dot{\varkappa}} u}{\partial_{\dot{\varkappa}} X^{\nu}} = u. \tag{4.14}$$

Thus, in the $\dot{\varkappa}$-representation we see that equation (1.1) transforms to a linear equation in the algebraic field from Section *3*.

(5) Let us consider $\tilde{f}(x) \in B_N^{N_0}$ in $\ddot{\varkappa}$-representation (3.26). We form an alternative increment of the function $\tilde{f}(x)$

$$\hat{\Delta}_{\ddot{\varkappa}}\tilde{f}(x) = \tilde{f}(x)\ddot{\ominus}\tilde{f}(x_0) = \ddot{\varkappa}_{\Omega}\left(..., g_{\alpha}(x)\ddot{-}g_{\alpha}(x_0), ...\right). \tag{4.15}$$

By the same calculations as in the preceding subsections we obtain an alternative derivative in $B_N^{N_0}$

$$\lim_{x\to x_0}(\ddot{\varkappa})\,\frac{\hat{\Delta}_{\ddot{\varkappa}}\tilde{f}(x)}{\hat{\Delta}_{\ddot{\varkappa}}\tilde{X}^{\sigma}} = \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}(x_0)}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\sigma}}, \tag{4.16}$$

where

$$\hat{\Delta}_{\ddot{\varkappa}}\tilde{X}^{\sigma} = \ddot{\varkappa}_{\Omega}\left(..., \hat{\Delta}x^{\sigma}, ...\right), \tag{4.17}$$

$$\hat{\Delta}x^{\sigma} = x^{\sigma}\ddot{-}x_0^{\sigma}.$$

It is not difficult to verify that

$$\frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}(x)}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\sigma}} = \ddot{\varkappa}_{\Omega}\left(..., \frac{\hat{\partial}g_{\alpha}(x)}{\hat{\partial}x^{\sigma}}, ...\right) \tag{4.18}$$

is fulfilled, where

$$\frac{\hat{\partial}g_{\alpha}(x)}{\hat{\partial}x^{\sigma}}$$

is the alternative derivative studied in [2].

By analogy with (3) it is easy to show the validity of the following equalities:

a)

$$\frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{c}}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\sigma}} = h,$$

where $\tilde{c}$ is an arbitrary constant element of the manifold $B_N^{N_0}$:

$$\tilde{c} = \ddot{\varkappa}_{\Omega}\left(..., c_{\alpha}, ...\right).$$

b)

$$\frac{\hat{\partial}_{\ddot{\varkappa}}}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}}\left(\tilde{f}_1 \ddot{\oplus} \tilde{f}_2\right) = \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_1}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}} \ddot{\oplus} \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_2}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}},$$

c)

$$\frac{\hat{\partial}_{\ddot{\varkappa}}}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}}\left(\tilde{f}_1 \ddot{\otimes} \tilde{f}_2\right) = \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_1}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}} \ddot{\otimes} \tilde{f}_2 \ddot{\oplus} \tilde{f}_1 \ddot{\otimes} \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_2}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}},$$

d)

$$\frac{\hat{\partial}_{\ddot{\varkappa}}}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}}\left((\ddot{\varkappa})\,\frac{\tilde{f}_1}{\tilde{f}_2}\right) = (\ddot{\varkappa})\,\frac{\left[\frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_1}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}} \ddot{\otimes} \tilde{f}_2 \ddot{\ominus} \tilde{f}_1 \ddot{\otimes} \frac{\hat{\partial}_{\ddot{\varkappa}}\tilde{f}_2}{\hat{\partial}_{\ddot{\varkappa}}\tilde{X}^{\nu}}\right]}{\tilde{f}_2 \ddot{\otimes} \tilde{f}_2}.$$

(6) We call the first term in the right-hand part of (4.6) a total differential. Now we can write

$$d_{\dot{\varkappa}} f(x) = \frac{\partial_{\dot{\varkappa}} f(x)}{\partial_{\dot{\varkappa}} X^{\nu}} \dot{\otimes} d_{\dot{\varkappa}} X^{\nu}, \tag{4.19}$$

where the $\dot{\oplus}$-summation is performed over the identical indexes ν. From (4.8) follows

$$d_{\dot{\varkappa}} X^{\nu} = \dot{\varkappa}_{\Omega}\left(..., dx^{\nu}, ...\right). \tag{4.20}$$

By virtue of (4.4–5) a total differential $d_{\dot{\varkappa}} f(x)$ can be represented as

$$d_{\dot{\varkappa}} f(x) = \dot{\varkappa}_{\Omega}\left(..., dg_{\alpha}(x), ...\right) \tag{4.21}$$

where $dg_{\alpha}(x)$ is defined in a standard manner as

$$dg_{\alpha}(x) = \frac{\partial g_{\alpha}(x)}{\partial x^{\nu}} dx^{\nu} \tag{4.22}$$

Equality (4.19) can, in principle, be obtained from (4.21–22) by using (3.18) and (3.27). Quite analogously, we define the alternative differentials

$$\hat{d}_{\ddot{\varkappa}} \tilde{f}(x) = \frac{\hat{\partial}_{\ddot{\varkappa}} \hat{f}(x)}{\hat{\partial}_{\ddot{\varkappa}} \tilde{X}^{\nu}} \ddot{\otimes} \hat{d}_{\ddot{\varkappa}} \tilde{X}^{\nu}, \tag{4.23}$$

where the $\ddot{\oplus}$-summation is performed over the indexes ν and

$$\hat{d}_{\ddot{\varkappa}} \tilde{X}^{\nu} = \ddot{\varkappa}_{\Omega}\left(..., \hat{d}x^{\nu}, ...\right). \tag{4.24}$$

(4.23) can be written in the expanded form

$$\hat{d}_{\ddot{\varkappa}} \tilde{f}(x) = \ddot{\varkappa}_{\Omega}\left(..., \hat{d}g_{\alpha}(x), ...\right), \tag{4.25}$$

where

$$\hat{d}g_{\alpha}(x) = \frac{\hat{\partial}g_{\alpha}(x)}{\hat{\partial}x^{\hat{\nu}}}\hat{d}x^{\hat{\nu}}. \tag{4.26}$$

In (4.26), the alternative summation is performed over the indexes ν [2].

(7) It is not difficult to write the differential calculus in the χ -representation. This is done by means of equalities (3.2) which relate the χ- and $\varkappa$-mappings to each other. According to (9) of Section *3*, $\dot{\chi}_{\Omega}$ and $\ddot{\chi}_{\Omega}$ realize the mapping of the trivial fiber manifold $P(\Gamma_{N_0},\ Q_{N\alpha},\ \pi)$ into the manifold $B_N^{N_0}$. Then

$$\exp\varphi_{\alpha} : Q_{N\alpha} \to B_{N\alpha}.$$

Let $q_{\alpha}(x) \in Q_{N\alpha}$, $g_{\alpha}(x) \in B_{N\alpha}$ which are related by

$$g_{\alpha}(x) = \exp\varphi_{\alpha}(q_{\alpha}(x)). \tag{4.27}$$

Using (7.7) [2], we can write the φ_{α}-derivative of the function $q_{\alpha}(x)$ as

$$\frac{\partial_{\varphi_{\alpha}}q_{\alpha}(x)}{\partial_{\varphi_{\alpha}}X_{\alpha}^{\sigma}} = \varphi_{\alpha}^{-1}\left(\ln\frac{\partial g_{\alpha}(x)}{\partial x^{\sigma}}\right), \tag{4.28}$$

where X_{α}^{σ} is (4.1). Analogously, for the alternative φ_{α}-derivative from (7.11) [2] we find

$$\frac{\hat{\partial}_{\varphi_{\alpha}}q_{\alpha}(x)}{\hat{\partial}_{\varphi_{\alpha}}X_{\alpha}^{\sigma}} = \varphi_{\alpha}^{-1}\left(\ln\frac{\hat{\partial}g_{\alpha}(x)}{\hat{\partial}x^{\sigma}}\right), \tag{4.29}$$

Substituting $\partial g_{\alpha}(x)/\partial x^{\sigma}$ from (4.28) into (4.7), by equality (3.2) we have

$$\frac{\partial_{\dot{\varkappa}}f(x)}{\partial_{\dot{\varkappa}}X^{\sigma}} = \dot{\chi}_{\Omega}\left(\ldots,\frac{\partial_{\varphi_{\alpha}}q_{\alpha}(x)}{\partial_{\varphi_{\alpha}}X_{\alpha}^{\sigma}},\ldots\right) \tag{4.30}$$

Since the right-hand side of this equality is $\frac{\partial_{\dot{\chi}}f(x)}{\partial_{\dot{\chi}}X^{\sigma}}$, we can write

$$\frac{\partial_{\dot{\varkappa}}f(x)}{\partial_{\dot{\varkappa}}X^{\sigma}} = \frac{\partial_{\dot{\chi}}f(x)}{\partial_{\dot{\chi}}X^{\sigma}}. \tag{4.31}$$

For the alternative χ-derivative we have

$$\frac{\hat{\partial}_{\ddot{\chi}}\tilde{f}(x)}{\hat{\partial}_{\ddot{\chi}}\tilde{X}^{\sigma}} = \ddot{\chi}_{\Omega}\left(\ldots,\frac{\hat{\partial}_{\varphi_{\alpha}}q_{\alpha}(x)}{\hat{\partial}_{\varphi_{\alpha}}X_{\alpha}^{\sigma}},\ldots\right). \tag{4.32}$$

Thus we have established the basic aspects of the differential calculus generated by equation (1.1).

5. Algebraic conjugation

As has been repeatedly said, the double numerical field consists of two fields with mutually alternative operations of addition. The transition from one field to the other is called the algebraic conjugation.

(1) Let us return to the alternative algebra of matrices [2]. Suppose we have the square N-matrices A, B and C with matrix elements A_n^k, B_n^k and C_n^k. Then the matrix equality $C = AB$ can be written in the form

$$C_n^k = A_m^k B_n^m \tag{5.1}$$

where the summation is performed over the index m from 1 to N.

Let us write the a-conjugate (the algebraic conjugation introduced in [2]) matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$. This means that the matrix elements of these matrices are related to the matrix elements A, B and C through the equalities

$$\hat{A}^k_n = 1/A^k_n, \quad \hat{B}^k_n = 1/B^k_n, \quad \hat{C}^k_n = 1/C^k_n. \tag{5.2}$$

Using (5.2), from (5.1) we obtain

$$\hat{C}^k_n = \hat{A}^k_{\hat{m}} \hat{B}^{\hat{m}}_n, \tag{5.3}$$

where the alternative summation is performed over the index m. In other words, we have the equality

$$(AB)\hat{} = \hat{A}\hat{\cdot}\hat{B},$$

where $\hat{\cdot}$ stands for an alternative multiplication of matrices [2]. In particular, from these arguments it follows that if A and B are reciprocal matrices, then the a-conjugate matrixes $\hat{A}$ and $\hat{B}$ are alternative reciprocal matrices:

$$\hat{A}\hat{\cdot}\hat{B} = \hat{1}.$$

Recall that the matrix elements $\hat{\delta}^k_n$ of an a -conjugate matrix $\hat{1}$ have the form [2]

$$\hat{\delta}^k_n = \begin{cases} 1, & k = i, \\ \infty, & k \neq i \ . \end{cases}$$

(2) Let us consider a complex number

$$z = x\dot{+}iy, \tag{5.4}$$

where $x\dot{+}iy = x + iy$; and x, y are real numbers. Let us calculate an a-conjugate number $\hat{z}$

$$\hat{z} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{\frac{1}{\hat{x}} + i\frac{1}{\hat{y}}} = \hat{x}\ddot{-}i\hat{y},$$

where $x\hat{x} = 1$, $y\hat{y} = 1$. Thus, an a-conjugate complex number to (5.4) has the form

$$\hat{z} = \hat{x}\ddot{-}i\hat{y}. \tag{5.5}$$

(3) Let $\hat{e}^{\hat{z}}$ be the a-conjugate function to the function e^z, i.e. both functions satisfy the equality

$$\hat{e}^{\hat{z}} \cdot e^z = 1,$$

where $z\hat{z} = 1$. We can also write

$$\hat{e}^{\hat{z}} = 1/e^{\frac{1}{\hat{z}}}.$$

It is obvious that for $z = 1$ we have $\hat{e} = 1/e$.

Let us calculate

$$\hat{e}^{\hat{z}_1 \ddot{+} \hat{z}_2} = \frac{1}{e^{\frac{1}{\hat{z}_1 \ddot{+} \hat{z}_2}}} = \frac{1}{e^{z_1+z_2}} = \frac{1}{e^{z_1}} \cdot \frac{1}{e^{z_2}} = \hat{e}^{\hat{z}_1} \cdot \hat{e}^{\hat{z}_2}.$$

Finally, we have

$$\hat{e}^{\hat{z}_1 \ddot{+} \hat{z}_2} = \hat{e}^{\hat{z}_1} \cdot \hat{e}^{\hat{z}_2}. \tag{5.6}$$

(4) If $\hat{\ln}\hat{z}$ and $\ln z$ are a-conjugate functions, then

$$\hat{\ln}\hat{z} = \frac{1}{\ln \frac{1}{\hat{z}}} = \frac{1}{\ln z}.$$

Let us calculate

$$\hat{\ln}\left(\hat{z}_1\hat{z}_2\right) = \frac{1}{\ln\left(z_1 z_2\right)} = \frac{1}{\ln z_1 + \ln z_2} = \frac{1}{\frac{1}{\hat{\ln}\hat{z}_1} + \frac{1}{\hat{\ln}\hat{z}_2}} = \hat{\ln}\hat{z}_1 \ddot{+} \hat{\ln}\hat{z}_2,$$

i.e. we have

$$\hat{\ln}\hat{z}_1\hat{z}_2 = \hat{\ln}\hat{z}_1 \ddot{+} \hat{\ln}\hat{z}_2. \qquad (5.7)$$

(5) Let $w(t)$ and $\hat{w}\left(\hat{t}\right)$ be a -conjugate functions, i.e. they satisfy the equalities

$$\hat{w}^k\left(\hat{t}\right) w^k(t) = 1, \qquad (5.8)$$
$$(k = 1, ..., N)$$

where

$$\hat{t}t = 1. \qquad (5.9)$$

Consider an alternative derivative of the function $\hat{w}\left(\hat{t}\right)$ [2]. It is shown in [2] that the following equalities are valid:

$$\frac{dw^k(t)}{dt}\frac{\hat{d}\hat{w}^k\left(\hat{t}\right)}{\hat{d}\hat{t}} = 1, \qquad (5.10)$$
$$(k = 1, ..., N).$$

Let $w^k(t)$ be a solution of equation (2.11). Then (2.11), (5.8) and (5.10) imply

$$\frac{\hat{d}\hat{w}^k\left(\hat{t}\right)}{\hat{d}\hat{t}} = \hat{w}^k\left(\hat{t}\right). \qquad (5.11)$$

Note that (5.11) is not equation (2.11) rewritten in terms of alternative derivatives. To verify that this is so, we differentiate, in a standard manner, equality (5.8) with respect to t

$$\frac{dw^k(t)}{dt}\hat{w}^k + w^k\frac{d\hat{w}^k(t)}{dt} = 0.$$

By (2.11) we have

$$\frac{d\hat{w}^k}{dt} = -\hat{w}^k. \qquad (5.12)$$

As we have seen in [1-2], if $w_1(t)$ and $w_2(t)$ are solutions of equation (2.11), then so are

$$w = w_1 \dot{+} w_2 = w_1 + w_2, \qquad (5.13)$$

$$\tilde{w} = w_1 \ddot{+} w_2 = 1/\left(\frac{1}{w_1} + \frac{1}{w_2}\right).$$

For simplicity, we have omitted the index k for $w(t)$. Let us now clarify what algebra equation

(5.11) has. Since $\hat{w}\left(\hat{t}\right) = 1/w\left(t\right)$ is a solution of (5.11), we can write

$$\hat{w}\left(\hat{t}\right) = \frac{1}{w\left(t\right)} = \frac{1}{w_1\left(t\right) + w_2\left(t\right)} = \left[\frac{1}{\hat{w}_1\left(\hat{t}\right)} + \frac{1}{\hat{w}_2\left(\hat{t}\right)}\right]^{-1} = \hat{w}_1\left(\hat{t}\right) \ddot{+} \hat{w}_2\left(\hat{t}\right),$$

i.e.

$$\hat{w}\left(\hat{t}\right) = \hat{w}_1\left(\hat{t}\right) \ddot{+} \hat{w}_2\left(\hat{t}\right), \tag{5.14}$$

where $\hat{w}_1\left(\hat{t}\right)$, $\hat{w}_2\left(\hat{t}\right)$ are solutions of (5.11) and a-conjugate functions of $w_1\left(t\right)$ and $w_2\left(t\right)$, respectively. Analogously,

$$\hat{\tilde{w}}\left(\hat{t}\right) = \frac{1}{\tilde{w}\left(t\right)} = \frac{1}{w_1\left(t\right) \ddot{+} w_2\left(t\right)} = \frac{1}{w_1\left(t\right)} + \frac{1}{w_2\left(t\right)} = \hat{w}_1\left(\hat{t}\right) \dot{+} \hat{w}_2\left(\hat{t}\right),$$

i.e.

$$\hat{\tilde{w}}\left(\hat{t}\right) = \hat{w}_1\left(\hat{t}\right) \dot{+} \hat{w}_2\left(\hat{t}\right) \tag{5.15}$$

The neutral elements (5.13) are also neutral elements for (5.14–15). At that, we have

$$\hat{0} = \infty, \ \hat{\infty} = 0.$$

Finally, we conclude that between the algebras of equations (2.11) and (5.11) there exist the following conjugate relations:

$$\begin{gathered}
\left(w_1\left(t\right) \dot{+} w_2\left(t\right)\right)^{\hat{}} = \hat{w}_1\left(\hat{t}\right) \ddot{+} \hat{w}_2\left(\hat{t}\right), \\
\left(w_1\left(t\right) \ddot{+} w_2\left(t\right)\right)^{\hat{}} = \hat{w}_1\left(\hat{t}\right) \dot{+} \hat{w}_2\left(\hat{t}\right), \\
\left(w\left(t\right)\right)^{\hat{}} = \left(w\left(t\right) \dot{+} 0\right)^{\hat{}} = \hat{w}\left(\hat{t}\right) \ddot{+} \hat{0} = \hat{w}\left(\hat{t}\right) \ddot{+} \infty = \hat{w}\left(\hat{t}\right), \\
\left(\infty\right)^{\hat{}} = \left(w\left(t\right) \dot{+} \infty\right)^{\hat{}} = \hat{w}\left(\hat{t}\right) \ddot{+} \hat{\infty} = \hat{w}\left(\hat{t}\right) \ddot{+} 0 = 0.
\end{gathered} \tag{5.16}$$

(5.13) and (5.14-15) clearly show that the algebras generated by the equations (2.11) and (5.11) coincide. In the sequel, (2.11) and (5.11) will be called a-conjugate equations.

(6) Let us consider equation (2.1). Let $u\left(t\right)$ be its solution. The a-conjugate function $\hat{u}\left(\hat{t}\right)$ is related to $u\left(t\right)$ by the equality

$$\hat{u}^k\left(\hat{t}\right) u^k\left(t\right) = 1, \quad \left(k = 1, ..., N\right), \tag{5.17}$$

where $\hat{t}t = 1$. Then we have [2]

$$\frac{du^k\left(t\right)}{dt} \frac{\hat{d}\hat{u}^k\left(\hat{t}\right)}{\hat{d}\hat{t}} = 1, \quad \left(k = 1, ..., N\right). \tag{5.18}$$

From (2.1) and (5.17-18) we easily obtain

$$\frac{\hat{d}\hat{u}^k\left(\hat{t}\right)}{\hat{d}\hat{t}} = \hat{F}^k\left(\hat{u}\left(\hat{t}\right)\right), \tag{5.19}$$

where

$$\hat{F}^k(\hat{u})\, F^k(u) = 1. \tag{5.20}$$

In the sequel, (2.1) and (5.19) will be called a-conjugate equations.

As has been stated in [1], an equation of characteristic functions $\varphi^k(u)$ for (2.1) has the form

$$\frac{\partial \varphi^k(u)}{\partial u^n} F^n(u) = b^k. \tag{5.21}$$

Following §2 [2], we assume $b^k = 1$, $k = 1, ..., N$. Then an equation of characteristic functions for (5.19) takes the form

$$\frac{\hat{\partial} \hat{\varphi}^k(\hat{u})}{\hat{\partial} \hat{u}^{\hat{n}}} \hat{F}^{\hat{n}}(\hat{u}) = b^k, \tag{5.22}$$

where the alternative summation is performed over the index n from 1 to N. Since (5.21) and (5.22) are obviously a-conjugate equations, the following equalities are valid:

$$\hat{\varphi}^k(\hat{u})\, \varphi^k(u) = 1, \tag{5.23}$$
$$(k = 1, ..., N)$$

(7) Keeping in mind that (2.1) and (5.19) are a-conjugate equations, let us investigate how the algebras generated by them are related to each other.

If u_1, u_2 are solutions of equation (2.1), then by (2.7) and (5.17) we can write

$$\hat{u}\left(\hat{t}\right) = \frac{1}{u(t)} = 1/\left(u_1 \underset{\varphi}{\dot{+}} u_2\right) = 1/\varphi^{-1}\left[\ln\left(e^{\varphi(u_1)} \dot{+} e^{\varphi(u_2)}\right)\right].$$

Since (5.23) is fulfilled, we have

$$\hat{u}\left(\hat{t}\right) = 1/\varphi^{-1}\left[\ln\left(e^{\frac{1}{\hat{\varphi}(\hat{u}_1(\hat{t}))}} \dot{+} e^{\frac{1}{\hat{\varphi}(\hat{u}_2(\hat{t}))}}\right)\right]. \tag{5.24}$$

Using the results of (3) and (4) of this section, we find

$$\ln\left[e^{\frac{1}{\hat{\varphi}(\hat{u}_1)}} \dot{+} e^{\frac{1}{\hat{\varphi}(\hat{u}_2)}}\right] = \ln\left[\frac{1}{\hat{e}^{\hat{\varphi}(\hat{u}_1)}} \dot{+} \frac{1}{\hat{e}^{\hat{\varphi}(\hat{u}_2)}}\right]$$
$$= \ln\frac{1}{\hat{e}^{\hat{\varphi}(\hat{u}_1)} \ddot{+} \hat{e}^{\hat{\varphi}(\hat{u}_2)}} = 1/\hat{\ln}\left(\hat{e}^{\hat{\varphi}(\hat{u}_1)} \ddot{+} \hat{e}^{\hat{\varphi}(\hat{u}_2)}\right).$$

The a-conjugation of φ^{-1} and $\hat{\varphi}^{-1}$ immediately follows from (5.17) and (5.23). Then by virtue of (5.24) we can write

$$\hat{u}\left(\hat{t}\right) = \hat{\varphi}^{-1}\left[\hat{\ln}\left(\hat{\exp}\hat{\varphi}(\hat{u}_1) + \hat{\exp}\hat{\varphi}(\hat{u}_2)\right)\right] = \hat{u}_1\left(\hat{t}\right) \underset{\hat{\varphi}}{\ddot{+}} \hat{u}_2\left(\hat{t}\right).$$

Finally, we have

$$\left(u_1(t) \underset{\varphi}{\dot{+}} u_2(t)\right)^{\hat{}} = \hat{u}_1\left(\hat{t}\right) \underset{\hat{\varphi}}{\ddot{+}} \hat{u}_2\left(\hat{t}\right). \tag{5.25}$$

It can be shown analogously that

$$\left(u_1(t) \underset{\varphi}{\ddot{+}} u_2(t)\right)^{\hat{}} = \hat{u}_1\left(\hat{t}\right) \underset{\hat{\varphi}}{\dot{+}} \hat{u}_2\left(\hat{t}\right). \tag{5.26}$$

For the neutral elements we find

$$(e)\hat{} = \hat{h},$$
$$(h)\hat{} = \hat{e}, \tag{5.27}$$

where the components of the elements e, h, $\hat{e}$, $\hat{h}$ satisfy the equalities

$$\hat{e}^k e^k = 1,$$

$$\hat{h}^k h^k = 1.$$

As has been shown in [2], a φ-product of the elements u_1, $u_2 \in J_N^1$ has the form

$$u_1 \underset{\varphi}{\odot} u_2 = \varphi^{-1}\left(\varphi\left(u_1\right) \dot{+} \varphi\left(u_2\right)\right). \tag{5.28}$$

It is not difficult to show that equation (5.19) gives rise to the $\hat{\varphi}$-product

$$\hat{u}_1 \underset{\hat{\varphi}}{\hat{\odot}} \hat{u}_2 = \hat{\varphi}^{-1}\left(\hat{\varphi}\left(\hat{u}_1\right) \ddot{+} \hat{\varphi}\left(\hat{u}_2\right)\right). \tag{5.29}$$

(5.28) and (5.29) readily imply

$$\left(u_1 \underset{\varphi}{\odot} u_2\right)\hat{} = \hat{u}_1 \underset{\hat{\varphi}}{\hat{\odot}} \hat{u}_2. \tag{5.30}$$

The unit elements E and $\hat{E}$ of products (5.28) and (5.29) are a-conjugate:

$$\hat{E}^k E^k = 1 \tag{5.31}$$

Thus we have established that the double fields formed by equations (2.1) and (5.19) are a-conjugate, i.e. they satisfy equations (5.25–31).

(8) Let us return to equation (1.1) and write its a-conjugate equation

$$\hat{a}_{\hat{n}}^{\hat{\nu}k}\left(\hat{u}\right) \frac{\hat{\partial} \hat{u}^{\hat{n}}}{\hat{\partial} \hat{x}^{\hat{\nu}}} = \hat{F}^k\left(\hat{u}\right), \tag{5.32}$$
$$\left(k = 1, \ldots, N\right),$$

where the alternative summation is performed over the indexes ν and n from 1 to N_0 and N, respectively. Hence the equalities

$$\hat{x}^\nu x^\nu = 1,$$
$$\hat{u}^k\left(\hat{x}\right) u^k\left(x\right) = 1,$$
$$\hat{a}_n^{\nu k}\left(\hat{u}\right) a_n^{\nu k}\left(u\right) = 1, \tag{5.33}$$
$$\hat{F}^k\left(\hat{u}\right) F^k\left(u\right) = 1$$
$$\frac{\hat{\partial} \hat{u}^k}{\hat{\partial} \hat{x}^\nu} \frac{\partial u^k}{\partial x^\nu} = 1$$

are fulfilled.

(9) In Section *1* we have introduced $z_\alpha = 1/l_\alpha \alpha_\nu x^\nu$. The equality $\hat{z}_\alpha = 1/z_\alpha$ readily implies

$$\hat{z}_{\hat{\alpha}} = \frac{1}{\hat{l}_{\hat{\alpha}}} \hat{\alpha}_{\hat{\nu}} \hat{x}^{\hat{\nu}},$$

where $\hat{\alpha}_\nu = 1/\alpha_\nu$, and the alternative summation is performed over the index ν. It is obvious that $\hat{l}_\alpha = \hat{\alpha}_{\hat{\nu}}\hat{l}^{\hat{\nu}}$, $\hat{l}^\nu = 1/l^\nu$.

By analogy with Section *1*, a solution of equation (5.32) is to be sought in terms of plane waves

$$\hat{u}_{\hat{\alpha}} = \hat{u}_{\hat{\alpha}}\left(\hat{z}_{\hat{\alpha}}\right). \tag{5.34}$$

The substitution of (5.34) into (5.32) gives the equation of alternative plane waves

$$\hat{a}_{\hat{\alpha}}\left(\hat{u}_{\hat{\alpha}}\right)\hat{\cdot}\frac{\hat{d}\hat{u}_{\hat{\alpha}}}{\hat{d}\hat{z}_{\hat{\alpha}}} = \hat{F}\left(\hat{u}_{\hat{\alpha}}\right). \tag{5.35}$$

The matrix $\hat{a}_{\hat{\alpha}}$ has the form

$$\hat{a}_{\hat{\alpha}} = \frac{1}{\hat{l}_{\hat{\alpha}}}\hat{\alpha}_{\hat{\nu}}\hat{a}^{\hat{\nu}}\left(\hat{u}_{\hat{\alpha}}\right),$$

where the alternative summation is performed over the index ν.

A general solution of equation (5.35) can be written as

$$\hat{\varphi}^k_{\hat{\alpha}}\left(\hat{u}_{\hat{\alpha}}\right) = b^k\hat{z}_{\hat{\alpha}}\ddot{+}\hat{c}^k_{\hat{\alpha}},$$

where $\hat{\varphi}^k_{\hat{\alpha}}\left(\hat{u}_{\hat{\alpha}}\right)$ are the characteristic functions which are a-conjugate to $\varphi^k_\alpha\left(u_\alpha\right)$.

Omitting the details, in the case of (5.32) we can write (2.16) and (2.20) in the form

$$\sum_{\hat{\alpha}\in\hat{\Omega}} q_\alpha \hat{\exp}\left[\hat{\varphi}^k_{\hat{\alpha}}\left(\hat{\dot{\chi}}\right) - \hat{\varphi}^k_{\hat{\alpha}}\left(\hat{u}_{\hat{\alpha}}\right)\right] = 1,$$

$$\sum_{\hat{\alpha}\in\hat{\Omega}} q_\alpha \hat{\exp}\left[\hat{\varphi}^k_{\hat{\alpha}}\left(\hat{u}_{\hat{\alpha}}\right) - \hat{\varphi}^k_{\hat{\alpha}}\left(\hat{\ddot{\chi}}\right)\right] = 1, \tag{5.36}$$

$$(k = 1, \ldots, N).$$

Then the following equalities are valid:

$$\left(\dot{\chi}_\Omega\right)^{\hat{}} = \hat{\ddot{\chi}}_{\hat{\Omega}},\ \left(\ddot{\chi}_\Omega\right)^{\hat{}} = \hat{\dot{\chi}}_{\hat{\Omega}}.$$

At this stage of investigation there is no need to dwell on those results for (5.32) which have been obtained in the case of equations (1.1). We want only to mention the fact that the double algebraic fields formed by means of equations (1.1) and (5.32) are inter-related by the Λ-operation and they are isomorphic to each other .

(10) Let us consider equation (2.1). As stated in Section *2*, the space W^1_N of solutions of equation (2.11) is the base space of the discrete fiber space $J^1_N = J^1_N\left(W^1_N,\ \exp\varphi\right)$, where $\exp\varphi : J^1_N \to W^1_N$ is the projector. By analogy with Section *3*, we extend the base space W^1_N to the manifold B^1_N filled with all kinds of smooth N -vector functions $g\left(t\right)$. Then the space J^1_N of solutions of equation (2.1) extends to some manifold $\tilde{J}^1_N = J^1_N\left(B^1_N,\ \exp\varphi\right)$.

Definition. *We call functions* $f\left(t\right)$, $\overset{\varphi}{\hat{f}}\left(\hat{t}\right) \in \tilde{J}^1_N$ *φ-conjugate functions if the equalities*

$$\varphi^k\left(f\left(t\right)\right) + \varphi^k\left(\overset{\varphi}{\hat{f}}\left(\hat{t}\right)\right) = 0, \tag{5.37}$$

$$(k = 1, ..., N)$$

are fulfilled and $\hat{t}t = 1$.

As has been shown in [2], (5.37) immediately implies

$$f(t) \odot \overset{\varphi}{\hat{f}}(\hat{t}) = E,$$

$$\frac{d_\varphi f(t)}{d_\varphi \tau} \odot \frac{\hat{d}_\varphi \overset{\varphi}{\hat{f}}(\hat{t})}{\hat{d}_\varphi \hat{\tau}} = E, \tag{5.38}$$

$$\tau \odot \hat{\tau} = E,$$

where

$$\tau = \varphi^{-1}(\ln t, ..., \ln t), \quad \hat{\tau} = \varphi^{-1}\left(\ln \hat{t}, ..., \ln \hat{t}\right),$$

$$\varphi(E) = 0.$$

Since

$$\varphi_0^k = \ln w^k$$

are the characteristic functions for equation (2.11), from (5.38) we immediately obtain (5.8–10). This means that the a-conjugation is a particular case of φ-conjugation.

(11) Let $u(t) \in J_N^1$, then

$$w^k(t) = \exp \varphi^k(u(t)), \tag{5.39}$$

where $w(t) \in W_N^1$. Substituting (5.39) into (2.11) and reducing by $\exp \varphi^k(u)$, we obtain

$$\frac{\partial \varphi^k(u)}{\partial u^n} \frac{du^n}{dt} = b^k,$$

where the summation is performed over the index n and b^k is (1.51). From this equality we have

$$\frac{du^k}{dt} = \left(\frac{\partial \varphi(u)}{\partial u}\right)^{-1\ k}_{n} b^n. \tag{5.40}$$

Comparing (5.40) with (2.1) we conclude that

$$F(u) = \left(\frac{\partial \varphi(u)}{\partial u}\right)^{-1} b. \tag{5.41}$$

Note that for the function $F(u)$, (5.41) is an equation for characteristic functions $\varphi^k(u)$ [1].

Let $u(t)$ and $\overset{\varphi}{\hat{u}}(\hat{t})$ be the φ-conjugate functions, where $u \in J_N^1$ and $t\hat{t} = 1$. Then by virtue of (5.39) and (5.8) we obtain

$$\hat{w}^k(\hat{t}) = \exp \varphi^k\left(\overset{\varphi}{\hat{u}}(\hat{t})\right). \tag{5.42}$$

Substituting (5.42) into (5.11) and using the properties of alternative derivatives [2] we have

$$e^{\varphi^k\left(\overset{\varphi}{\hat{u}}\right)} \frac{1}{\left[\varphi^k\left(\overset{\varphi}{\hat{u}}\right)\right]^2} \frac{\hat{\partial}\varphi^k\left(\overset{\varphi}{\hat{u}}\right)}{\hat{\partial}\overset{\varphi}{\hat{u}}{}^{\hat{n}}} \frac{\hat{d}\overset{\varphi}{\hat{u}}{}^{\hat{n}}}{\hat{d}\hat{t}} = e^{\varphi^k\left(\overset{\varphi}{\hat{u}}\right)},$$

where the alternative summation is performed over the index n. The reduction of the obtained equality by $\exp \varphi^k$ gives

$$\frac{\hat{\partial} \tilde{\varphi}^k \left(\overset{\varphi}{\hat{u}}\right)}{\hat{\partial} \overset{\varphi}{\hat{u}}{}^{\hat{n}}} \frac{\hat{d} \overset{\varphi}{\hat{u}}{}^{\hat{n}}}{\hat{d}\hat{t}} = -b^k, \tag{5.43}$$

where

$$\tilde{\varphi}^k \left(\overset{\varphi}{\hat{u}}\right) = 1/\varphi^k \left(\overset{\varphi}{\hat{u}}\right). \tag{5.44}$$

Using the alternative matrix algebra [2], we have

$$\frac{\hat{d} \overset{\varphi}{\hat{u}}{}^k}{\hat{d}\hat{t}} = -\left(\frac{\hat{\partial} \tilde{\varphi} \left(\overset{\varphi}{\hat{u}}\right)}{\hat{\partial} \overset{\varphi}{\hat{u}}}\right)^{-\hat{1}\ k}_{\hat{n}} b^{\hat{n}}, \tag{5.45}$$

where $A^{-\hat{1}} \hat{\cdot} A = \hat{1}$. Let us introduce the notation

$$\tilde{F}^k \left(\overset{\varphi}{\hat{u}}\right) = -\left(\frac{\hat{\partial} \tilde{\varphi} \left(\overset{\varphi}{\hat{u}}\right)}{\hat{\partial} \overset{\varphi}{\hat{u}}}\right)^{-\hat{1}\ k}_{n} \hat{b}^n.$$

Then (5.45) takes the final form

$$\frac{\hat{d} \overset{\varphi}{\hat{u}}}{\hat{d}\hat{t}} = \tilde{F} \left(\overset{\varphi}{\hat{u}}\right). \tag{5.46}$$

(12) Let us establish a direct relationship between equations (2.1) and (5.46). To this end, we introduce the functions $v^k(t)$ satisfying the equalities:

$$\overset{\varphi}{\hat{u}}{}^k (\hat{t})\, v^k(t) = 1, \quad (k = 1, ..., N) \tag{5.47}$$

It immediately follows from (5.44) and (5.47) that the functions $\varphi(1/v)$ and $\tilde{\varphi}\left(\overset{\varphi}{\hat{u}}\right)$ are the a-conjugate functions [2]

$$\varphi^k \left(\frac{1}{v}\right) \tilde{\varphi}^k \left(\overset{\varphi}{\hat{u}}\right) = 1, \quad (k = 1, ..., N).$$

Since standard and alternative derivatives are inter-related [2], equality (5.43) can be rewritten as

$$\frac{\partial \varphi^k(y)}{\partial y^n}\Big|_{y=\frac{1}{v}} \left(-\frac{1}{(v^n)^2}\right) \frac{dv^n}{dt} = -b^k,$$

where the summation is performed over the index n. By (5.41) we readily find

$$\frac{dv^k}{dt} = \left(v^k\right)^2 F^k \left(\frac{1}{v}\right). \tag{5.48}$$

Since (5.47) is fulfilled, the final form of equation (5.48) in terms of alternative derivatives is

$$\frac{\hat{d}\overset{\varphi}{\hat{u}}{}^{k}}{\hat{d}\hat{t}} = \frac{\left(\overset{\varphi}{\hat{u}}{}^{k}\right)^{2}}{F^{k}\left(\overset{\varphi}{\hat{u}}\right)}. \tag{5.49}$$

Comparing (5.46) and (5.49), we can write the relation between $\tilde{F}$ and F as follows:

$$\tilde{F}^{k}\left(\overset{\varphi}{\hat{u}}\right) = \frac{\left(\overset{\varphi}{\hat{u}}{}^{k}\right)^{2}}{F^{k}\left(\overset{\varphi}{\hat{u}}\right)}. \tag{5.50}$$

(13) Let us show that the algebra generated by system (5.49) coincides with algebra (2.1). Indeed, for equation (2.1) we have (2.7–10). The φ-conjugate elements $u\,(t)$ and $\overset{\varphi}{\hat{u}}\left(\hat{t}\right)$ satisfy equations (5.37). Then we can write

$$\overset{\varphi}{\hat{u}}\left(\hat{t}\right) = \varphi^{-1}\left(-\varphi\left(u\left(t\right)\right)\right).$$

If $u\,(t)$ is a solution of equation (2.1), then $\overset{\varphi}{\hat{u}}\left(\hat{t}\right)$ is a solution of (5.49).

Let $u = u_1 \underset{\varphi}{\dot{+}} u_2$. The φ-conjugation immediately implies

$$\overset{\varphi}{\hat{u}} = \varphi^{-1}\left(-\varphi\left(u\right)\right) = \varphi^{-1}\left(-\varphi\left(u_1 \underset{\varphi}{\dot{+}} u_2\right)\right) = \varphi^{-1}\left(-\ln\left(\exp\varphi\left(u_1\right) + \exp\varphi\left(u_2\right)\right)\right).$$

By (5.37) we can write

$$\begin{aligned}\overset{\varphi}{\hat{u}} &= \varphi^{-1}\left(-\ln\left(\exp\left(-\varphi\left(\overset{\varphi}{\hat{u}}_1\right)\right) + \exp\left(-\varphi\left(\overset{\varphi}{\hat{u}}_2\right)\right)\right)\right) \\ &= \varphi^{-1}\left(\ln\left(\exp\varphi\left(\overset{\varphi}{\hat{u}}_1\right) \ddot{+} \exp\varphi\left(\overset{\varphi}{\hat{u}}_2\right)\right)\right) = \overset{\varphi}{\hat{u}}_1 \underset{\varphi}{\ddot{+}} \overset{\varphi}{\hat{u}}_2.\end{aligned}$$

This means the fulfillment of

$$\left(u_1 \underset{\varphi}{\dot{+}} u_2\right)^{\overset{\varphi}{\wedge}} = \overset{\varphi}{\hat{u}}_1 \underset{\varphi}{\ddot{+}} \overset{\varphi}{\hat{u}}_2. \tag{5.51}$$

Analogously, we can prove

$$\left(u_1 \underset{\varphi}{\ddot{+}} u_2\right)^{\overset{\varphi}{\wedge}} = \overset{\varphi}{\hat{u}}_1 \underset{\varphi}{\dot{+}} \overset{\varphi}{\hat{u}}_2. \tag{5.52}$$

From the equality $\varphi\left(\overset{\varphi}{\hat{u}}\right) = -\varphi\left(u\right)$ and (6.6) [2] we conclude that

$$\left(e\right)^{\overset{\varphi}{\wedge}} = h, \quad \left(h\right)^{\overset{\varphi}{\wedge}} = e. \tag{5.53}$$

Using the φ-product of $u_1 \underset{\varphi}{\odot} u_2 = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$, it is not difficult to prove

$$\left(u_1 \underset{\varphi}{\odot} u_2\right)^{\overset{\varphi}{\hat{}}} = \overset{\varphi}{\hat{u}}_1 \underset{\varphi}{\odot} \overset{\varphi}{\hat{u}}_2, \tag{5.54}$$

$$(E)^{\overset{\varphi}{\hat{}}} = E,$$

where E is the unit element of multiplication (recall that $\varphi(E) = 0$).

Comparing (2.7–10) and (5.51–54), we see that the algebraic fields of equations (2.1) and (5.49) coincide. Based on this fact, we call (2.1) and (5.49) the φ-conjugate equations. Note that in this sense equations (2.11) and (5.11) are also φ-conjugate.

(14) Let $u(t)$, $\overset{\varphi}{\hat{u}}(\hat{t})$ be solutions of equations (2.1) and (5.49), respectively. Since they are φ-conjugate, we have

$$\varphi(u(t)) + \varphi\left(\overset{\varphi}{\hat{u}}(\hat{t})\right) = 0. \tag{5.55}$$

Differentiating this equality with respect to t and using (5.40–41), in the standard calculus we obtain an equation for $\overset{\varphi}{\hat{u}}$

$$\frac{d\overset{\varphi}{\hat{u}}}{dt} = -F\left(\overset{\varphi}{\hat{u}}\right). \tag{5.56}$$

This equation is analogous to (5.12).

Let us return to equation (5.41) and rewrite it as

$$\frac{\partial\varphi(u)}{\partial u} F(u) = b \tag{5.57}$$

As has already been said, (5.57) is a differential equation for φ. In turn, $\varphi(u)$ are the characteristic functions for the given equation (2.1).

To obtain a differential equations for the characteristic functions of equation (5.56), it is necessary to make a replacement in (5.57):

$$F \to -F. \tag{5.58}$$

Then (5.57) becomes the equation

$$-\frac{\partial\psi}{\partial u} F(u) = b, \tag{5.59}$$

where ψ are the characteristic functions of equation (5.56). From (5.57) and (5.59) we can write

$$\psi = -\varphi. \tag{5.60}$$

Using equalities (5.60) and (5.55), the operation of φ-conjugation $\overset{\varphi}{\hat{}}$ for φ takes the form

$$(\varphi(u))^{\overset{\varphi}{\hat{}}} = -\varphi\left(\overset{\varphi}{\hat{u}}\right), \tag{5.61}$$

where u is a solution of equation (2.1), while $\overset{\varphi}{\hat{u}}$ is a solution of equation (5.56).

(15) Let us return to equation (5.49) and make a replacement

$$\overset{\varphi}{\hat{u}}{}^{k}\left(\hat{t}\right) = 1/\hat{v}^{k}\left(\hat{t}\right), \qquad (k = 1, ..., N). \tag{5.62}$$

As a result, (5.49) takes the form

$$\frac{\hat{d}\hat{v}^{k}\left(\hat{t}\right)}{\hat{d}\hat{t}} = -\hat{F}^{k}\left(\hat{v}\left(\hat{t}\right)\right), \tag{5.63}$$

where

$$\hat{F}^{k}\left(\hat{v}\left(\hat{t}\right)\right) = 1/F^{k}\left(\frac{1}{\hat{v}\left(\hat{t}\right)}\right),$$

which agrees with definition (5.20) of the function $\hat{F}$.

After carrying out the operation of a-conjugation in (5.63), we obtain

$$\frac{dv^{k}\left(t\right)}{dt} = -F^{k}\left(v\left(t\right)\right), \tag{5.64}$$

where $v\left(t\right)$ and $\hat{v}\left(\hat{t}\right)$ are a-conjugate functions, i.e. they satisfy the equalities

$$\hat{v}^{k}\left(\hat{t}\right)v^{k}\left(t\right) = 1, \qquad (k = 1, ..., N). \tag{5.65}$$

Thus we have obtained the a-conjugate equations (5.63) and (5.64). It is obvious that if we solve equation (5.63), then from equality (5.62) we can find φ-conjugate solutions $\overset{\varphi}{\hat{u}}\left(\hat{t}\right)$ to a solution $u\left(t\right)$ of (2.1). Therefore in the sequel we will call (2.1) and (5.63) φ-conjugate equations.

(16) Let us consider equation (1.2) and analyze it in terms of φ-conjugation. For this we fix $\alpha \in \Gamma_{N_0}$.

Let $u_\alpha\left(z_\alpha\right)$ and $\overset{\varphi}{\hat{u}}_\alpha\left(\hat{z}_\alpha\right)$ be φ-conjugate functions, i.e. they satisfy the equalities

$$\varphi_\alpha^{k}\left(u_\alpha\left(z_\alpha\right)\right) + \varphi_\alpha^{k}\left(\overset{\varphi}{\hat{u}}_\alpha\left(\hat{z}_\alpha\right)\right) = 0, \qquad (k = 1, ..., N) \tag{5.66}$$

By (1) of Section *1*, $z_\alpha = \left(1/l_\alpha\right)\alpha_\nu x^\nu$, and by (9) of Section *5*, $\hat{z}_{\hat{\alpha}} = \left(1/\hat{l}_{\hat{\alpha}}\right)\hat{\alpha}_{\hat{\nu}}\hat{x}^{\hat{\nu}}$. However, for simplicity, in the sequel we will replace $\hat{z}_{\hat{\alpha}}$ by the notation $\hat{z}_\alpha = \left(1/\hat{l}_{\hat{\alpha}}\right)\hat{\alpha}_{\hat{\nu}}\hat{x}^{\hat{\nu}}$.

In (5.66), u_α is a solution of equation (1.2). Then, by analogy with (5.56), the function $\overset{\varphi}{\hat{u}}_\alpha$ is a solution of the equation

$$a_\alpha\left(\overset{\varphi}{\hat{u}}_\alpha\right)\frac{d\overset{\varphi}{\hat{u}}_\alpha}{dz_\alpha} = -F\left(\overset{\varphi}{\hat{u}}_\alpha\right). \tag{5.67}$$

Equation (5.67) suggests the idea that along with (1.1) we should consider the equation

$$a^{\nu}\left(U\right)\frac{\partial U}{\partial x^{\nu}} = -F\left(U\right). \tag{5.68}$$

Obviously, (5.67) is an equation of plane waves for the quasilinear system (5.68).

As has been established in (14), the relation between the characteristic functions φ_α and ψ_α of equations (1.2) and (5.67), respectively, is (5.60):

$$\psi_\alpha = -\varphi_\alpha. \tag{5.69}$$

Then, by virtue of (5.61), the operation of conjugation $\overset{\varphi}{\hat{}}$ can be written in the form

$$(\varphi_\alpha(u_\alpha))^{\overset{\varphi}{\hat{}}} = -\varphi_\alpha\left(\overset{\varphi}{\hat{u}_\alpha}\right). \tag{5.70}$$

(17) As has been stated above, ψ_α are the characteristic functions of equation (5.67). Then for equation (5.67) we rewrite (2.16) and (2.20) as

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\psi_\alpha^k\left(\overset{\varphi}{\hat{u}_\alpha}\right) - \psi_\alpha^k\left(\bar{\dot{\chi}}\right)\right] = 1, \tag{5.71}$$

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\psi_\alpha^k\left(\bar{\ddot{\chi}}\right) - \psi_\alpha^k\left(\overset{\varphi}{\hat{u}_\alpha}\right)\right] = 1.$$

In view of (5.69–70), equalities (5.71) take the form

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\varphi_\alpha^k\left(\bar{\dot{\chi}}\right) - \varphi_\alpha^k\left(\overset{\varphi}{\hat{u}_\alpha}\right)\right] = 1,$$

$$\sum_{\alpha\in\Omega} q_\alpha \exp\left[\varphi_\alpha^k\left(\overset{\varphi}{\hat{u}_\alpha}\right) - \varphi_\alpha^k\left(\bar{\ddot{\chi}}\right)\right] = 1.$$

Comparing these equalities with (2.16) and (2.20), we observe the coincidence of the functional structures $\bar{\dot{\chi}}$ and $\dot{\chi}$, and, accordingly, $\bar{\ddot{\chi}}$ and $\ddot{\chi}$. Hence for the operation $\overset{\varphi}{\hat{}}$ we obtain

$$[\dot{\chi}(\ldots, u_\alpha, \ldots)]^{\overset{\varphi}{\hat{}}} = \ddot{\chi}\left(\ldots, \overset{\varphi}{\hat{u}_\alpha}, \ldots\right), \tag{5.72}$$

$$[\ddot{\chi}(\ldots, u_\alpha, \ldots)]^{\overset{\varphi}{\hat{}}} = \dot{\chi}\left(\ldots, \overset{\varphi}{\hat{u}_\alpha}, \ldots\right),$$

where $\dot{\chi}(\ldots, u_\alpha, \ldots)$, $\ddot{\chi}(\ldots, u_\alpha, \ldots)$ are solutions of equation (1.1), and $\dot{\chi}\left(\ldots, \overset{\varphi}{\hat{u}_\alpha}, \ldots\right)$, $\ddot{\chi}\left(\ldots, \overset{\varphi}{\hat{u}_\alpha}, \ldots\right)$ are solutions of (5.68).

(18) Let u_1, $u_2 \in J_N^{N_0}$, where $J_N^{N_0}$ is the space of solutions of equation (1.1). Let us represent these solutions as (2.28) and (2.29). Using the results of Section *3* and equalities (5.51–52) and (5.72), we can write

$$\begin{aligned}(u_1 \dot{\oplus} u_2)^{\overset{\varphi}{\hat{}}} &= \left(\dot{\chi}_\Omega\left(\ldots, u_{\alpha 1} \underset{\varphi}{\dot{+}} u_{\alpha 2}, \ldots\right)\right)^{\overset{\varphi}{\hat{}}} \\ &= \ddot{\chi}_\Omega\left(\ldots, \overset{\varphi}{\hat{u}_{\alpha 1}} \underset{\varphi}{\ddot{+}} \overset{\varphi}{\hat{u}_{\alpha 2}}, \ldots\right) = U_1 \ddot{\oplus} U_2,\end{aligned}$$

$$\left(u_1\ddot{\oplus}u_2\right)^{\overset{\varphi}{\wedge}} = \left(\ddot{\chi}_\Omega\left(...,u_{\alpha1}\underset{\varphi}{\ddot{+}}u_{\alpha2},...\right)\right)^{\overset{\varphi}{\wedge}}$$
$$= \dot{\chi}_\Omega\left(...,\overset{\varphi}{\hat{u}}_{\alpha1}\underset{\varphi}{\dot{+}}\overset{\varphi}{\hat{u}}_{\alpha2},...\right) = U_1\dot{\oplus}U_2,$$

where $U_1, U_2 \in \overset{\varphi}{\hat{J}}{}_N^{N_0}$ is the space of solutions of equation (5.68). We thereby come to the equalities

$$\left(u_1\dot{\oplus}u_2\right)^{\overset{\varphi}{\wedge}} = U_1\ddot{\oplus}U_2, \tag{5.73}$$
$$\left(u_1\ddot{\oplus}u_2\right)^{\overset{\varphi}{\wedge}} = U_1\dot{\oplus}U_2.$$

Analogously, we show that

$$\left(u_1\dot{\otimes}u_2\right)^{\overset{\varphi}{\wedge}} = U_1\ddot{\otimes}U_2, \tag{5.74}$$
$$\left(u_1\ddot{\otimes}u_2\right)^{\overset{\varphi}{\wedge}} = U_1\dot{\otimes}U_2.$$

It is likewise easy to prove

$$(e)^{\overset{\varphi}{\wedge}} = h,$$
$$(h)^{\overset{\varphi}{\wedge}} = e, \tag{5.75}$$
$$(E_\Omega)^{\overset{\varphi}{\wedge}} = E_\Omega.$$

Thus we have shown that when equation (1.1) transforms to equation (5.68), the algebras of these equations do not change. It is only the algebraic fields contained in the double field that change their polarity. In the sequel, we will call (1.1) and (5.68) the φ-conjugate equations.

(19) Let u and U be solutions of equations (1.1) and (5.68), respectively, which in the χ-representation look like

$$u = \dot{\chi}\left(...,u_\alpha,...\right), \tag{5.76}$$
$$U = \dot{\chi}\left(...,\overset{\varphi}{\hat{u}}_\alpha,...\right).$$

Here u_α and $\overset{\varphi}{\hat{u}}_\alpha$ are the φ-conjugate solutions of equations (1.2) and (5.67), respectively.

Using rule (3.41), we multiply u by U to obtain

$$u\dot{\otimes}U = \dot{\chi}_\Omega\left(...,u_\alpha\odot\overset{\varphi}{\hat{u}}_\alpha,...\right). \tag{5.77}$$

As has been shown in [2],

$$\varphi_\alpha\left(u_\alpha\odot\overset{\varphi}{\hat{u}}_\alpha\right) = \varphi_\alpha\left(u_\alpha\right) + \varphi_\alpha\left(\overset{\varphi}{\hat{u}}_\alpha\right).$$

Using equality (5.66), we obtain

$$\varphi_\alpha\left(u_\alpha\odot\overset{\varphi}{\hat{u}}_\alpha\right) = 0.$$

Then (3.43–44) immediately imply

$$u\dot{\otimes}U = E_\Omega. \tag{5.78}$$

By (5.66) it is not difficult to show that

$$\left(\overset{\varphi}{\hat{u}}_{\alpha}\right)^{\overset{\varphi}{\wedge}} = u_{\alpha}, \tag{5.79}$$

$$\left(E_{\alpha}\right)^{\overset{\varphi}{\wedge}} = E_{\alpha}.$$

For the φ-conjugation, using properties (5.74), for $u\dot{\otimes}U$ we obtain

$$\left(u\dot{\otimes}U\right)^{\overset{\varphi}{\wedge}} = U\ddot{\otimes}u = u\ddot{\otimes}U.$$

Applying the operation of φ-conjugation to equality (5.78) and taking into account (5.75), it is not difficult to obtain the validity of the equality

$$u\ddot{\otimes}U = E_{\Omega}. \tag{5.80}$$

(20) Let a solution of equation (1.1) be written in form (3.20)

$$u = \dot{\varkappa}_{\Omega}\left(..., w_{\alpha}\left(z_{\alpha}\right), ...\right), \tag{5.81}$$

where $w_{\alpha}\left(z_{\alpha}\right)$ is a solution of equation (3.1).

Note that for the φ-conjugation, (5.39) transforms to (5.42). By analogy one can show, that the following takes place:

$$\left(w_{\alpha}\left(z_{\alpha}\right)\right)^{\wedge} = \hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right). \tag{5.82}$$

Since $w_{\alpha}\left(z_{\alpha}\right)$ is a solution of (3.1), $\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)$ is a solution of the equation

$$\frac{\hat{d}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)}{\hat{d}\hat{z}_{\alpha}} = \hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right). \tag{5.83}$$

For the φ-conjugation, a solution of equation (5.81) transforms to a solution of equation (5.68):

$$U = \ddot{\varkappa}_{\Omega}\left(..., \hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right), ...\right). \tag{5.84}$$

Using (4.18), we find the alternative $\ddot{\varkappa}$-derivative of (5.84)

$$\frac{\hat{\partial}_{\ddot{\varkappa}}U}{\hat{\partial}_{\ddot{\varkappa}}\hat{X}^{\nu}} = \ddot{\varkappa}_{\Omega}\left(..., \frac{\hat{\partial}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)}{\hat{\partial}\hat{x}^{\nu}}, ...\right),$$

where

$$\hat{X}^{\nu} = \ddot{\varkappa}_{\Omega}\left(..., \hat{x}^{\nu}, ..., \hat{x}^{\nu}, ...\right). \tag{5.85}$$

Applying the results obtained in [2], we can easy calculate

$$\frac{\hat{\partial}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)}{\hat{\partial}\hat{x}^{\nu}} = \frac{\hat{d}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)}{\hat{d}\hat{z}_{\alpha}}\frac{\hat{d}\hat{z}_{\alpha}}{\hat{d}\hat{x}^{\nu}} = \frac{\hat{\alpha}_{\nu}}{\hat{l}_{\alpha}}\frac{\hat{d}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right)}{\hat{d}\hat{z}_{\alpha}}.$$

Taking into account (5.83), we find

$$\frac{\hat{\partial}_{\ddot{\varkappa}}U}{\hat{\partial}_{\ddot{\varkappa}}\hat{X}^{\nu}} = \ddot{\varkappa}_{\Omega}\left(..., \frac{\hat{\alpha}_{\nu}}{\hat{l}_{\alpha}}\hat{w}_{\alpha}\left(\hat{z}_{\alpha}\right), ...\right). \tag{5.86}$$

By analogy with (4.13), let us introduce the constant values

$$\hat{L}^{\nu} = \ddot{\varkappa}_{\Omega}\left(..., \hat{l}^{\nu}, ..., \hat{l}^{\nu}, ...\right), \tag{5.87}$$

where $\hat{l}^{\nu} = 1/l^{\nu}$. Multiplying equality (5.86) by (5.87) according to rule (3.27_2) and performing the $\ddot{\oplus}$-summation over the index ν, we eventually obtain

$$\hat{L}^{\hat{\nu}}\ddot{\otimes}\frac{\hat{\partial}_{\ddot{\varkappa}}U}{\hat{\partial}_{\ddot{\varkappa}}\hat{X}^{\hat{\nu}}} = U. \tag{5.88}$$

We call (4.14) and (5.88) the φ-conjugate equations.

6. Process and the antiprocess

(1) Let us consider the equation

$$\frac{du}{dt} = u(1-u). \tag{6.1}$$

As is known, equation (6.1) is called the logistic equation [7] describing a biological population in the condition of competition for food (for instance, $u(t)$ can be interpreted as a quantity of fish in the pond). The function $u(t)$ in (6.1) is normed.

The characteristic function of equation (6.1) has the form

$$\varphi(u) = \ln\frac{u}{1-u}.$$

Let us write the φ-conjugation condition for u and $\overset{\varphi}{\hat{u}}$:

$$\ln\frac{u}{1-u} + \ln\frac{\overset{\varphi}{\hat{u}}}{1-\overset{\varphi}{\hat{u}}} = 0.$$

Hence it readily follows that

$$u(t) + \overset{\varphi}{\hat{u}}\left(\hat{t}\right) = 1. \tag{6.2}$$

A general solution of (6.1) has the form

$$u(t) = 1/\left(1 + \exp\left[-(t+c)\right]\right), \tag{6.3}$$

where c is an arbitrary integration constant. Using (6.2), we obtain

$$\overset{\varphi}{\hat{u}}\left(\hat{t}\right) = 1/\left(1 + \exp(t+c)\right). \tag{6.4}$$

This equality can be rewritten in the form

$$\overset{\varphi}{\hat{u}}\left(\hat{t}\right) = 1\ddot{+}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right), \tag{6.5}$$

where we have used the equalities $\hat{t}t = 1$, $\hat{c}c = 1$.

Using (5.49), we write the φ-conjugate equation to (6.1) as follows:

$$\frac{\hat{d}\overset{\varphi}{\hat{u}}}{\hat{d}\hat{t}} = \frac{\overset{\varphi}{\hat{u}}{}^2}{\overset{\varphi}{\hat{u}}\left(1-\overset{\varphi}{\hat{u}}\right)}.$$

The right-hand part of this equation can be represented as

$$\frac{\overset{\varphi}{\hat{u}}}{1-\overset{\varphi}{\hat{u}}} = \frac{1}{\frac{1}{\overset{\varphi}{\hat{u}}}-1} = \overset{\varphi}{\hat{u}}\ddot{-}1.$$

Then the φ-conjugate equation takes the form

$$\frac{\hat{d}\overset{\varphi}{\hat{u}}}{\hat{d}\hat{t}} = \overset{\varphi}{\hat{u}}\ddot{-}1. \tag{6.6}$$

Let us show by direct calculations that (6.5) is a solution of equation (6.6). For this, we have to calculate an alternative derivative for (6.5). Using the properties of alternative derivatives stated in [2], we can write

$$\begin{aligned}\frac{\hat{d}\overset{\varphi}{\hat{u}}}{\hat{d}\hat{t}} &= \frac{\hat{d}}{\hat{d}\hat{t}}\left[1\ddot{+}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right)\right] = \frac{\hat{d}}{\hat{d}\hat{t}}1\ddot{+}\frac{\hat{d}}{\hat{d}\hat{t}}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right)\\ &= \infty\ddot{+}\frac{\hat{d}}{\hat{d}\hat{t}}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right) = \frac{\hat{d}}{\hat{d}\hat{t}}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right)\\ &= \frac{1}{\frac{d}{dt}\exp\left(t+c\right)} = \exp\left(-\left(t+c\right)\right).\end{aligned}$$

Eventually,

$$\frac{\hat{d}\overset{\varphi}{\hat{u}}}{\hat{d}\hat{t}} = \exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right). \tag{6.7}$$

Substituting (6.5) and (6.7) into (6.6), we obtain the identity

$$\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right) = \left[1\ddot{+}\exp\left(-\frac{1}{\hat{t}\ddot{+}\hat{c}}\right)\right]\ddot{-}1.$$

Recall that in the algebra of equation (6.1) given in [1], $e = 0$, and $h = 1$ are identity elements. On the other hand, the equations $u = 0$, $u = 1$ are called the logistic curves [7]. As follows from (6.3), with an increase of time t, $u(t)$ tends to the logistic curve $u = 1$.

If $u(t)$ is the quantity of fish in the pond at a moment of time t, then $\overset{\varphi}{\hat{u}}$ should be regarded as the quantity of vacant places, i.e. not occupied by fish. This conclusion follows from equality (6.2). From (6.4) we immediately conclude that with an increase of time t, the quantity of vacant places tends to zero. Thus the φ-conjugate equation (6.6) describes the availability of vacant places not occupied by fish in the pond at a given moment of time.

It should be said that such an interpretation of a φ-conjugate equation is in parallel to the theory of holes introduced by Dirac in the quantum theory of an electron.

In [2] we have established that in the process described by (6.1) the proper time τ is

$$\tau = \frac{t}{1+t}. \tag{6.8}$$

As t increases, the proper time τ tends to 1. If $0 \leq t \leq \infty$, then, obviously,

$$0 \leq \tau \leq 1. \tag{6.9}$$

The value $\tau = 1$ is called the process lifetime, since upon the expiration of this time the reproduction of fish in the pond comes to an end and the process stops.

In the considered process, t is the time of an external observer sitting on the bank and counting t by means of some dynamic process which is in no way connected with what is taking place in the pond. As to τ, it is the time which is directly connected with the fish population and counted taking into account its dynamics.

In addition to (6.8), there is another proper time [2]

$$\hat{\tau} = \frac{\hat{t}}{1+\hat{t}} \tag{6.10}$$

where $t\hat{t} = 1$. From (6.8) and (6.10) it is easy to find

$$\tau + \hat{\tau} = 1. \tag{6.11}$$

If τ is interpreted as proper time of the process of fish reproduction in the pond, which is described by equation (6.1), then $\hat{\tau}$ should be regarded as the time of the process during which the quantity of vacant places changes and which is described by equation (6.6).

It immediately follows from (6.11) that if τ increases from 0 to 1, then $\hat{\tau}$ decreases from 1 to 0, i.e. the proper times τ and $\hat{\tau}$ change in the opposite direction.

Thus, if the fish population in the pond is considered to be the process, then the change in the quantity of vacant places should be called the antiprocess.

(2) Let us return to equation (2.1) and assume that the system of differential equations (2.1) defines the population of N different fish species competing for food in the pond. Some of the fish are predators, others are victims. The pond can, in principle, be connected with another pond from which (or, on the contrary, to which) some part of fish of different species can migrate. In equation (2.1) the unknown function $u^k(t)$ is the quantity of fish of the k-th species in the pond at the moment of time t.

In [2], when constructing the differential calculus with the aid of a φ-double field, we formulated the notion of proper time τ^k ([2], §7.1 and §7.9). Its extended form is

$$\tau^k = \varphi^{-1\ k}(\ln t, ..., \ln t), \quad (k = 1, ..., N), \tag{6.12}$$

where φ^{-1} is the inverse function to the characteristic function φ of equation (2.1). After eliminating t from (6.12), we obtain the equalities

$$\varphi^1\left(\tau^1, ..., \tau^N\right) = ... = \varphi^N\left(\tau^1, ..., \tau^N\right). \tag{6.13}$$

To clarify result (6.12), let us return to the characteristic functions. In §2 [1] it is shown that solutions of equation (2.1) and characteristic functions are related by the equality

$$\varphi^k(u) = b^k t + c^k, \quad (k = 1, ..., N), \tag{6.14}$$

which immediately implies

$$u^k = \varphi^{-1\ k}\left(b^1 t + c^1, ..., b^N t + c^N\right), \tag{6.15}$$

where $u^k(t)$ is the quantity of fish of the k-th species in the pond at the moment of time t. Then, comparing (6.12) and (6.15), we conclude that τ^k is the proper time for the fish of the k-th species. In other words, each fish species in the pond has its own clock that conforms to its reproduction dynamics. From (6.12) it follows that all proper clocks work simultaneously when the external time t changes. We remark that the external time t has no advantage over proper times. The variable t can be removed from equation (6.12) if we assume that one of $\tau^1, ..., \tau^N$, say, τ^1, is an independent variable. Then $\tau^2, ..., \tau^N$ are the functions of τ^1.

The foregoing arguments need some clarification. For this, we will consider the mapping $\exp\varphi : J_N^1 \to W_N^1$, where J_N^1 is the space of solutions of (2.1), and W_N^1 is the space of solutions of (2.11). As is shown in §1 [1], the inverse mappings $\varphi^{-1}(\ln) : W_N^1 \to J_N^1$ are, generally speaking, not unique. Let us choose, in $J_{N(m)}^1$, a subspace J_N^1 such that the mapping $J_{N(m)}^1$ in J_N^1 defined by $\exp\varphi$ be a diffeomorphism. As we have noted in §1 [1], $J_{N(m)}^1$ is a leaf of the covering, where m runs through some discrete set. Therefore, the collection of N clocks works in the leaf $J_{N(m)}^1$ and shows the proper times defined by equations (6.12). In other words, every leaf has its own collection of N clocks. Example 7.1 in [2] can serve as an illustration of the above-said. For the equation

$$\frac{du}{dt} = \sin u$$

the proper time τ is defined by the equation

$$\tau = 2\arctan t + 2\pi m,$$

where $m = 0, \pm 1, \pm 2, ...$ is the number of the leaf $J_{N(m)}^1$.

Now let us pass to discussing the φ-conjugate equation (5.49). By (1) of Section *6*, the solution $\overset{\varphi}{\hat{u}}{}^k$ of equation (5.49) can be interpreted as the quantity of the remaining vacant places for the fish of the k-th species at the moment of time t. By (5.37) and (6.14), a solution of equation (5.49) has the form

$$\overset{\varphi}{\hat{u}}{}^k = \varphi^{-1\ k}\left(-\left(b^1 t + c^1\right), ..., -\left(b^N t + c^N\right)\right). \tag{6.16}$$

Since $\left(b^k t + c^k\right)\left(\hat{b}^k \hat{t} \ddot{+} \hat{c}^k\right) = 1$, where $t\hat{t} = 1$, $\hat{b}^k b^k = 1$, $\hat{c}^k c^k = 1$, (6.16) can be rewritten as

$$\overset{\varphi}{\hat{u}}{}^k = \varphi^{-1\ k}\left(-\frac{1}{\hat{b}^1 \hat{t} \ddot{+} \hat{c}^1}, ..., -\frac{1}{\hat{b}^N \hat{t} \ddot{+} \hat{c}^N}\right).$$

Based on the fact for $u(t)$ and $\overset{\varphi}{\hat{u}}{}^k\left(\hat{t}\right)$ we have (5.37), an analogous equality is fulfilled for τ^k

and $\hat{\tau}^k$, too:

$$\varphi^k(\tau) + \varphi^k(\hat{\tau}) = 0, \quad (k = 1, ..., N). \tag{6.17}$$

Using the φ-product [2] for τ and $\hat{\tau}$ and also taking into account (6.17), we obtain

$$\hat{\tau} \odot \tau = E,$$

where E is a unit element in the double algebraic field of equation (2.1). This result is an analogue of the equality $\hat{t}t = 1$.

From (6.12) it immediately follows that

$$\varphi^k(\tau) = \ln t. \tag{6.18}$$

Substituting (6.18) into (6.17) and using $\hat{t}t = 1$, we obtain

$$\varphi^k(\hat{\tau}) = \ln \hat{t}, \quad (k = 1, ..., N), \tag{6.19}$$

whence we readily derive an analogue of (6.12)

$$\hat{\tau}^k = \varphi^{-1\ k}\left(\ln \hat{t}, ..., \ln \hat{t}\right). \tag{6.20}$$

Equality (6.19) immediately implies

$$\varphi^1\left(\hat{\tau}^1, ..., \hat{\tau}^N\right) = \cdots = \varphi^N\left(\hat{\tau}^1, ..., \hat{\tau}^N\right). \tag{6.21}$$

It is obvious that (6.13) and (6.21) do not contradict (6.17).

From (6.18) and (6.6)[2] we conclude that the following equalities are fulfilled:

$$\lim_{t\to 0} \tau^k = e^k,$$

$$\lim_{t\to \infty} \tau^k = h^k,$$

$$(k = 1, ..., N).$$

Analogously, from (6.19), (6.6) [2], taking into account $\hat{t}t = 1$, we obtain

$$\lim_{t\to 0} \hat{\tau}^k = h^k,$$

$$\lim_{t\to \infty} \hat{\tau}^k = e^k,$$

$$(k = 1, ..., N).$$

Note that e and h are logistic curves.

Thus if τ^k is the proper time for the reproduction process of fish of the k-th species, then $\hat{\tau}^k$ should be interpreted as the proper time for the process during which a change occurs in the number of vacant places in the pond for the same fish species.

7. The Dirac equation

(1) Let us consider the relativistic motion of an electron in the electromagnetic field. This process is described by the Dirac equation [8]

$$\gamma^{\nu}\left(\frac{\partial}{\partial x^{\nu}}-iA_{\nu}\right)\psi=-im\psi, \qquad (7.1)$$

where m is the electron mass and A_{ν} are the potentials of the electromagnetic field; x^{ν} and m are assumed to be dimensionless elements.

The Dirac 4-matrices γ^{ν} satisfy the equalities

$$\gamma^{\nu}\gamma^{\tau}+\gamma^{\tau}\gamma^{\nu}=2\mathring{g}^{\nu\tau}1,$$

where 1 is the unit 4-matrix and $\mathring{g}^{\nu\tau}=diag\,(1,1,1,-1)$ is the metric tensor of the Minkovski space. The matrices γ^{1}, γ^{3}, γ^{4} are constructed of values $0,\ 1,\ -1$, and γ^{2} – of $0,\ i,\ -i$, where i is the imaginary unity [8].

Using the results of Section *5*, we can write the a-conjugate equation to the Dirac equation (7.1) as follows:

$$\tilde{\gamma}^{\hat{\nu}}\hat{\cdot}\left(\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\hat{\nu}}}\ddot{+}i\hat{A}_{\hat{\nu}}\right)\hat{\psi}=i\hat{m}\hat{\psi}, \qquad (7.2)$$

where

$$\begin{aligned}\hat{x}^{\nu}&=1/x^{\nu},\\ \hat{A}_{\nu}&=1/A_{\nu},\\ \hat{m}&=1/m\\ \tilde{\gamma}^{\nu k}_{n}&=1/\gamma^{\nu k}_{n},\\ \hat{\psi}^{k}&=1/\psi^{k}.\end{aligned} \qquad (7.3)$$

Recall that an alternative summation [2] is performed over the identical indexes ν from 1 to 4.

Now let us introduce the matrixes $\hat{\gamma}^{\nu}$ which are derived from the matrices γ^{ν} if we replace the zero elements by the nonproper value ∞. It is not difficult to prove that the following equalities are valid:

$$\hat{\gamma}^{\nu}\hat{\cdot}\hat{\gamma}^{\tau}\ddot{+}\hat{\gamma}^{\tau}\hat{\cdot}\hat{\gamma}^{\nu}=2\mathring{g}^{\nu\tau}\hat{1}, \qquad (7.4)$$

where $\hat{\cdot}$ means an alternative multiplication of the matrices [2]. The matrix $\hat{1}$ has been considered in (1) of Section *5*. Then (7.3) implies

$$\begin{aligned}\tilde{\gamma}^{1}&=\hat{\gamma}^{1},\\ \tilde{\gamma}^{2}&=-\hat{\gamma}^{2},\\ \tilde{\gamma}^{3}&=\hat{\gamma}^{3},\\ \tilde{\gamma}^{4}&=\hat{\gamma}^{4}.\end{aligned} \qquad (7.5)$$

Let us alternatively multiply (7.2) from the left by the matrix $\hat{\gamma}^{2}$. Using (7.4), we obtain

$$\hat{\gamma}^{\hat{\nu}}\hat{\cdot}\left(\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\hat{\nu}}}\ddot{+}i\hat{A}_{\hat{\nu}}\right)\hat{\psi}^{c}=-i\hat{m}\hat{\psi}^{c}, \qquad (7.6)$$

where

$$\hat{\psi}^{c}=\hat{\gamma}^{2}\hat{\cdot}\hat{\psi}. \qquad (7.7)$$

In the context of the obtained results it is appropriate to remind that in the case of charge conjugation, equation (7.1) transforms to

$$\gamma^{\nu}\left(\frac{\partial}{\partial x^{\nu}}+iA_{\nu}\right)\psi^{c}=-im\psi^{c}, \tag{7.8}$$

where $\psi^{c}=\gamma_2\gamma_4\bar{\psi}^{T}$, $\bar{\psi}$ is the Dirac conjugate spinor [8].

(2) In Section *5* we have established that, in addition to the a-conjugation there also exists the operation of φ-conjugation. For the φ-conjugation, equation (1.1) transforms to equation (5.68). Then equation (7.1) transforms to an equation which we write in the form

$$\gamma^{\nu}\left(-\frac{\partial}{\partial x^{\nu}}-iA_{\nu}\right)U=-imU.$$

Multiplying this equation from the left by γ_5, we obtain

$$\gamma^{\nu}\left(\frac{\partial}{\partial x^{\nu}}+iA_{\nu}\right)\tilde{\psi}=-im\tilde{\psi},$$

where $\tilde{\psi}=\gamma_5\psi$.

Thus we come to a conclusion that the a- and φ-conjugation operations are related to the charge conjugation and the space-time mapping.

8. Calculus relativity and algebraic objects

As we have seen, every system of algebraic equations generates a double algebraic field used in constructing a proper differential and an alternative calculus. Since the set of differential equations is infinite, systems of various calculi are also infinitely many. We are interested in finding out whether there exists a relation between different calculus systems and if it does, then what it is like.

Assume for simplicity that the considered autonomous systems of differential equations have one and the same dimension N. In the sequel, the processes described by these systems will be called the processes of the class N.

(1) Let us consider system (2.11):

$$\frac{dw^{k}}{dt}=w^{k}, \quad (k=1,\ldots,N). \tag{8.1}$$

It is obvious that the system consists of N independent linear equations. According to §1 [2], every equation of system (8.1) has its own proper double numerical field. These fields are independent because the equations do not overlap.

As has been mentioned, W_N^1 is the space of solutions of equation (8.1), where an arbitrary element $w(t)\in W_N^1$ is written in the form $w(t)=\left(w^1(t),...,w^N(t)\right)$. Let us extend the space W_N^1 replacing $w^k(t)$ by arbitrary smooth functions $q^k(t)$. We thus obtain the Euclidean space Γ^N on which the considered differential equations describing the processes of the class N are defined. It is assumed that each coordinate q^k of an element $q\in\Gamma^N$ is subjected to the action of its own double numerical field generated by the equation

$$\frac{dw^{k}}{dt}=w^{k}.$$

Then we have N separate copies of one-dimensional and independent mathematical objects. However, the time t, i.e. the value characterizing the dynamics of the process steps in here. Namely, the time t becomes the factor uniting all these one-dimensional objects into one whole N-dimensional object which we call a dynamic object.

(2) Let us assume that the observer is inside the dynamic process described by equation (8.1). He has at his disposal the double numerical field acting on the coordinates of the Euclidean space Γ^N. The double algebraic field (the collection of N double numerical fields) acting in Γ^N makes it possible to introduce standard differential and integral calculi with their alternative calculi in the space Γ^N. Therefore the observer can carry out calculations by the rules admitted by equation (8.1). Finally, since the process is dynamic, the counting of time can be directly connected with the process itself (for instance, through the introduction of the value $t = \ln |w^1|$). In other words, the observer has the clock.

The above reasoning thus implies that the dynamic process described by equation (8.1) acquires, in the space Γ^N, its own calculus and clock.

(3) Let us assume that in the absence of external forces, in the space Γ^N, there occurs some dynamic process differing from process (8.1). The observer is inside process (8.1). We want to know what the investigated process looks like for the observer from the inside of (8.1). It is obvious that the only investigation tool for this must be the dynamic process inside which the observer is and which he can use without violating equation (8.1). Otherwise, some external processes might implicitly intervene. The observer could perform measurements only if there exists a possibility to establish the interaction between the processes. Otherwise, the observe would not be able to see the process. In the sequel, it will be assumed that the observer somehow uses process (8.1) to probe the process he investigates. It is understood that since the observer investigates the process, his calculations must be based on the algebra contained in process (8.1). During the probing he performs manipulations with solutions $w^k(t)$ of equation (8.1) and to remain in the framework of (8.1) he has to obey the rule of the algebraic game contained in equation (8.1). Otherwise, he would have to leave process (8.1). Moreover, the investigated process must be of the same class as the process inside which the observer is. Thus the observer establishes that $u^k(t)$, $(k = 1, ..., N)$ completely describes the investigated process. As to t, it is the time counted by process (8.1).

On the basis of his observation results, the observer begins to compose a differential equation of the process dynamics, but he discovers that in the equation there are terms with $w(t)$ which have appeared as a result of the probing of the investigated process. Analogous terms with $u(t)$ also appear in (8.1). Having finished the probing, the observer throws off these crossing terms and obtains an autonomous system of differential equations which describes the investigated process:

$$\frac{du^k}{dt} = F^k\left(u^1, ..., u^N\right), \qquad (k = 1, ..., N). \tag{8.2}$$

Though the problem of measurements has been presented here rather superficially, nevertheless (1) and (2) enable us to make the following important conclusion: equation (8.2) of the investigated process is written in terms of the calculus of process (8.1). It should also be noted that the final result having form (8.1) and (8.2) is already free of the interaction between processes.

(4) Let us assume that the observer is inside the process defined by equation (8.2). As is shown in [2], equation (8.2) in its system of calculus takes a linear form. From (1.5) it follows that the components of the vector b, which figures in (7.29) in [2], in this paper, have the form $b^k = 1$, $(k = 1, ..., N)$. Since $b^k = \exp \varphi^k(B)$, we obtain $B = E$, where E is the unit element of the operation of multiplication of the φ-double field. Then equation (8.2) in its system of

calculus takes the form (7.31) [2]

$$\frac{d_\varphi u}{d_\varphi \tau} = u, \tag{8.3}$$

where τ is the proper time discussed with sufficient completeness in Section *6*.

(5) Let us consider now the system of calculus connected with equations (8.2). This system consists of the Euclidean space Γ^N, the double field generated by equations (8.2), and the time τ counted by the dynamic process (8.2). Analyzing the results of §6 and §7 from [2], we see that algebraic operations can be performed over arbitrarily taken vectors of the space Γ^N both from standpoint of the double field of equation (8.1) and from the standpoint of the double field of equations (8.2). In passing over from the dynamic process (8.1) to the dynamic process (8.2), the space Γ^N remains unchanged, but the double field representation and the time counting change.

We will discuss these issues in detail. Consider the derivative of the given function. The finding of the standard derivative is in fact a chain of algebraic operations from the numerical field which are performed over the function and its arguments. Hence the derivative with all its properties can be interpreted as an algebraic object. Let $q(t) \in \Gamma^N$. We can differentiate $q(t)$ in a standard manner with respect to t and obtain $\frac{dq(t)}{dt}$. Analogously, using the double field of equations (8.2) and the results of [2], we can find the φ-derivative of the same vector $q \in \Gamma^N$.

$$\frac{d_\varphi q(t)}{d_\varphi \tau} = \varphi^{-1}\left(\ln \frac{d}{dt} \exp \varphi(q(t))\right), \tag{8.4}$$

where $\tau^k = \varphi^{-1\ k}(\ln t, ..., \ln t)$ is the proper time of process (8.2). Recall that $\varphi(u)$ are the characteristic functions of equations (8.2).

(6) It is obvious that

$$\mathring{\varphi}^k(w) = \ln w^k, \quad (k = 1, ..., N) \tag{8.5}$$

are the characteristic functions of equations (8.1). We introduce the new vector $g(t) \in \Gamma^N$ which is related to $q(t)$ by the equality

$$g(t) = \varphi^{-1} \circ \mathring{\varphi}(q(t)). \tag{8.6}$$

Then, by virtue of (8.4), the φ-derivative of the function $g(t)$ is written as

$$\frac{d_\varphi g(t)}{d_\varphi \tau} = \varphi^{-1} \circ \mathring{\varphi}\left(\frac{dq(t)}{dt}\right). \tag{8.7}$$

(7) If we introduce the N-dimensional vector $t = (t, ...t)$, then by analogy with (8.6) the proper time τ of process (8.2) can be written in the form

$$\tau = \varphi^{-1} \circ \mathring{\varphi}(t) \tag{8.8}$$

which immediately implies equation (6.13) for τ

$$\varphi^1\left(\tau^1, ..., \tau^N\right) = \cdots = \varphi^N\left(\tau^1, ..., \tau^N\right). \tag{8.9}$$

Since in (8.9) there is no external time t, equality (8.9) should be called an invariant definition of the proper time. As shown in Section *6*, τ^k is the proper time of the k-th channel of process (8.2). Analogously, we can introduce the proper time t^k of the k-th channel in (8.1). Then, like in the case of (8.9), by (8.5) we can write

$$\ln t^1 = \cdots = \ln t^N, \tag{8.10}$$

which immediately implies

$$t^k = t,$$
$$(k = 1, ..., N).$$

It is obvious that (8.10) is an invariant definition of the proper time of process (8.1).

(8) Let us now consider some third process of the same class N as processes (8.1) and (8.2). It is assumed that the equation of motion of this process written in terms of the calculus of (8.1) is

$$\frac{dv}{dt} = f(v), \tag{8.11}$$

where ν and f are N-vectors. Let $\psi^k(v)$ be the characteristic functions of equation (8.11). For solutions of (8.11) we have the equalities

$$\psi^k(v) = b^k t + c_1^k,$$

where $b^k = 1,\ (k = 1, ..., N)$. Then in the proper system of calculus the equation of motion of this dynamic process has the form

$$\frac{d_\psi v}{d_\psi \sigma} = v. \tag{8.12}$$

Here $\sigma = \psi^{-1}(\ln t, ..., \ln t)$ is the proper time of process (8.11).

Let us introduce the N-vector $\theta(t)$ related to $q(t)$ by the equality

$$\theta(t) = \psi^{-1} \circ \mathring{\varphi}(q(t)). \tag{8.13}$$

Then, analogously to (8.7), the ψ-derivative of the vector $\theta(t)$ takes the form

$$\frac{d_\psi \theta(t)}{d_\psi \sigma} = \psi^{-1} \circ \mathring{\varphi}\left(\frac{dq(t)}{dt}\right). \tag{8.14}$$

From (8.6) and (8.13), having eliminated $q(t)$, we can write

$$\theta(t) = \psi^{-1} \circ \varphi(g(t)). \tag{8.15}$$

Analogously, from (8.7) and (8.14) we obtain

$$\frac{d_\psi \theta(t)}{d_\psi \sigma} = \psi^{-1} \circ \varphi\left(\frac{d_\varphi g(t)}{d_\varphi \tau}\right). \tag{8.16}$$

We define $g(t)$ from (8.15) and substitute it into (8.16) to obtain

$$\frac{d_\psi \theta}{d_\psi \sigma} = \psi^{-1} \circ \varphi\left(\frac{d_\varphi \varphi^{-1} \circ \psi(\theta)}{d_\varphi \tau}\right). \tag{8.17}$$

It is obvious that (8.17) is a full analogue of equality (8.4) whose left-and part contains the derivative of the function θ in the system of calculus of equation (8.12), while the right-hand part contains the derivative of this function θ in the system of calculus of equation (8.2).

(9) Let us return to proper times. In processes (8.2) and (8.11) the proper times are respec-

tively defined by

$$\tau = \varphi^{-1} \circ \mathring{\varphi}\,(t, ..., t)\,, \qquad (8.18)$$
$$\sigma = \psi^{-1} \circ \mathring{\varphi}\,(t, ..., t)\,.$$

After eliminating $\mathring{\varphi}\,(t)$ in these equalities, we obtain

$$\psi\,(\sigma) = \varphi\,(\tau)\,. \qquad (8.19)$$

From (8.19) and (8.9) it immediately follows that the proper time σ of process (8.11) satisfies the equalities

$$\psi^1\left(\sigma^1, ..., \sigma^N\right) = \cdots = \psi^N\left(\sigma^1, ..., \sigma^N\right). \qquad (8.20)$$

Equalities (8.9) and (8.20) obviously imply that the proper time is defined by the process itself independently of other processes. As to the interrelation of the proper times of dynamic processes (8.2) and (8.11), by virtue of (8.19) it is defined by

$$\sigma = \psi^{-1} \circ \varphi\,(\tau)\,. \qquad (8.21)$$

In particular, if in (8.21) ψ is replaced by the characteristic functions $\mathring{\varphi}$ from (8.5), then we obtain the proper time of process (8.1)

$$t^k = \exp \varphi^k\,(\tau)\,,$$
$$(k = 1, ..., N)\,.$$

But since (8.9) are fulfilled, t^k satisfies equalities (8.10). Certainly, τ is the external time for process (8.1), and t the proper time.

(10) Let us consider process (8.1) as it is viewed by the observer who is inside process (8.2). For this, in (8.4) we should replace $q\,(t)$ by $w\,(t)$. Taking into account (8.1) we obtain

$$\frac{d_\varphi w\,(t)}{d_\varphi \tau} = \varphi^{-1}\left(\ln\left(\exp\left(\varphi\,(w)\right)\frac{\partial \varphi\,(w)}{\partial w}w\right)\right). \qquad (8.22)$$

It is obvious that this is a differential equation of the unknown functions $w\,(t)$ in the calculus of (8.2). In other words, from the inside of (8.2) the observer views process (8.1) as nonlinear (if the equation (8.2) is nonlinear).

(11) In view of (8.5–6), equality (2.12) can be rewritten as

$$g_1 \underset{\varphi}{\dot{+}} g_2 = \varphi^{-1} \circ \mathring{\varphi}\left(q_1 \dot{+} q_2\right),$$

$$g_1 \underset{\varphi}{\ddot{+}} g_2 = \varphi^{-1} \circ \mathring{\varphi}\left(q_1 \ddot{+} q_2\right).$$

Analogous equalities are valid in the ψ-representation as well. But then these equalities and (8.13) imply

$$\theta_1 \underset{\psi}{\dot{+}} \theta_2 = \psi^{-1} \circ \varphi\left(g_1 \underset{\varphi}{\dot{+}} g_2\right), \qquad (8.23)$$

$$\theta_1 \underset{\psi}{\ddot{+}} \theta_2 = \psi^{-1} \circ \varphi\left(g_1 \underset{\varphi}{\ddot{+}} g_2\right).$$

It is not difficult to verify that the following equality is true:

$$\theta_1 \underset{\psi}{\odot} \theta_2 = \psi^{-1} \circ \varphi\left(g_1 \underset{\varphi}{\odot} g_2\right), \qquad (8.24)$$

where $\underset{\varphi}{\odot}$ and $\underset{\psi}{\odot}$ stand for the multiplication in the calculus systems of (8.2) and (8.11), respectively. Along with (8.20–21), we have

$$\begin{aligned} e_\psi &= \psi^{-1} \circ \varphi\left(e_\varphi\right), \\ h_\psi &= \psi^{-1} \circ \varphi\left(h_\varphi\right), \\ E_\psi &= \psi^{-1} \circ \varphi\left(E_\varphi\right), \end{aligned} \tag{8.25}$$

where e_φ, h_φ, E_φ and e_ψ, h_ψ, E_ψ are neutral and unit elements of the algebraic double fields of equations (8.2) and (8.11), respectively.

(12) Let us return to algebraically conjugate values. By virtue of definition (5.37), we will call the elements $g, \overset{\varphi}{\hat{g}} \in \Gamma^N$ the φ-conjugate elements of process (8.2) if the following equality is fulfilled:

$$\varphi\left(g\right) + \varphi\left(\overset{\varphi}{\hat{g}}\right) = 0. \tag{8.26}$$

Using equality (8.15), we associate $g, \overset{\varphi}{\hat{g}} \in \Gamma^N$ with the elements $\theta, \overset{\psi}{\hat{\theta}}$ of the same space:

$$\theta = \psi^{-1} \circ \varphi\left(g\right), \tag{8.27}$$

$$\overset{\psi}{\hat{\theta}} = \psi^{-1} \circ \varphi\left(\overset{\varphi}{\hat{g}}\right).$$

Expressing $g, \overset{\varphi}{\hat{g}}$ from (8.27) through θ and $\overset{\psi}{\hat{\theta}}$ and substituting the obtained values into (8.26), we have

$$\psi\left(\theta\right) + \psi\left(\overset{\psi}{\hat{\theta}}\right) = 0. \tag{8.28}$$

Thus we see that in passing from one system of calculus to the other, the algebraically conjugate values transform to the algebraically conjugate values.

Let us multiply equation (8.3) by $\frac{\hat{d}_\varphi \overset{\varphi}{\hat{u}}}{\hat{d}_\varphi \hat{\tau}}$ in the calculus of (8.2). Using equalities (5.38), we obtain

$$E_\varphi = u \underset{\varphi}{\odot} \frac{\hat{d}_\varphi \overset{\varphi}{\hat{u}}}{\hat{d}_\varphi \hat{\tau}}. \tag{8.29}$$

Since (5.37), (5.38) are fulfilled, we can multiply (8.29) by $\overset{\varphi}{\hat{u}}$. Finally, we have

$$\frac{\hat{d}_\varphi \overset{\varphi}{\hat{u}}}{\hat{d}_\varphi \hat{\tau}} = \overset{\varphi}{\hat{u}}. \tag{8.30}$$

It is obvious that if (8.3) describes the process, then (8.30) describes the antiprocess.

Let us establish the connection between solutions of equations (9.2) and (9.11). From equalities $w = \exp\varphi\left(u\right)$ and $w = \exp\psi\left(v\right)$, we can write $\varphi\left(u\right) = \psi\left(v\right)$, or

$$v = \psi^{-1} \circ \varphi\left(u\right). \tag{8.31}$$

As has been shown above, for (8.30) we have

$$\overset{\psi}{\hat{v}} = \psi^{-1} \circ \varphi \left(\overset{\varphi}{\hat{u}} \right).$$

By (5.38) and equality (8.24), it can be easily verified that equation (8.30) transforms to

$$\frac{\hat{d}_{\psi} \overset{\psi}{\hat{v}}}{\hat{d}_{\psi} \hat{\sigma}} = \overset{\psi}{\hat{v}}. \tag{8.32}$$

It is likewise easy to show the validity of the following equalities:

$$\begin{aligned} \hat{\sigma} &= \psi^{-1} \circ \varphi \left(\hat{\tau} \right), \\ \hat{\tau} \underset{\varphi}{\odot} \tau &= E_{\varphi}, \\ \hat{\sigma} \underset{\psi}{\odot} \sigma &= E_{\psi}. \end{aligned} \tag{8.33}$$

To clarify the obtained results, let us consider equation (8.1). Using the characteristic functions $\mathring{\varphi}(w)$ of (8.1), we can represent equality (5.8) as

$$\mathring{\varphi}(w) + \mathring{\varphi}(\hat{w}) = 0. \tag{8.34}$$

Equalities (5.8–10) can be rewritten as follows:

$$\begin{aligned} \hat{t} \underset{\mathring{\varphi}}{\odot} t &= E_{\mathring{\varphi}}, \\ \hat{w}\left(\hat{t}\right) \underset{\mathring{\varphi}}{\odot} w(t) &= E_{\mathring{\varphi}}, \\ \frac{\hat{d}\hat{w}}{\hat{d}\hat{t}} \underset{\mathring{\varphi}}{\odot} \frac{dw}{dt} &= E_{\mathring{\varphi}}, \end{aligned} \tag{8.35}$$

where $E_{\mathring{\varphi}} = (1, ..., 1)$. Recall that $E_{\mathring{\varphi}}$ is a solution of the equation $\mathring{\varphi}(E_{\mathring{\varphi}}) = 0$, where $\mathring{\varphi}$ is (8.5). If (8.1) describes the process, then equality (5.11) describes antiprocess.

(13) From (8.2) and (8.3) it immediately follows that $u(t)$ is differentiated in the calculus systems of both (8.1) and (8.2). This means that in passing from one observer to another observer who is inside the other process, the space Γ^N experiences no transformation. One and the same element $g \in \Gamma^N$ is viewed by the observers from processes (8.2) and (8.11), generally speaking, in different ways. For instance, if e_{φ} is a neutral element in the calculus of process (8.2), it is not necessary that the same element be also neutral for calculus (8.11). The observer from (8.11) can view e_{φ} as an ordinary element of the space Γ^N. As to g and θ inter-related by equalities (8.15) and (8.16), both observers from (8.2) and (8.25) algebraically regard them in the same way. Equalities (8.25) may serve as an example.

The differential equations bring us to a conclusion that when the algebraic operations (addition, multiplication) and their combinations (differentiation and others) are applied to the elements of the space Γ^N, this leads to the formation of algebraic objects. In passing from one calculus to another, these objects experience transformation (8.15–16), (8.23–25). In particular, (8.21) directly implies that the proper time is an algebraic object.

Let us assume $g = u$ in equality (8.15), where u is a solution of equation (8.3). Then it is not difficult to show that θ is a solution of (8.12), i.e. we have

$$v = \psi^{-1} \circ \varphi\,(u)\,. \tag{8.36}$$

This equality does not mean in any way that the processes described by equations (8.2) and (8.11) are identical. They are simply solutions exhibiting identical algebraic properties in their calculi.

Thus we conclude that when one process passes over to another, the system of calculus and its related algebraic objects undergo transformation, but the space Γ^N remains unchanged. A special mention should be made of one important property of such transformations. If we have a set of processes and pass over from one process to the second one and, next, to the third one and so on until we come to the last process, then we obtain the successive multiplication of transformations of calculus systems. It is not difficult to verify that in the final transformation there are no transformations connected with all intermediate processes and there remains only the transformation connected with the first and the last process.

(14) Let W_N^1, $J_N^1\,(u)$ and $J_N^1\,(v)$ be respectively the spaces of solutions of (8.1), (8.2) and (8.11). $J_N^1\,(u)$ and $J_N^1\,(v)$ can be discrete fiber spaces and the discrete groups acting in them do not necessarily coincide.

Let us represent (8.36) as

$$v = \psi^{-1} \circ \mathring{\varphi} \circ \mathring{\varphi}^{-1} \circ \varphi\,(u)\,, \tag{8.37}$$

where $\mathring{\varphi}$ are the characteristic functions of the equation (8.1) and are defined by (8.5). As has been mentioned several times, $\mathring{\varphi}^{-1} \circ \varphi = \exp \varphi$ is the mapping $J_N^1\,(u) \to W_N^1$ and $\psi^{-1} \circ \mathring{\varphi} : W_N^1 \to J_N^1\,(v)\,$. Then, by virtue of (13), we conclude that (8.36–37) is the mapping $\psi^{-1} \circ \varphi : J_N^1\,(u) \to J_N^1\,(v)$.

9. On the frame of reference and the relativity of processes.

In this section we investigate the relationship existing between various systems of calculus generated by quasilinear partial differential equations. For this, we have to consider a set of various equations of form (1.1). Like in Section *1*, it is assumed that in these equations the vector fields $F\,(y)$ from (1.1) and $\det a_\alpha\,(y)$ can have isolated zeros and infinities. It is also assumed that the dimension N of the system and the number N_0 of independent variables are the same for all equations in the considered set. We call the processes described by these equations the processes of the same class.

By virtue of the results of the preceding paragraph, all equations of the considered set are implicitly assumed to be written in the calculus of the external observer.

(1) Let there exist some dynamic process described by the equation

$$l^\nu \frac{\partial w^k}{\partial x^\nu} = w^k, \qquad (k = 1, ..., N)\,, \tag{9.1}$$

where all l^ν, $(\nu = 1, ..., N_0)$, are different from zero. Suppose that $w_\alpha = w\,(z_\alpha)$, where

$$z_\alpha = \frac{1}{l_\alpha} \alpha_\nu x^\nu, \tag{9.2}$$

and $l_\alpha = \alpha_\nu l^\nu$. But then the plane wave equation takes the form

$$\frac{dw_\alpha^k}{dz_\alpha} = w_\alpha^k, \quad (k = 1, ..., N). \tag{9.3}$$

It is obvious that the characteristic functions of equation (10.3) can be written in the form

$$\mathring{\varphi}_\alpha^k (w_\alpha) = \ln w_\alpha^k. \tag{9.4}$$

A solution of equation (10.1) is represented as

$$\dot{\chi}_\Omega^k = \dot{\sum_{\alpha\in\Omega}} q_\alpha w_\alpha^k,$$

$$\ddot{\chi}_\Omega^k = \ddot{\sum_{\alpha\in\Omega}} \frac{1}{q_\alpha} w_\alpha^k, \quad (k = 1, ..., N). \tag{9.5}$$

As is known, w_α is the irreducible state of process (10.1). An arbitrary state of the same process can be written in form (10.5) (certainly, provided that the set Ω has been chosen appropriately).

Furthermore, it is assumed that, like (1.1), equation (10.1) is written in the system of calculus of some external observer who is inside a certain dynamic process. Suppose now that the other observer is inside the dynamic process (10.1). It is from the position of the latter observer that we will study the process we are interested in.

Let the process be in the α-state, which means that it is described by equation (10.3). By analogy with the preceding section, the invariant definition of the proper time must satisfy the equalities

$$\mathring{\varphi}_\alpha^1 \left(z_\alpha^1, ..., z_\alpha^N\right) = \cdots = \mathring{\varphi}_\alpha^N \left(z_\alpha^1, ..., z_\alpha^N\right), \tag{9.6}$$

where $\mathring{\varphi}_\alpha^k$ is (10.4) or, which is the same,

$$\ln z_\alpha^1 = \cdots = \ln z_\alpha^N.$$

If in these equations, we assume that one of z_α^k is independent and denote it by z_α, then we can write

$$z_\alpha^k = z_\alpha, \quad (k = 1, ..., N). \tag{9.7}$$

Let us return to (4.1). By analogy with (9.6), from (4.1) we can write the invariant definition of $X_\alpha^{k\nu}$:

$$\mathring{\varphi}_\alpha^1 \left(X_\alpha^{1\nu}, ..., X_\alpha^{N\nu}\right) = \cdots = \mathring{\varphi}_\alpha^N \left(X_\alpha^{1\nu}, ..., X_\alpha^{N\nu}\right). \tag{9.8}$$

$X_\alpha^{k\nu}$ are the coordinates of the proper external space, i.e. of the space of independent variables. The values $X_\alpha^{k\nu}$ are interpreted as coordinates of the proper external space from the position of the observer being in the α-state. According to the results of (7) and (9), for the fixed index k the variables $X_\alpha^{k\nu}$ are interpreted as coordinates of the proper external space as the observer sees them through the k-th channel of the process.

Using (9.4), from (4.1) we obtain

$$X_{\alpha}^{k\nu} = x^{\nu}, \qquad (k = 1, ..., N) \tag{9.9}$$

and thereby establish that the coordinates of the external space (the space seen by the external observer) and the coordinates of the proper external space are inter-related when process (9.1) is in the α -state.

Let us consider (4.2). By (9.5), we can write $X^{k\nu} = \dot{\sum}_{\alpha\in\Omega} q_{\alpha} X_{\alpha}^{k\nu}$. Taking into account (9.9), we have

$$X^{k\nu} = \dot{\sum_{\alpha\in\Omega}} q_{\alpha} x^{\nu}. \tag{9.10}$$

This result does not change if we use equality (4.3).

Let us discuss (9.10). Using (3.62) we can write

$$\dot{\sum_{\alpha\in\Omega}} q_{\alpha} x^{\nu} = x^{\nu} \sum_{\alpha\in\Omega} q_{\alpha} = x^{\nu}.$$

We calculate (4.11)

$$\frac{\partial_{\dot{\varkappa}} w}{\partial_{\dot{\varkappa}} X^{\nu}} = \dot{\sum_{\alpha\in\Omega}} q_{\alpha} \frac{\partial w_{\alpha}}{\partial x^{\nu}},$$

or

$$\frac{\partial_{\dot{\varkappa}} w}{\partial_{\dot{\varkappa}} X^{\nu}} = \dot{\sum_{\alpha\in\Omega}} q_{\alpha} \frac{1}{l_{\alpha}} \alpha_{\nu} w_{\alpha},$$

where $l_{\alpha} = \alpha_{\nu} l^{\nu}$. We form (4.13)

$$L^{\nu} = \dot{\sum_{\alpha\in\Omega}} q_{\alpha} l^{\nu} = l^{\nu}.$$

Using (4.7), we calculate the left-hand part of (4.14)

$$\dot{\bigoplus_{\nu}} L^{\nu} \dot{\otimes} \frac{\partial_{\dot{\varkappa}} w}{\partial_{\dot{\varkappa}} X^{\nu}} = \dot{\bigoplus_{\nu}} \sum_{\alpha\in\Omega} q_{\alpha} l^{\nu} \dot{\otimes} \dot{\sum_{\alpha\in\Omega}} q_{\alpha} \frac{\partial w_{\alpha}}{\partial x^{\nu}}.$$

To the right-hand part of this equation we apply operation (3.27_1) and then operation (3.18). As a result we obtain

$$\dot{\bigoplus_{\nu}} L^{\nu} \dot{\otimes} \frac{\partial_{\dot{\varkappa}} w}{\partial_{\dot{\varkappa}} X^{\nu}} = \sum_{\alpha\in\Omega} q_{\alpha} l^{\nu} \frac{\partial w_{\alpha}}{\partial x^{\nu}}, \tag{9.11}$$

where the usual summation is performed over the index ν in the right-hand part. On the other hand, $\dot{\sum}$ is a standard sum. Thus we can write

$$\dot{\sum_{\alpha\in\Omega}} q_{\alpha} l^{\nu} \frac{\partial w_{\alpha}}{\partial x^{\nu}} = l^{\nu} \frac{\partial}{\partial x^{\nu}} \dot{\sum_{\alpha\in\Omega}} q_{\alpha} w_{\alpha} = l^{\nu} \frac{\partial w}{\partial x^{\nu}}. \tag{9.12}$$

Let us calculate

$$\dot{\sum_{\alpha\in\Omega}} q_{\alpha} l^{\nu} \frac{\partial w_{\alpha}}{\partial x^{\nu}}.$$

Since by (9.3)

$$l^{\nu}\frac{\partial w_{\alpha}}{\partial x^{\nu}}=l^{\nu}\frac{1}{l_{\alpha}}\alpha_{\nu}\frac{dw_{\alpha}}{dz_{\alpha}}=w_{\alpha},$$

we obtain

$$\sum_{\alpha\in\Omega}^{\cdot}q_{\alpha}l^{\nu}\frac{\partial w_{\alpha}}{\partial x^{\nu}}=w. \tag{9.13}$$

Equating (9.11) to (9.13), we obtain (4.14). Along with this, (9.12) and (9.3) give equation (9.1). By virtue of this result and equalities (9.7) and (9.9), we conclude that the external observer and the one inside process (9.1) have identical systems of calculus. Hence in the sequel it will be assumed that all equations discussed at the beginning of this paragraph are written in the calculus of equation (9.1).

(2) In the external space Γ^{N_0}, we introduce the coordinate system $\left(x^1,...,x^{N_0}\right)$ obtained on the basis of process (9.1). In the sequel this system will be called the frame of reference from the position of the observer inside process (9.1).

Let us consider the transformation group acting in the space Γ^{N_0}

$$\bar{x}^{\nu}=\xi^{\nu}_{\sigma}x^{\sigma}+\xi^{\nu}, \tag{9.14}$$

where $\det\xi\neq 0$. Along with (9.14), we require that

$$\bar{\alpha}_{\nu}=\tilde{\xi}^{\sigma}_{\nu}\alpha_{\sigma}, \tag{9.15}$$

$$\bar{l}^{\nu}=\xi^{\nu}_{\sigma}l^{\sigma},$$

where ξ and $\tilde{\xi}$, be reciprocal matrices. Then it is not difficult to show that under the action of the linear group (9.14), equation (9.1) remains invariant. We thereby arrive at inertial frames of reference which, for definiteness, will be referred to as inertial frames of reference of process (9.1).

(3) Let us investigate the plane wave equation (1.2). We consider the algebraic objects connected with (1.2) from the position of the observer being inside the process described by equation (1.1).

It is obvious that the phase variable z_{α} plays the role of "time" in equation (1.2) written, like (1.1), in the system of calculus of the observer being inside process (9.1). Then

$$\tau^{k}_{\alpha}=\varphi^{-1k}_{\alpha}\left(\ln z_{\alpha},...,\ln z_{\alpha}\right) \tag{9.16}$$

are the proper "times", where φ_{α} are the characteristic functions of equation (1.2). Equality (9.16) establishes the relation between τ_{α} and z_{α}. Note that in its turn τ_{α} is

$$\varphi^{1}_{\alpha}\left(\tau_{\alpha}\right)=\cdots=\varphi^{N}_{\alpha}\left(\tau_{\alpha}\right). \tag{9.17}$$

In (8.23–24) we replace $\psi\rightarrow\varphi_{\alpha}$, $\varphi\rightarrow\mathring{\varphi}_{\alpha}$, where φ_{α} are the characteristic functions of equation (1.2), and $\mathring{\varphi}_{\alpha}$ is (9.4). Then τ_{α} in the algebra of the observer who is inside process (1.2) can be represented as

$$\tau_{\alpha}=\tilde{L}_{\alpha}\underset{\varphi_{\alpha}}{\odot}A_{\alpha_{\nu}}\underset{\varphi_{\alpha}}{\odot}X^{\nu}_{\alpha}, \tag{9.18}$$

where the $\underset{\varphi_{\alpha}}{\dot{+}}$-summation is performed over the index ν from 1 to N_0 and

$$\tilde{L}_{\alpha}=\varphi^{-1}_{\alpha}\left(\ln\frac{1}{l_{\alpha}},...,\ln\frac{1}{l_{\alpha}}\right),$$

$$A_{\alpha_\nu} = \varphi_\alpha^{-1}\left(\ln\alpha_\nu, ..., \ln\alpha_\nu\right), \tag{9.19}$$
$$X_\alpha^\nu = \varphi_\alpha^{-1}\left(\ln x^\nu, ..., \ln x^\nu\right).$$

As follows from (9.18), in the algebra generated by equation (1.2) τ_α has an algebraic structure analogous to

$$z_\alpha = \frac{1}{l_\alpha}\alpha_\nu x^\nu$$

in a standard numerical field.

Now let us consider the expression

$$L_\alpha = A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} L_\alpha^\nu, \tag{9.20}$$

where

$$L_\alpha^\nu = \varphi_\alpha^{-1}\left(\ln l^\nu, ..., \ln l^\nu\right), \tag{9.21}$$

and the $\underset{\varphi_\alpha}{\dot{+}}$-summation is performed over the index ν in (9.20). One can easily verify that the equality

$$L_\alpha \underset{\varphi_\alpha}{\odot} \tilde{L}_\alpha = E_\alpha \tag{9.22}$$

is valid. Using the mathematical means presented in §7 of [2], we write the φ_α-differential of the function u_α:

$$d_{\varphi_\alpha}u_\alpha = \frac{d_{\varphi_\alpha}u_\alpha}{d_{\varphi_\alpha}\tau_\alpha} \underset{\varphi_\alpha}{\odot} d_{\varphi_\alpha}\tau_\alpha. \tag{9.23}$$

From (9.18) we find

$$d_{\varphi_\alpha}\tau_\alpha = \tilde{L}_\alpha \underset{\varphi_\alpha}{\odot} A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} d_{\varphi_\alpha}X_\alpha^\nu. \tag{9.24}$$

The substitution of (9.24) into (9.23) gives

$$d_{\varphi_\alpha}u_\alpha = \tilde{L}_\alpha \underset{\varphi_\alpha}{\odot} A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} \frac{d_{\varphi_\alpha}u_\alpha}{d_{\varphi_\alpha}\tau_\alpha} \underset{\varphi_\alpha}{\odot} d_{\varphi_\alpha}X_\alpha^\nu. \tag{9.25}$$

By the results of Section *4*, from (9.25) we conclude that

$$\frac{\partial_{\varphi_\alpha}u_\alpha}{\partial_{\varphi_\alpha}X_\alpha^\nu} = \tilde{L}_\alpha \underset{\varphi_\alpha}{\odot} A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} \frac{d_{\varphi_\alpha}u_\alpha}{d_{\varphi_\alpha}\tau_\alpha}. \tag{9.26}$$

Note that (9.26) is a full analogue of the equality

$$\frac{\partial w_\alpha}{\partial x^\nu} = \frac{1}{l_\alpha}\alpha_\nu\frac{dw_\alpha}{dz_\alpha}$$

written in the standard calculus, i.e. in the calculus of equation (9.3).

As is shown in [2], in its respective calculus, equation (1.2) has the form

$$\frac{d_{\varphi_\alpha}u_\alpha}{d_{\varphi_\alpha}\tau_\alpha} = u_\alpha, \tag{9.27}$$

where τ_α plays the role of a proper "time".

Consider the last equality in (9.19). This equality immediately implies the invariant definition of X_α^ν

$$\varphi_\alpha^1\left(X_\alpha^{1\nu},...,X_\alpha^{N\nu}\right)=\cdots=\varphi_\alpha^N\left(X_\alpha^{1\nu},...,X_\alpha^{N\nu}\right),\quad(\nu=1,...,N_0)\,. \tag{9.28}$$

It is obvious that for the fixed ν, only one variable among the variables $X_\alpha^{1\nu},...,X_\alpha^{N\nu}$ is independent, and all other variables are expressed through it. This is absolutely analogous to what we have discussed in (2) of Section *6*, in relation with the interpretation of the proper "time" τ^k. Though, as different from the other variables $X_\alpha^{k\nu}$, the independent variable that has arisen among them surely does not depend on α, we all the same write it with the subscript α, thus emphasizing that X_α^ν are the coordinates of the proper external space when process (1.1) is in state (1.2).

(4) The variables x^ν in (9.19) do not depend on α. Then, along with (9.28), we have the equality

$$\varphi_\alpha\left(X_\alpha^\nu\right)=\varphi_\beta\left(X_\beta^\nu\right) \tag{9.29}$$

for any $\alpha,\beta\in\Omega$.

If X_α^ν is a solution of equation (9.28), then (9.29) immediately implies that X_β^ν is a solution of an analogous equation to (9.28), where α is replaced by the variable β.

(5) Let us go back to (4.2). Obviously, (4.2) is a solution of the equation

$$\sum_{\alpha\in\Omega}q_\alpha\exp\left[\varphi_\alpha^k\left(X_\alpha^\nu\right)-\varphi_\alpha^k\left(X^\nu\right)\right]=1,\quad(k=1,\ldots,N\ ),\quad(\nu=1,\ldots,N_0)\,. \tag{9.30}$$

By virtue of (9.29), equation (9.30) can be rewritten as

$$\sum_{\alpha\in\Omega}q_\alpha\exp\left[-\varphi_\alpha^k\left(X^\nu\right)\right]=\exp\left[-\varphi_\beta^k\left(X_\beta^\nu\right)\right] \tag{9.31}$$

for any $\beta\in\Omega$.

From (9.31), (9.29) and (9.28) we conclude that X^ν are solutions of the equation

$$\sum_{\alpha\in\Omega}q_\alpha\exp\left[-\varphi_\alpha^1\left(X^\nu\right)\right]=\cdots=\sum_{\alpha\in\Omega}q_\alpha\exp\left[-\varphi_\alpha^N\left(X^\nu\right)\right],\quad(\nu=1,...,N_0)\,. \tag{9.32}$$

By analogy with (9.28), we call (9.32) the invariant equation for defining the variables X^ν.

(6) Let us fix some point α_0 of the set Ω and assume that for $\alpha=\alpha_0$, the solution of equation (9.28) is $X_{\alpha_0}^{k\nu}$ which we denote by $X_0^{k\nu}$, i.e.

$$X_{\alpha_0}^{k\nu}=X_0^{k\nu},\quad(k=1,\ldots,N)\,,\quad(\nu=1,\ldots,N_0)\,. \tag{9.33}$$

As has been mentioned in (3), among $X_\alpha^{k\nu}$ there are N_0 independent variables, while the other variables X_0^ν are expressed through them.

Set $\beta=\alpha_0$ in the right-hand part of (9.29). This equality with (9.33) taken into account gives

$$X^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \circ \varphi_{\alpha_0} \left(X^{\nu}_0 \right), \quad (\nu = 1, \ldots, N_0). \tag{9.34}$$

By the properties of (9.29) and (9.33), from equality (9.31) it follows that X^{ν} has the form

$$X^{\nu} = \dot{\chi}_{\Omega} \left(..., X^{\nu}_0, ..., X^{\nu}_0, ... \right). \tag{9.35}$$

Therefore we come to a conclusion that if X^{ν}_0 is defined from (9.28) (with the fixed α), then the rest of the variables X^{ν}_{α} and X^{ν} are uniquely defined from equalities (9.34) and (9.35), respectively. Alternatively, X^{ν} can be defined from the invariant equation (9.32). Then from equality (9.31) we immediately obtain the other variables X^{ν}_{α} (with α running through Ω).

We concluded (1) by stating that x^{ν} are the coordinates of the proper external space as represented by the observer from process (9.1). Equalities (9.34) and (9.35) are complete analogues of equalities (9.9) and (9.10), respectively. Therefore X^{ν}_0 can be interpreted as the coordinates of the frame of reference of the proper external space as represented by the observer, who is inside process (1.2), when $\alpha = \alpha_0$.

(7) Let us consider the cofactor $A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha}$ in (9.18). As different from (9.18), for the time being we do not perform the φ_α-summation over ν in $A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha}$. By (6.12) in [2], we can write

$$A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \left(\varphi_\alpha \left(A_{\alpha_\nu} \right) + \varphi_\alpha \left(X^{\nu}_{\alpha} \right) \right). \tag{9.36}$$

We introduce the new variables $\mathring{A}^{k}_{\alpha_\nu}$ related to α_ν by the equality

$$\mathring{A}_{\alpha_\nu} = \varphi^{-1}_{\alpha_0} \left(\ln \alpha_\nu \right), \tag{9.37}$$

where the point $\alpha_0 \in \Omega$ is fixed. From (9.19) it immediately follows that $\varphi_\alpha \left(A_{\alpha_\nu} \right) = \ln \alpha_\nu$. Then in view of (9.37) we have

$$\varphi_\alpha \left(A_{\alpha_\nu} \right) = \varphi_{\alpha_0} \left(\mathring{A}_{\alpha_\nu} \right).$$

By (9.34) and (9.37), product (9.36) can be rewritten as

$$A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \left(\varphi_{\alpha_0} \left(\mathring{A}_{\alpha_\nu} \right) + \varphi_{\alpha_0} \left(X^{\nu}_0 \right) \right).$$

Again using (6.12) [2], we finally obtain

$$A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \circ \varphi_{\alpha_0} \left(\mathring{A}_{\alpha_\nu} \underset{\varphi_{\alpha_0}}{\odot} X^{\nu}_0 \right). \tag{9.38}$$

Applying rule (8.23), we perform the operation of summation in (9.38) over ν from 1 to N_0 :

$$\underset{\nu,\ \varphi_\alpha}{\dot{+}} A_{\alpha_\nu} \underset{\varphi_\alpha}{\odot} X^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \circ \varphi_{\alpha_0} \left(\underset{\nu,\ \varphi_{\alpha_0}}{\dot{+}} \mathring{A}_{\alpha_\nu} \underset{\varphi_{\alpha_0}}{\odot} X^{\nu}_0 \right). \tag{9.39}$$

In the sequel the notation $\underset{\nu,\ \varphi}{\dot{+}}$ will be omitted.

Since l^{ν} does not depend on α, L^{ν}_{α} from (9.21) satisfies the same equalities as X^{ν}_{α}, i.e. (9.28) and (9.29). Let us introduce the notation

$$L^{\nu}_{\alpha_0} = L^{\nu}_{0}, \tag{9.40}$$

where $L^{\nu}_{0} = \varphi^{-1}_{\alpha}\left(\ln l^{\nu}\right)$. Thus we can write

$$L^{\nu}_{\alpha} = \varphi^{-1}_{\alpha} \circ \varphi_{\alpha_0}\left(L^{\nu}_{0}\right).$$

Applying (8.23–24) and the above reasoning, we obtain the equality

$$\tau_{\alpha} = \varphi^{-1}_{\alpha} \circ \varphi_{\alpha_0}\left(\mathring{\tau}_{\alpha}\right), \tag{9.41}$$

where

$$\mathring{\tau}_{\alpha} = \tilde{\mathring{L}}_{\alpha} \underset{\varphi_{\alpha_0}}{\odot} \mathring{A}_{\alpha_\nu} \underset{\varphi_{\alpha_0}}{\odot} X^{\nu}_{0}. \tag{9.42}$$

Here the φ_{α_0}-summation is performed over ν and

$$\tilde{\mathring{L}}_{\alpha} \underset{\varphi_{\alpha_0}}{\odot} \mathring{L}_{\alpha} = E_0$$

where E_0 is the unit element of the $\underset{\varphi_{\alpha_0}}{\odot}$-product (i.e. it satisfies the equality $\varphi_{\alpha_0}\left(E_0\right) = 0$). By analogy with $l_{\alpha} = \alpha_{\nu} l^{\nu}$, the object $\mathring{L}_{\alpha}$ has the form

$$\mathring{L}_{\alpha} = \mathring{A}_{\alpha_\nu} \underset{\varphi_{\alpha_0}}{\odot} L^{\nu}_{0}, \tag{9.43}$$

where the φ_{α_0}-summation is performed over ν. The question arises: what do α_{ν} and $\mathring{A}_{\alpha_\nu}$ have in common? The observer who is inside process (9.1) describes process (1.1) in the calculus of process (9.1). He introduces the parameters α_{ν} and obtains the plane wave equation (1.2). On the other hand, the observer, who is inside process (1.2) which is in the state of $\alpha = \alpha_0$, can introduce the parameters $\mathring{A}_{\alpha_\nu}$ in his calculus. The transition from α_{ν} to $\mathring{A}_{\alpha_\nu}$ is realized by means of equality (9.37).

Thus we come to a conclusion that the variables τ_{α}, $\mathring{\tau}_{\alpha}$ defined from (9.18) and (9.42), respectively, are identical by their algebraic structure with the phase variable

$$z_{\alpha} = \frac{1}{l_{\alpha}} \alpha_{\nu} x^{\nu},$$

where $l_{\alpha} = \alpha_{\nu} l^{\nu}$. Moreover, equality (9.16) is fulfilled and, in particular,

$$\mathring{\tau}_{\alpha} = \varphi^{-1}_{\alpha_0}\left(\ln z_{\alpha}\right). \tag{9.44}$$

These equalities immediately imply (9.41).

(8) Let α, $\beta \in \Omega$ and $\alpha \neq \beta$. Then, along with (1.2), we have

$$a_{\beta}\left(u_{\beta}\right) \frac{du_{\beta}}{dz_{\beta}} = F\left(u_{\beta}\right). \tag{9.45}$$

By the reasoning of the preceding paragraph we have

$$u_{\beta} = \varphi^{-1}_{\beta} \circ \varphi_{\alpha}\left(u_{\alpha}\right). \tag{9.46}$$

This equality can be written in the form $\varphi_\beta(u_\beta) = \varphi_\alpha(u_\alpha)$. Since (1.5) is valid, we obtain

$$bz_\beta + c_\beta = bz_\alpha + c_\alpha.$$

Since x^ν are independent variables, and c_α and c_β are independent of x^ν, we conclude that

$$\frac{1}{l_\beta}\beta_\nu = \frac{1}{l_\alpha}\alpha_\nu, \quad c_\beta = c_\alpha.$$

Using these equalities, we easily verify that, under assumption (9.46), equations (1.2) and (9.45) identically coincide and therefore $u_\beta = u_\alpha$. As should have been expected, we come to a conclusion that u_α and u_β are independent solutions of equation (1.1) for $\beta \neq \alpha$ and their values cannot be reduced to each other. In other words, we are faced with the irreducible states of process (1.1).

(9) Let the first observer be in process (9.1) and the second one in (1.1). Assume that these processes are in the irreducible α-states. As shown in (1), equations (9.1) and (1.1) are written in the calculus of the first observer. As follows from (2), (9.9) is the frame of reference obtained on the basis of process (9.1).

Let us consider group (9.14) acting in the external space Γ^{N_0}. If w^k are treated as scalar values, then for (9.14–15) equation (9.1) is invariant with respect to the considered group. Since group (9.14) is linear and the linear equation (9.1) remains invariant, from the standpoint of process (9.1), system (9.9) should be interpreted as an invariant frame of reference. (9.14) is the transition from one inertial frame of reference to another inertial frame.

Further we assume that under the action of group (9.14) equation (1.1) remains invariant. Then there arises the rule of transformation of solutions $u(x)$ of equation (1.1), which brings us to the representation of group (9.14). The group properties of differential equations are well known and therefore we do not discuss them here. We only note that the invariance of equation (1.1) with respect to group (9.14) implies the invariance not only of (1.2), but of (1.6) too.

Under the action of (9.14), equality (4.1) takes the form

$$\bar{X}^{\nu k}_{\bar{\alpha}} = \bar{\varphi}^{-1k}_{\bar{\alpha}}\left(\ln \bar{x}^\nu, ..., \ln \bar{x}^\nu\right),$$

where $\bar{\varphi}_{\bar{\alpha}}$ is the transformed solution of equation (1.6). Using (9.14), we obtain

$$\bar{X}^{\nu k}_{\bar{\alpha}} = \bar{\varphi}^{-1k}_{\bar{\alpha}}\left(\ln\left(\xi^\nu_\sigma x^\sigma + \xi^\nu\right), ..., \ln\left(\xi^\nu_\sigma x^\sigma + \xi^\nu\right)\right). \tag{9.47}$$

Since $\bar{\varphi}_{\bar{\alpha}}$ are the characteristic functions of the transformed equation (1.2), we can use equalities (8.23–24). For this, in (8.23–24) we make the replacement $\psi \to \bar{\varphi}_{\bar{\alpha}}$, and $\varphi \to \ln$. Then (9.47) takes the form

$$\bar{X}^\nu_{\bar{\alpha}} = \eta^\nu_{\bar{\alpha}\sigma} \underset{\bar{\varphi}_{\bar{\alpha}}}{\odot} X^\sigma_{\bar{\alpha}} \underset{\bar{\varphi}_{\bar{\alpha}}}{\dot{+}} \eta^\nu_{\bar{\alpha}}, \tag{9.48}$$

where

$$\begin{aligned} X^\sigma_{\bar{\alpha}} &= \bar{\varphi}^{-1}_{\bar{\alpha}}\left(\ln x^\nu\right), \\ \eta^\nu_{\bar{\alpha}\sigma} &= \bar{\varphi}^{-1}_{\bar{\alpha}}\left(\ln \xi^\nu_\sigma\right), \\ \eta^\nu_{\bar{\alpha}} &= \bar{\varphi}^{-1}_{\bar{\alpha}}\left(\ln \xi^\nu\right). \end{aligned} \tag{9.49}$$

In (9.48), the $\underset{\bar{\varphi}_{\bar{\alpha}}}{\dot{+}}$-summation is performed over σ from 1 to N_0.

It is obvious that transformation (9.48) in the calculus of the second observer is the linear transformation acting on X^ν. Using the terminology of Section *8*, the transformation of the frame of reference is an algebraic object, since in the calculus of the first observer, to the linear group (9.14) there corresponds the linear group (9.48) in the calculus of the second observer.

Under the action of the linear group (9.48), equations (9.1) and (1.1) remain, as has been mentioned above, invariant. This means that equation (1.1), which in the calculus of the second observer has the linear form (4.14), preserves its form under the action of group (9.48). Then the frame of reference X_α^ν arising from process (1.1) should be interpreted as an inertial frame of reference from the standpoint of the second observer. As to (9.48), it is the passage from one inertial frame of reference (X^ν) to the other frame $\left(\bar{X}^\nu\right)$.

Thus each process described by autonomous systems of differential equations generates its own set of inertial frames, in which we have the acting linear group. However, the inertial frame of one process should not necessarily be the inertial frame of the other process. Between the inertial frames of various processes there establishes a noninertial, i.e. nonlinear relation. An example is (4.1) which transforms the inertial frame (x_α^ν) of process (9.1) to the inertial frame (X_α^ν) of process (1.1).

(10) Now, along with the process let us consider its antiprocess. As mentioned in (1), the characteristic functions φ_α of equation (9.1) are (9.4). But then the algebraic operations of a -conjugation and $\mathring{\varphi}_\alpha$-conjugation identically coincide. Hence the antiprocess of process (9.1) is described by equation (5.32) as follows:

$$\hat{l}^{\hat{\nu}} \frac{\hat{\partial} \hat{w}^k}{\hat{\partial} \hat{x}^{\hat{\nu}}} = \hat{w}^k, \qquad (k = 1, \dots, N), \tag{9.50}$$

where the alternative summation is performed over the index ν from 1 to N_0.

From (5.33) we obtain

$$\begin{aligned} \hat{x}^\nu &= 1/x^\nu, \\ \hat{l}^\nu &= 1/l^\nu, \\ \hat{w}^k &= 1/w^k. \end{aligned} \tag{9.51}$$

Under the action of group (9.14), $\hat{x}^\nu$ transforms by the rule

$$\bar{\hat{x}}^\nu = \hat{\xi}^\nu_{\hat{\sigma}} \hat{x}^{\hat{\sigma}} \ddot{+} \hat{\xi}^\nu, \tag{9.52}$$

where $\hat{\xi}^\nu_\sigma = 1/\xi^\nu_\sigma$, $\hat{\xi}^\nu = 1/\xi^\nu$. Note that (9.52) is derived from (9.14) by the operation of a-conjugation. Therefore $\hat{x}^\nu$'s in antiprocess (9.50) form the inertial frame of reference. On the other hand, from (9.51) we conclude that (x) and $(\hat{x})$ are the noninertial frames of reference for processes (9.50) and (9.1), respectively.

Let us consider process (1.1). In its respective calculus this process is written in form (4.14). But each process has its antiprocess. In our case the equation of the antiprocess in its own calculus has form (5.88). The variables $\hat{X}^\nu$ in (5.88) are defined from (5.85), where $\hat{x}^\nu$ and x^ν are related by (9.51).

As has been shown, the variables X^ν in (4.14) are defined from (4.2). Then $\hat{X}^\nu$ can be written as

$$\hat{X}^\nu = \ddot{\chi}_\Omega \left(..., \hat{X}^\nu_\alpha, ... \right), \tag{9.53}$$

where X_α^ν and $\hat{X}_\alpha^\nu$ are related by

$$\varphi_\alpha \left(X_\alpha^\nu \right) + \varphi_\alpha \left(\hat{X}_\alpha^\nu \right) = 0. \tag{9.54}$$

Using (4.1) and (9.51), from (9.54) we readily have

$$\hat{X}_\alpha^{\nu k} = \varphi_\alpha^{\nu k} \left(\ln \hat{x}^\nu, ..., \ln \hat{x}^\nu \right). \tag{9.55}$$

Applying a reasoning analogous to that we have used in deriving (9.48), from (9.52) and (9.55) we obtain

$$\bar{\hat{X}}^{\nu}_{\bar{\alpha}} = \hat{\eta}^{\nu}_{\bar{\alpha}\hat{\sigma}} \underset{\bar{\varphi}_{\bar{\alpha}}}{\odot} \hat{X}^{\hat{\sigma}}_{\bar{\alpha}} \underset{\bar{\varphi}_{\bar{\alpha}}}{\ddot{+}} \hat{\eta}^{\nu}_{\bar{\alpha}}, \tag{9.56}$$

where

$$\hat{\eta}^{\sigma}_{\bar{\alpha}\nu} = \bar{\varphi}^{-1}_{\bar{\alpha}} \left(\ln \hat{\xi}^{\nu}_{\sigma} \right), \tag{9.57}$$

$$\hat{\eta}^{\nu}_{\bar{\alpha}} = \bar{\varphi}^{-1}_{\bar{\alpha}} \left(\ln \hat{\xi}^{\nu} \right).$$

In (9.56) the alternative $\bar{\varphi}_{\bar{\alpha}}$-summation is performed over the index σ. Expressions (9.48–49) and (9.56–57) are the φ-conjugate values. This conclusion agrees with the fact that (4.14) and (5.88) are φ-conjugate equations.

Thus the antiprocess has its own inertial frames of reference which do not coincide with those of the process. However the inertial frames of the process and its antiprocess are φ-conjugate.

10. On the matrix representation

In order to have a complete algebraic picture we wish to draw attention to some problems which have remained outside the scope of our investigation.

(1) Let us go back to equation (2.1). As has already been noted, the characteristic functions have the form

$$\varphi^{k}(u) = b^{k} t + c^{k}, \tag{10.1}$$

and satisfy the equation

$$\frac{\partial \varphi^{k}(u)}{\partial u^{n}} F^{n}(u) = b^{k}. \tag{10.2}$$

Recall (§1 [1]) that c^k in equality (10.1) are the integration constants of (2.1).

It is obvious that if $\varphi(u)$ is some solution of (10.2), then the functions u^k defined from (10.1) are general solutions of (2.1). Also note that if the vector $b = 0$, then from (10.1) we conclude that arbitrary solutions of (2.1) are constant values. This means that $F(u)$ is identically zero and the characteristic functions take the form $\varphi^{k}(u) = u^{k}$. Thus in the sequel it will be assumed that $F(u)$ is a nonzero vector and therefore in (10.1-2) the vector $b \neq 0$.

By the reasoning of §2 in [1], the components b^k of a nonzero vector b are, generally speaking, arbitrary values. To two different vectors b_1 and b_2 there correspond different characteristic functions $\varphi_1(u)$ and $\varphi_2(u)$ defined as solutions of (10.2). In other words, the functions $\varphi(u)$ may have different representations for one and the same equation (2.1).

(2) Let us now consider the mapping

$$w^{k} = \exp \varphi^{k}(u), \tag{10.3}$$
$$(k = 1, ..., N)$$

which plays a fundamental role in the algebraic theory of differential equations. From (10.1) it immediately follows that w is a solution of the equation

$$\frac{dw}{dt} = \mathring{A} w, \tag{10.4}$$

where the matrix $\mathring{A}$ has the form

$$\mathring{A} = diag\left(b^{1}, ..., b^{N}\right). \tag{10.5}$$

It is obvious that the processes described by equations (2.1) and (10.4) belong to the same class.

In (10.4) we transform $w \to Sw$, where S is an arbitrary nonsingular matrix. Then the matrix $\mathring{A}$ transforms by the rule

$$A = S\mathring{A}S^{-1}. \tag{10.6}$$

Therefore, by virtue of mapping (10.3) and equation (10.4), the components b^k figuring in (10.1–2) are the elements of the diagonal matrix. Then from equality (10.1) we conclude that it is not only b^k that are the elements of the diagonal matrix $\mathring{A}$, but φ^k and c^k also form the diagonal matrices

$$\mathring{\Phi} = diag\left(\varphi^1, ..., \varphi^N\right), \tag{10.7}$$

$$\mathring{C} = diag\left(c^1, ..., c^N\right).$$

Equalities (10.3) and (10.7) immediately imply that, along with $\mathring{A}$, $\mathring{\Phi}$ and $\mathring{C}$, the solutions w of equation (10.4) are also diagonal matrices. Then using (10.6) we obtain the transformations

$$\Phi = S\mathring{\Phi}S^{-1}, \tag{10.8}$$

$$C = S\mathring{C}S^{-1},$$

$$W = SwS^{-1}.$$

It is appropriate to recall that in [1] we have considered the linear transformation (2.5) [1] of the objects $\varphi(u)$ and b in their vector interpretation. As a result, we have obtained the nonlinear transformation (2.6) [1] of w. However, in the case of the matrix representation, the linear transformation (10.6) of object (10.5) brings us to the same kind of linear transformations (10.8) of the objects Φ, C and W. This means that, as different from the vector representation, the matrix representation of the considered objects is intrenally coordinated.

Let the observer be inside the process described by (10.4). The latter equation generates the calculus by means of which the observer carries out his investigation. In studying process (2.1) from the standpoint of (10.4), the observer is forced to admit that the objects A, Φ and C should be interpreted as matrix ones. He thereby comes to a conclusion that the solution w of equation (10.4) is also a matrix object. The second observer, who is inside the process described by equation (2.1), reveals in his calculus that equation (2.1) takes the linear form (7.31) [2]. In studying process (10.4) from the standpoint of (2.1), objects similar to those arisen in (10.1-4) have the matrix representation in the calculus of equation (2.1). A solution of equation (7.31) [2] or, which is the same, of equation (2.1) is also a matrix object. It is obvious that in (2.1) the passage from the vector representation to the matrix one is not an easy problem, since it is rooted in the algebraic crannies of the equations. So, at the present stage of our investigation we are not concerned with it.

Thus, in establishing the algebraic relation between various differential equations of the same class, we find that their solutions begin to manifest the matrix properties.

(3) Let us dwell on the matrix representation of solutions and investigate the algebraic regularities arising in this context. Assume that the observer is in some process and the dynamic equation of this process is represented as follows

$$\frac{dw}{dt} = w, \tag{10.9}$$

where w is a square matrix of order N.

It is obvious that a general solution of equation (10.9) can be written as $w = \exp 1_N t \cdot c$, where 1_N is the unit N-matrix. Hence we easily obtain the rule of addition in the space of solutions

$$w_1 \dot{+} w_2 = w_1 + w_2, \tag{10.10}$$

$$w_1 \ddot{+} w_2 = \left(w_1^{-1} + w_2^{-1}\right)^{-1}.$$

These operations are commutative and associative.

The neutral elements are the zero N-matrix e_0 and the N-matrix h_0, all elements of which are equal to ∞. The following equalities are valid:

$$w \dot{+} e_0 = w,$$

$$w \dot{+} h_0 = h_0, \tag{10.11}$$

$$w \ddot{+} e_0 = e_0,$$

$$w \ddot{+} h_0 = w.$$

Along with (10.10), we introduce the operation of multiplication of two arbitrary solutions w_1 and w_2 in the form of matrix multiplication

$$(w_1 \cdot w_2)^k_n = w_1{}^k{}_i w_2{}^i{}_n, \tag{10.12}$$
$$(k,\ n = 1, ..., N).$$

It is not difficult to show that operations (10.10) and (10.12) have distributive properties.

Thus we obtain an associative algebraic body [5] which is double since it consists of two alternative bodies.

It might seem that we apply operation (10.12) to the considered equation from the outside, but this is not so. If w is a solution of equation (10.9), then the matrix products Aw and wA, where A is an arbitrary constant N-matrix, will be a solution. This means that the matrix multiplication rule already exists in the equation itself.

All these facts stimulate us to introduce the space $\Gamma_0^{N\times N}$ whose elements are N-matrices and where operations (10.10–12) act. These operations allow the observer from process (10.9) to construct his own mathematical analysis.

Now let us introduce the space $\hat{\Gamma}_0^{N\times N}$ whose arbitrary elements $\hat{A}$, $\hat{B}$, $\hat{C}$ are related to A, B, $C \in \Gamma_0^{N\times N}$ by equalities (5.2). In other words, the elements of the spaces $\Gamma_0^{N\times N}$ and $\hat{\Gamma}_0^{N\times N}$ are a-conjugate. Omitting details, we wish only to note that the following equalities are true:

$$\left(A \dot{+} B\right)^{\hat{}} = \hat{A} \ddot{+} \hat{B},$$

$$\left(A \ddot{+} B\right)^{\hat{}} = \hat{A} \dot{+} \hat{B},$$

$$(A \cdot B)^{\hat{}} = \hat{A} \hat{\cdot} \hat{B},$$

$$(e_0)^{\hat{}} = h_0,$$

$$(h_0)^{\hat{}} = e_0,$$

where $\hat{\cdot}$ denotes the alternative matrix multiplication (5.3).

After applying the operation of a-conjugation, equation (10.9) transforms to the antiprocess equation

$$\frac{\hat{d}\hat{w}}{\hat{d}\hat{t}} = \hat{w},$$

where $t \cdot \hat{t} = 1$.

Now let us consider the other process which the observer studies from the standpoint of his process (10.9). Assume that in the observer's system of calculus the equation of the processes studied by the observer has the form

$$\frac{du_n^k}{dt} = F_n^k\left(u_1^1, u_2^1, ..., u_N^N\right), \tag{10.13}$$

where the functions F_n^k are defined on the entire N^2-dimensional Euclidean space $\Gamma_1^{N\times N}$. This process belongs to the same class as (10.9). Let $\varphi_n^k(u)$ be the characteristic functions of (10.13) equation. Then we can write

$$\varphi_n^k(u) = A_n^k t + c_n^k,$$

where c_n^k are the constants of integration. The matrix A is assumed to be nonsingular. Let us multiply this equality by A^{-1}. Without introducing the new notation, in the matrix form we obtain

$$\varphi(u) = 1_N t + c. \tag{10.14}$$

For simplicity, it is assumed that for arbitrary N-matrices $y \in \Gamma_1^{N\times N}$, the matrix $\varphi(y)$ is equivalent to a diagonal matrix.

Let us form the matrix mapping

$$\exp\varphi : \Gamma_1^{N\times N} \to \Gamma_0^{N\times N}. \tag{10.15}$$

Since φ is equivalent to a diagonal matrix, the inverse mapping to (10.15) has the form

$$\varphi^{-1}(\ln) : \Gamma_0^{N\times N} \to \Gamma_1^{N\times N}. \tag{10.16}$$

Speaking in general, under these mappings, $\Gamma_1^{N\times N}$ may look like a discrete fiber space. It is obvious that mappings (10.15) and (10.16) transform the spaces of solutions of equations (10.9) and (10.13) to each other.

Therefore, using (10.10–12) we can introduce the following algebraic operations in the $\Gamma_1^{N\times N}$:

$$\begin{gathered} y_1 \underset{\varphi}{\dot{+}} y_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(y_1) \dot{+} \exp\varphi(y_2)\right]\right), \\ y_1 \underset{\varphi}{\ddot{+}} y_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(y_1) \ddot{+} \exp\varphi(y_2)\right]\right), \\ y_1 \underset{\varphi}{\odot} y_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(y_1) \cdot \exp\varphi(y_2)\right]\right). \end{gathered} \tag{10.17}$$

Without going into details, we set

$$e = \varphi^{-1}(\ln e_0), \; h = \varphi^{-1}(\ln h_0), \; E = \varphi^{-1}(\ln 1_N). \tag{10.18}$$

From (10.11) and (10.17) we find:

$$\begin{gathered} y \underset{\varphi}{\dot{+}} e = y, \\ y \underset{\varphi}{\dot{+}} h = h, \\ y \underset{\varphi}{\ddot{+}} e = e, \\ y \underset{\varphi}{\ddot{+}} h = y, \\ E \underset{\varphi}{\odot} y = y \underset{\varphi}{\odot} E = y. \end{gathered} \tag{10.19}$$

Thus operations (10.17) form an associative double algebraic body acting in the space $\Gamma_1^{N\times N}$.

If each discrete fiber from $\Gamma_1^{N\times N}$ is assumed to be one element, then from (10.15–16) follows an isomorphism between the double bodies (10.10–12) and (10.17–19).

Two matrices y and $\overset{\varphi}{\hat{y}}$ are assumed to be φ-conjugate if they satisfy the equality

$$y \underset{\varphi}{\odot} \overset{\varphi}{\hat{y}} = E.$$

which, in view of (10.17) and (10.18), immediately implies

$$\varphi(y) + \varphi\left(\overset{\varphi}{\hat{y}}\right) = 0. \tag{10.20}$$

The latter equality is analogous to (5.55). We thereby come to a conclusion that if u describes the process, then $\overset{\varphi}{\hat{u}}$ describes its antiprocess. Moreover, the characteristic functions of these process coincide.

Suppose there is one more process along with (10.13). The equation of this process from the position of the observer who is in (10.9), has the following matrix form:

$$\frac{dv}{dt} = f(v), \tag{10.21}$$

where v and f are N-matrices.

Let $\psi(z)$ be the characteristic function of process (10.21), where the N-matrix $z \in \Gamma_2^{N\times N}$. By analogy with (10.14) the solutions of (10.21) satisfy the equality $\psi(v) = 1_N t + \tilde{c}$. Omiting the detailed discussion, we write: $\exp\varphi(u) = w = \exp\psi(v)$. From this immediately follows:

$$\psi(v) = \varphi(u).$$

Then we can form the mapping

$$\psi^{-1} \circ \varphi : \Gamma_1^{N\times N} \to \Gamma_2^{N\times N}. \tag{10.22}$$

The conclusions obtained in (13) of Section *9*, completely apply to mapping (10.22).

At this stage we restrict ourselves to the above mentioned investigation.

(4) Let a linear homogeneous system of the form

$$\frac{du}{dt} = Au, \tag{10.23}$$

be given, where A is a square N-matrix. By virtue of the above reasoning it can be assumed that a solution of (10.23) is a nonsingular N-matrix. Obviously, the general solution of the equation (10.23) is $u = \exp At \cdot C$, where C is an arbitrary square matrix. If (10.23) describes a certain process, then the columns of the non-singular matrix u represent the irreducible states of this process. By the appropriate selection of constants of integration the given concrete evolution of the process can occur in any one of these well-defined states. Note that the process cannot pass from one state to another without external disturbance.

(5) In [1], using the algebraic properties of differential equations, we discussed the questions connected with the metric of a space. We want now to return to these questions in the context of the matrix representation of solutions.

We can write equation (10.23) in the form

$$du \cdot u^{-1} = Adt. \tag{10.24}$$

For the Hermit-conjugate equation we obtain

$$u^{+\ -1} du^{+} = A^{+} dt. \tag{10.25}$$

On multiplying matrices (10.24) and (10.25), and we form the matrix trace

$$sp\left[u^{+\ -1} du^{+} du\ u^{-1}\right] = sp\left(A^{+}A\right) dt^{2}.$$

By analogy with [1], by the latter equality we introduce the metric

$$ds^{2} = dt^{2} - 2H sp\left[\left(u^{+}\ u\right)^{-1} du^{+} du\ \right], \tag{10.26}$$

where H is a constant of dimension t^2.

From (10.26) we shall construct the action integral

$$S = \int_{t_1}^{t_2} L\left(t\right) dt,$$

where L is the Lagvangian

$$L = \sqrt{1 - 2H sp\left[\left(u^{+}\ u\right)^{-1} \frac{du^{+}}{dt}\frac{du}{dt}\ \right]}. \tag{10.27}$$

Applying the variational principle, we obtain the Euler equation. From the latter equation we easily obtain the preserving integrals

$$\frac{1}{L}\left(u^{+}\ u\right)^{-1} \frac{du^{+}}{dt} u = B, \tag{10.28}$$

$$\frac{1}{L} u^{+} \frac{du}{dt}\left(u^{+}\ u\right)^{-1} = B^{+},$$

where B is an arbitrary constant matrix, while B and B^{+} are Hermit-conjugate matrices.

For the solutions of (10.24–25), Lagrangian (10.27) takes the constant value

$$L = \sqrt{1 - 2H sp\left(A^{+}A\right)}. \tag{10.29}$$

For simplicity, in the sequel it will be assumed that

$$2H sp\left(A^{+}A\right) \neq 1.$$

Using (10.24–25), from equation (10.28) we obtain

$$u^{-1} A^{+} u = LB, \tag{10.30}$$

$$u^{+} A u^{+\ -1} = LB^{+}.$$

Let us differentiate (10.30) with respect to t. Using (10.24–25) and taking (10.29) into account we easily find

$$\left[A^{+},\ A\right] = 0, \tag{10.31}$$

where [,] is a commutator. In other words, the matrix A is unitarily equivalent to a diagonal matrix.

Solutions of equations (10.24-25) are written as follows: $u = \exp\left(At\right) u_0$, $u^{+} = u_0^{+} \exp\left(A^{+}t\right)$, where u_0 is an arbitrary constant matrix which we assume to be nonsingular. Substituting these solutions into (10.30) and taking into account (10.31), we find that the left-hand side of (10.30) takes the constant value. This means that if the matrices A and A^+ in equations (10.24) and (10.25) commute with each other, then the matrix solutions u and u^+ are geodesics in the space with metric (10.26).

If (10.24) A runs through the set of N-matrices satisfying condition (10.31), then the solutions of the corresponding equations (10.24–25) generate the set of all geodesics of metric (10.26).

Thus, when equation (10.23) is a given one, it generates metric (10.26). The set of all geodesics filling up the space generates the set of all possible equations (10.23) satisfying condition (10.31). Thus the geometry of the space with metric (10.26) and the set of equations (10.23) become inter-related.

(6) Let us consider an equation of the form

$$a^\nu \frac{\partial u}{\partial x^\nu} = mu, \tag{10.32}$$

where a^ν are constant N-matrices satisfying the condition

$$\det\left(\alpha_\nu a^\nu\right) \neq 0, \tag{10.33}$$

when $\alpha \neq 0$. The mass $m \neq 0$ is measured in the inverse units of length. Like in (5), we assume that the solution $u\,(x)$ is stretched over onto the algebra of N-dimensional square matrices.

For (10.32) and the Hermit-conjugate equation, the plane wave equation is written in the form

$$a_\alpha \frac{du_\alpha}{dz_\alpha} = mu_\alpha, \qquad \frac{du_\alpha^+}{dz_\alpha} a_\alpha^+ = mu_\alpha^+. \tag{10.34}$$

Assume that the constants of integration of the general solution of (10.34) are arbitrary functions of α, $\beta \in \Omega$. Without going into details assume that the matrix U with elements $u_{\alpha\beta}$ is nonsingular. Obviously, $U_\beta = \sum_{\alpha\in\Omega} q_\alpha u_{\alpha\beta}$ is a solution of equation (10.32). Besides, when β runs through the set Ω, we obtain the set of all irreducible states of process (10.32) for the given Ω. It should be noted that we arbitrarily fix β in the solution U_β for the given concrete state of process (10.32), and as we know, constants of integration are determined from the process itself. As we have pointed out in (8) of Section *2* , the given process defines the set Ω. But then, by analogy with section *4*, the given state is accompanied by the set of irreducible states U_β, where β runs through Ω.

(7) Using the reasoning of [1] and the results of (6) we can introduce the metric

$$ds^2 = \sum_{\alpha\in\Omega} q_\alpha \left[\mathring{g}_\alpha dz_\alpha^2 - 2H_\alpha sp\left(\left(u_\alpha^+\ u_\alpha\right)^{-1} du_\alpha^+ du_\alpha\right)\right]$$

or, which is the same,

$$ds^2 = \mathring{g}_{\nu\tau} dx^\nu dx^\tau - 2\sum_{\alpha\in\Omega} q_\alpha H_\alpha sp\left[\left(u_\alpha^+\ u_\alpha\right)^{-1} du_\alpha^+ du_\alpha\right], \tag{10.35}$$

where H_α are real constants. It should be noted that equation (10.32) does not directly define H_α.

By (10.31) and (10.34), if the restrictions

$$\left[a_\alpha^+,\ a_\alpha\right] = 0 \tag{10.36}$$

are imposed on the matrices a_α, then the solutions $u(x)$ of the corresponding equation (10.32) become the geodesics of metric (10.35).

We intend to discuss these problems in greater detail in our next paper.

(8) We wish now to focus attention on the relation between the solutions of equations of the process and the antiprocess. For this, we consider the plane waves u_α, $\alpha \in \Omega$, of equation (1.1), which are the set of irreducible states of the process defined by a given equation. As has been shown, $\dot{\chi}_\Omega(..., u_\alpha, ...)$ is a solution of the quasilinear system (1.1), i.e. $\dot{\chi}_\Omega(..., u_\alpha, ...)$ is the superposition of irreducible states of u_α, $\alpha \in \Omega$. Our investigation has shown that along with $\dot{\chi}_\Omega(..., u_\alpha, ...)$ there always exists an alternative superposition $\ddot{\chi}_\Omega(..., u_\alpha, ...)$ of the same states of u_α.

Let us now consider the antiprocess of a given process. If u_α are the plane waves of the process, then $\overset{\varphi}{\hat{u}}_\alpha$ are the plane waves of its antiprocess. Superpositions in the antiprocess are formed by the same functions $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ as in the process, while $\overset{\varphi}{\hat{u}}_\alpha$ serves as their argument. We have established that u_α and $\overset{\varphi}{\hat{u}}_\alpha$ are related by

$$u_\alpha \odot \overset{\varphi}{\hat{u}}_\alpha = E_\alpha, \tag{10.37}$$

where $\odot$ is the multiplication in the algebraic field generated by the plane wave equation (1.2), and E_α is the unit element of that algebra. As shown in (19) of Section *5*, an analogous relationship exists between the superpositions of the process and the antiprocess:

$$\dot{\chi}_\Omega(..., u_\alpha, ...)\,\dot{\otimes}\,\dot{\chi}_\Omega\left(..., \overset{\varphi}{\hat{u}}_\alpha, ...\right) = \dot{\chi}_\Omega(..., E_\alpha, ...) = E_\Omega, \tag{10.38}$$

$$\ddot{\chi}_\Omega(..., u_\alpha, ...)\,\ddot{\otimes}\,\ddot{\chi}_\Omega\left(..., \overset{\varphi}{\hat{u}}_\alpha, ...\right) = \ddot{\chi}_\Omega(..., E_\alpha, ...) = E_\Omega.$$

Recall that $\dot{\otimes}$, $\ddot{\otimes}$ are the operations of multiplication generated by equation (1.1), while E_Ω is the unit element. Equalities (10.37–38) arising from the algebraic properties of the process and the antiprocess are certain analogues of the algebraic relations postulated in the quantum field theory.

Therefore, the above-mentioned arguments and conclusions of (4) and (6), surely bring us to the assumption that the quantification of a field is related to the existence of interference between the process and antiprocess. Undoubtedly, the justification of this assumption needs more detailed investigation.

11. To the Questions Concerning the Fourier Method

(1) Assume in (3.17) that

$$g_\alpha^k = w^k, \tag{11.1}$$

for all $\alpha \in \Omega$, and equate f to the solution of equation (1.1):

$$u = \dot{\varkappa}_\Omega(..., w, ...). \tag{11.2}$$

Applying (4.7) to equality (11.2), we obtain

$$\frac{\partial_\varkappa u}{\partial_\varkappa X^\nu} = \dot{\varkappa}_\Omega\left(..., \frac{\partial w}{\partial x^\nu}, ...\right). \tag{11.3}$$

After substituting (11.2) and (11.3) into equation (4.14) and using (3.27) we come to the equality

$$\dot{\varkappa}_\Omega\left(..., l^\nu\frac{\partial w}{\partial x^\nu}, ...\right) = \dot{\varkappa}_\Omega\left(..., w, ...\right),$$

from which we conclude that w must be a solution of the equation

$$l^\nu\frac{\partial w}{\partial x^\nu} = w. \tag{11.4}$$

Thus, if $w(x)$ is a solution of (11.4), then $u(x)$ defined from (11.2) is a solution of equation (1.1).

(2) Let us return to equality (2.16). Using (3.3) and notation (3.2) we have

$$\sum_{\alpha\in\Omega} q_\alpha w_\alpha^k \exp\left[-\varphi_\alpha^k\left(\dot{\varkappa}\right)\right] = 1, \tag{11.5}$$
$$(k = 1, \ldots, N).$$

Solving this equation for $\dot{\varkappa}$, we obtain (3.2). If however we make the replacement $w_\alpha^k \to g_\alpha^k$ for all $\alpha \in \Omega$, then we obtain the right-hand side of (3.17). Bu then, in view of (11.1–2), from (11.5) we find

$$w^k = \left[\sum_{\alpha\in\Omega} q_\alpha \exp\left[-\varphi_\alpha^k\left(u\right)\right]\right]^{-1}, \tag{11.6}$$
$$(k = 1, \ldots, N).$$

Using the alternative algebra, (11.6) can be rewritten in the form

$$w^k = \ddot{\sum_{\alpha\in\Omega}} \frac{1}{q_\alpha} \exp\varphi_\alpha^k\left(u\right),$$
$$(k = 1, \ldots, N).$$

We have thus obtained the mapping

$$w^k = M^k\left(u^1, ..., w^N\right), \tag{11.7}$$
$$(k = 1, \ldots, N)$$

of the space of solutions of equation (1.1) in the space of solutions of equation (11.4).

Let us discuss the properties of mapping (11.7). For this we consider the set of processes of one class Π which are described by equations like (1.1). It is assumed that, along with other processes, the set Π contains a process π_0 described by equation (11.4).

If π is a process described by equation (1.1), then mapping (11.7) can be written in the form

$$\pi_0 = M\left(\pi\right). \tag{11.8}$$

Let us assume that $\pi_1, \pi_2 \in \Pi$. By virtue of (11.8) we can write

$$\pi_0 = M_1\left(\pi_1\right),$$
$$\pi_0 = M_2\left(\pi_2\right).$$

From these equalities we find the mapping $\pi_1 \to \pi_2$:

$$\pi_2 = M_2^{-1} \circ M_1 (\pi_1) . \tag{11.9}$$

Let us now make the replacement $\pi_1 \to \pi_2$ and after that the replacement $\pi_2 \to \pi_3$. Then, using mapping (11.9), we obtain

$$\pi_3 = M_3^{-1} \circ M_2 (\pi_2) = M_3^{-1} \circ M_2 \circ M_2^{-1} \circ M_1 (\pi_1) = M_3^{-1} \circ M_1 (\pi_1) .$$

As we see, the intermediate mapping M_2 falls out.

(3) Proceeding from (9.5), we can represent a solution of equation (11.4) in the form

$$w = \sum_{\alpha \in \Omega} q_\alpha w_\alpha, \tag{11.10}$$

where w_α is a general solution of equation (9.3).

From (11.7) and (11.10) we can write

$$u = M^{-1} \left(\sum_{\alpha \in \Omega} q_\alpha w_\alpha \right), \tag{11.11}$$

where M and M^{-1} are one-to-one functions.

The found solutions of the nonlinear equations (3.45) and (3.63) may serve as an illustration of expansion (11.11).

Given the initial boundary conditions for equation (1.1), we can find by means of transformation (11.7) the corresponding initial boundary conditions for equation (11.4). Applying the classical Fourier method to series (11.10) we define the concrete function $\mathring{w}$. Simultaneously with the solution of this problem, we also define the set $\Omega \subset \Gamma_{N_0}$. In that case, (11.11) implies

$$\mathring{u} = M^{-1} (\mathring{w}) .$$

It is obvious that $\mathring{u}$ is a solution of equation (1.1) with the prescribed initial boundary conditions.

(4) Example.

Let us consider the equation of motion of a noncompressible viscous fluid under gravity [9]:

$$\frac{\partial u^\nu}{\partial t} + u^\sigma \frac{\partial u^\nu}{\partial x^\sigma} = -\frac{1}{\rho} \frac{\partial P}{\partial x^\nu} + \varepsilon \frac{\partial^2 u^\nu}{\partial x^\sigma \delta x^\sigma} + g^\nu,$$

$$\frac{\partial u^\sigma}{\delta x^\sigma} = 0, \tag{11.12}$$

$$(\nu = 1, 2, 3),$$

where ρ is the density and ε the viscosity of the fluid. It is assumed that

$$g^1 = g^2 = 0, \quad g^3 = g. \tag{11.13}$$

In order to use the algebraic theory of differential equations, system (11.12) should be rewritten as a system of first order

$$\frac{\partial P^\nu}{\partial t} + u^\sigma \frac{\partial u^\nu}{\partial x^\sigma} = -\frac{1}{\rho} \frac{\partial P}{\partial x^\nu} + \varepsilon \exists_{\nu\sigma\mu} \frac{\partial F^\mu}{\partial x^\sigma} + g^\nu,$$

$$\frac{\partial u^\sigma}{\delta x^\sigma} = 0, \tag{11.14}$$

$$\frac{\partial u^\nu}{\partial x^\sigma} - \frac{\partial u^\sigma}{\partial x^\nu} = \exists_{\nu\sigma\mu} F^\mu,$$

where $\exists_{\nu\sigma\mu}$ is the perfect antisymmetrical tensor ($\exists_{123} = 1$). It is obvious that system (11.14) contains seven unknowns: u^ν, F^ν, P.

Let us write a plane wave equation for system (11.14). For this we will consider the case where u^ν, F^ν, P are the functions of one argument

$$z_\alpha = \frac{1}{l_\alpha}(\alpha_0 t + \alpha_\sigma x^\sigma), \tag{11.15}$$

where

$$\ell_\alpha = \alpha_0 l^0 + \alpha_\sigma . l^\sigma .$$

Then system (11.14) takes the form

$$(\alpha_0 + \alpha_\sigma u_\alpha^\sigma)\frac{du_\alpha^\nu}{dz_\alpha} = -\frac{1}{P}\alpha_\nu \frac{dP_\alpha}{dz_\alpha} + \varepsilon \exists_{\nu\sigma\mu}\alpha_\sigma \frac{dF_\alpha^\mu}{dz_\alpha +} + \ell_\alpha g^\nu,$$

$$\alpha_\sigma \frac{du_\alpha^\sigma}{dz_\alpha} = 0, \tag{11.16}$$

$$\alpha_\sigma \frac{du_\alpha^\nu}{dz_\alpha} - \alpha_\nu \frac{du_\alpha^\sigma}{dz_\alpha} = \ell_\alpha \exists_{\nu\sigma\mu} F_\alpha^\mu .$$

In system (11.16), the first equation over the index ν is convolved with α_ν. Taking into account the second equation and the antisymmetry over all indexes $\exists_{\nu\sigma\mu}$, we easily find

$$\frac{\alpha^2}{\rho l_\alpha g_\alpha} P_\alpha = z_\alpha + A_\alpha, \tag{11.17}$$

where

$$\alpha^2 = \alpha_\sigma \alpha_\sigma, \quad g_\alpha = \alpha_\sigma g^\sigma, \tag{11.18}$$

A_α is the integration constant.

From the second equation of system (11.16) it follows that $\alpha_0 + \alpha_\sigma u_\alpha^\sigma$ is a constant value. Then from the first equation of system (11.16) we find

$$\frac{1}{g\ell_\alpha}\left[(\alpha_0 + \alpha_\sigma u_\alpha^\sigma)u_\alpha^\nu + \frac{1}{\rho}\alpha_\nu P_\alpha - \varepsilon \exists_{\nu\sigma\mu}\alpha_\sigma F_\alpha^\mu\right] = \frac{g^\nu}{g} z_\alpha + \widetilde{B}_\alpha^\nu, \tag{11.19}$$

where $\widetilde{B}_\alpha^\nu$ is the integration constant. Note that the summands $\exists_{\nu\sigma\mu}\alpha_\sigma F_\alpha^\mu$ in (11.19) are the components of the product of the vectors $(\alpha_1, \alpha_2, \alpha_3)$ and $(F_\alpha^1, F_\alpha^2, F_\alpha^3)$.

In system (11.16), the third equation over the index σ is convolved with α_σ. From the resulting equality we define $\frac{du_\alpha^\nu}{dz_\alpha}$ and substitute it into the first equation of (11.16). Then, with (11.17) taken into account, we obtain

$$\varepsilon \exists_{\nu\sigma\mu}\alpha_\sigma \frac{dF_\alpha^\mu}{dz_\alpha} - \frac{1}{\alpha^2}(\alpha_0 + \alpha_\tau u_\alpha^\tau)\ell_\alpha \exists_{\nu\sigma\mu}\alpha_\sigma F_\alpha^\mu = \ell_\alpha \left(\frac{1}{\alpha^2} g_\alpha \alpha_\nu - g^\nu\right).$$

The solution of this equation can be written in the form

$$\frac{\varepsilon\alpha^2}{\ell_\alpha(\alpha_0+\alpha_\sigma u_\alpha^\sigma)}\ln\left(F_\alpha^\nu-\frac{1}{\alpha_0+\alpha_\sigma u_\alpha^\sigma}\exists_{\nu\tau\mu}\alpha_\tau g^\mu\right)=b^\nu z_\alpha+c_\alpha^\nu, \tag{11.20}$$

where c_α^ν is the integration constant and

$$b^\nu=1,\quad (\nu=1,2,3). \tag{11.21}$$

On the basis of the theory expounded above, we conclude that the left-hand parts of equalities (11.17), (11.19) and (11.20) are a complete set of the characteristic functions of equation (11.16):

$$\begin{aligned}\overset{\circ}{\varphi}_\alpha(P_\alpha)&=\frac{\alpha^2}{\rho\ell_\alpha g_\alpha}P_\alpha,\\ \widetilde{\varphi}_\alpha^\nu(u_\alpha,P_\alpha,F_\alpha)&=\frac{1}{g\ell_\alpha}\left[(\alpha_0+\alpha_\sigma u_\alpha^\sigma)u_\alpha^\nu+\frac{1}{\rho}\alpha_\nu P_\alpha-\varepsilon\exists_{\nu\sigma\mu}\alpha_\sigma F_\alpha^\mu\right],\\ \psi_\alpha^\nu(u_\alpha,F_\alpha)&=\frac{\varepsilon\alpha^2}{\ell_\alpha(\alpha_0+\alpha_\sigma u_\alpha^\sigma)}\ln\left[F_\alpha^\nu-\frac{1}{\alpha_0+\alpha_\sigma u_\alpha^\sigma}\exists_{\nu\tau\mu}\alpha_\tau g^\mu\right].\end{aligned} \tag{11.22}$$

Equalities (11.17), (11.19) and (11.20) contain the integration constants as summands. This means that the characteristic functions (11.22) admit the shifts

$$\begin{aligned}\overset{\circ}{\varphi}_\alpha&\to\overset{\circ}{\varphi}_\alpha+\overset{\circ}{m}_\alpha,\\ \widetilde{\varphi}_\alpha^\nu&\to\widetilde{\varphi}_\alpha^\nu+\widetilde{m}_\alpha^\nu,\\ \psi_\alpha^\nu&\to\psi_\alpha^\nu+m_\alpha^\nu,\end{aligned}$$

where m_α are arbitrary constant values.

Let us return to equality (11.19). When ν runs through values 1, 2, 3, we obtain with (11.13) and (11.22) taken into account

$$\begin{aligned}\widetilde{\varphi}_\alpha^1(u_\alpha,P_\alpha,F_\alpha)&=\widetilde{B}_\alpha^1,\\ \widetilde{\varphi}_\alpha^2(u_\alpha,P_\alpha,F_\alpha)&=\widetilde{B}_\alpha^2,\\ \widetilde{\varphi}_\alpha^3(u_\alpha,P_\alpha,F_\alpha)&=z_\alpha+\widetilde{B}_\alpha^3.\end{aligned} \tag{11.23}$$

As we see, the first two equalities of the right-hand part of (11.23) do not contain the variable z_α. As it was shown in Section *1*, for the characteristic functions $\widetilde{\varphi}_\alpha^\nu$ the following properties are true:

a) along with $\widetilde{\varphi}_\alpha^1$ and $\widetilde{\varphi}_\alpha^2$, the functions

$$\widetilde{\phi}_\alpha^1(\widetilde{\varphi}_\alpha^1,\widetilde{\varphi}_\alpha^2),\quad \widetilde{\phi}_\alpha^2(\widetilde{\varphi}_\alpha^1,\widetilde{\varphi}_\alpha^2),$$

are also the characteristic ones, where $\widetilde{\phi}_\alpha^1$, $\widetilde{\phi}_\alpha^2$ are arbitrary functions of their arguments;

b) $\widetilde{\varphi}_\alpha^1+\widetilde{\varphi}_\alpha^2+\widetilde{\varphi}_\alpha^3$, $\widetilde{\varphi}_\alpha^2+\widetilde{\varphi}_\alpha^3$, $\widetilde{\varphi}_\alpha^3$ are characteristic functions.

We introduce the matrix

$$(a_\tau^\nu)=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}.$$

It is not difficult to verify that the following equality is valid:

$$a_\tau^\nu g^\tau=gb^\nu,$$

where g^ν and b^ν are respectively (11.13) and (11.21).

Multiplying equality (11.19) from the left by the matrix a^ν_τ, we obtain

$$\varphi^\nu_\alpha(u_\alpha, P_\alpha, F_\alpha) = b^\nu z_\alpha + B^\nu_\alpha, \tag{11.24}$$

where

$$\varphi^\nu_\alpha = \frac{1}{g\ell_\alpha} a^\nu_\tau \left[(\alpha_0 + \alpha_\sigma u^\sigma_\alpha) u^\tau_\alpha + \frac{1}{\rho}\alpha_\tau P_\alpha - \varepsilon \exists_{\tau\sigma\mu} \alpha_\sigma F^\mu_\alpha\right], \tag{11.25}$$

$B^\nu_\alpha = a^\nu_\tau \widetilde{B}^\tau_\alpha$. From the properties indicated in items a) and b) we conclude that (11.25) are the characteristic functions of equation (11.16).

Proceeding from the above-said, let us dwell on the complete set of characteristic functions written in form (11.22) and (11.25). From (11.17), (11.20) and (11.24) it follows that all characteristic functions

$$\overset{\circ}{\varphi}_\alpha(P), \quad \varphi^\nu_\alpha(u_\alpha, P_\alpha, F_\alpha), \quad \psi^\nu_\alpha(u_\alpha, F_\alpha),$$

are equal to $z_\alpha +$ integration constants. Then exact solutions of system (11.14) can be written implicitly as

$$\left[\sum_{\alpha\in\Omega} q_\alpha \exp\left(-\frac{\alpha^2}{\rho \ell_\alpha g_\alpha} P\right)\right]^{-1} = \overset{\circ}{w},$$

$$\left[\sum_{\alpha\in\Omega} q_\alpha \exp\left(-\frac{1}{g\ell_\alpha} a^\nu_\tau \left[(\alpha_0 + \alpha_\sigma u^\sigma) u^\tau + \frac{1}{\rho}\alpha_\tau P - \varepsilon \exists_{\tau\sigma\mu}\alpha_\sigma F^\mu\right]\right)\right]^{-1} = w^\nu, \tag{11.26}$$

$$\left[\sum_{\alpha\in\Omega} q_\alpha \exp\left(-\frac{\varepsilon\alpha^2}{\ell_\alpha(\alpha_0 + \alpha_\sigma u^\sigma)} \ln\left[F^\nu - \frac{1}{\alpha_0 + \alpha_\sigma u^\sigma} \exists_{\nu\tau\mu}\alpha_\tau g^\mu\right]\right)\right]^{-1} = v^\nu,$$

where $\nu = 1, 2, 3$. Note that the sought solutions P, u^ν, F^ν of equations (11.14) do not depend of α.

The properties of characteristic functions indicated in items a) and b) allow one to manipulate the value q_α present in (11.26). For example, if Ω is a continuous set, then instead of the sum we can write the integral over α.

In the right-hand part of equation (11.26), the functions $\overset{\circ}{w}$, w^ν and v^ν, are the solutions of the linear equations

$$\begin{aligned} \ell^0 \frac{\partial \overset{\circ}{w}}{\partial t} + \ell^\sigma \frac{\partial \overset{\circ}{w}}{\partial x^\sigma} &= \overset{\circ}{w}, \\ \ell^0 \frac{\partial w^\nu}{\partial t} + \ell^\sigma \frac{\partial w^\nu}{\partial x^\sigma} &= w^\nu, \\ \ell^0 \frac{\partial v^\nu}{\partial t} + \ell^\sigma \frac{\partial v^\nu}{\partial x^\sigma} &= v^\nu. \end{aligned} \tag{11.27}$$

If we substitute the initial boundary conditions of equations (11.14) into the left-hand parts of equalities (11.26), then we can define the initial boundary conditions for equations (11.27). Thus the solution of the problem connected with system (11.14) is reduced to the solution of the problem of equations (11.27).

Let us write the solutions of equations (11.27) in terms of plane waves:

$$\overset{\circ}{w} = \sum_{\alpha\in\Omega} q_\alpha \overset{\circ}{w}_\alpha(z_\alpha),$$

$$w^{\nu} = \sum_{\alpha\in\Omega} q_{\alpha} w^{\nu}_{\alpha}(z_{\alpha}), \tag{11.28}$$
$$v^{\nu} = \sum_{\alpha\in\Omega} q_{\alpha} v^{\nu}_{\alpha}(z_{\alpha}),$$

where z_{α} is (11.15), while $\overset{\circ}{w}_{\alpha}(z_{\alpha})$, $w^{\nu}_{\alpha}(z_{\alpha})$, $v^{\nu}_{\alpha}(z_{\alpha})$ are the solutions of the equations

$$\frac{d\overset{\circ}{w}}{dz_{\alpha}} = \overset{\circ}{w},$$
$$\frac{dw^{\nu}_{\alpha}}{dz_{\alpha}} = w^{\nu}_{\alpha}, \tag{11.29}$$
$$\frac{dv^{\nu}_{\alpha}}{dz_{\alpha}} = v^{\nu}_{\alpha}.$$

In equalities (11.28), the integration constants and the set ω are defined from the initial boundary conditions.

Conclusions

To sum up our investigation of the algebraic properties of differential equations, we make the following conclusions:

(1) Any dynamic process described by an autonomous system of first order quasilinear partial differential equations generates its own calculus. When written in this calculus, the equation of the considered process takes the linear form. We have established that the differential equation of any other process from the same class (consisting of processes for which spaces are of the same dimension) can also be written in this calculus. This means that the observer placed in some process treats all other processes from the standpoint of his process. If there are two processes of the same class and with their own calculi, then there always exists a transformation that changes one calculus to the other one. This transformation reveals algebraic objects analogous to tensor objects.

(2) Since each equation generates a double numerical field, we obtain an alternative calculus. In this connection there arises the operation of algebraic conjugation. It has turned out that each equation has its algebraic conjugate. The process described by the latter equation is called the antiprocess.

(3) Each process described by an equation of form (1.1) defines invariantly its frame of reference (system of independent variables). If there are two processes of the same class, then the calculus of one process transforms to the calculus of the second process, the frame of reference of the first process also transforms to the frame of reference of the second process. In other words, the frame of reference is an algebraic object.

To the frame of reference of some process we can always apply a group of linear transformations defined in the calculus of this very process. If we require that the entire class of differential equations written in the calculus of the considered process be invariant with respect to the action of that linear group, then such frames of reference are called inertial frames of reference. It is shown in the work that an inertial frame of reference is an algebraic object. This however does not mean that the inertial frame of reference of one process is simultaneously the inertial system of the second process. Moreover, the inertial frame of reference of one process passes over to the inertial frame of reference of the second process through a nonlinear transformation, i.e. non-inertially if and only if the equation of the second process written in the calculus of the first one is nonlinear. Therefore each process can have a lot of inertial frames of reference.

All of this brings us to reconsidering the concepts of process motion relativity. We will study these questions in the next paper.

(4) There have emerged a perspective of explaining the field quantization. The matrix representation of differential equations, which we is obtained due to the algebraic theory and the alternative nature of the process and its antiprocess, obviously allow us to conjecture that the quantization is a consequence of the existence of interference between the process and the antiprocess.

Thus the work presents the basic aspects of the algebraic theory of motion dictated by differential equations.

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Article received: 2009-07-07