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LEARNING OF GROUP STRUCTURES BY REINFORCEMENT LEARNING AND IMPLICATIONS FOR INTERDISCIPLINARY SCIENCES

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Abstract
Group theory is widely applied in physical sciences. In addition, there have been attempts to apply it to social sciences such as anthropology and psychology. Here, it is shown that some group structures can be acquired through reinforcement learning. A reduced residue class group emerged through learning, and the result is isomorphic to the Klein four-group, which is the structure taken up by some renowned social scientists. Extending this approach to cyclic groups is straightforward. This finding provides the basis for how the group can be implicitly learned by the human.

Group theory is a field of mathematics that provides the basis for many other mathematical fields. It is well known that it is also extensively applied in physics. Less well known is the fact that some to social sciences have also applied group theory. Anthropologist Levi-Strauss, also considered a giant in philosophical structuralism, used the Klein four-group (with a help of mathematician Weil) to explain social structures in a tribe. Psychologist Piaget also discussed the Klein four-group, and called it the INRC group, to propose that this structure underlies our rational, logical mind. (For some examples of applying group theory to fields outside physical sciences, see [1]).

A group is a structure that satisfies

$a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associatively)

$a\cdot e=e\cdot a=a$ (existence of identity element)

$a\cdot b=b\cdot a=e$ (existence of inverse element for each element)

In addition, if commutatively holds ($a\cdot b=b\cdot a$), it is called an Abelian group.

For a completely rigorous definition, see any textbook on algebra. In arithmetic of integers, the identity element is one in the case of multiplication, and zero in the case of addition. In this article, operation is multiplication.

Although bold applications of group theory to broad sciences are attractive, one must also consider how the human learns the group structure. It cannot be assumed that humans always knew group theory explicitly, independently from its inception by Galois. One must address how humans acquired the concept implicitly. Considering this mechanism will complement those social scientists' works.

In this article, it is shown that one such mechanism is possible with reinforcement learning [2]. Reinforcement learning is extensively studied in neuroscience as well as computer science. This is because there are some experimental indications that outputs of some neurons seem to correspond to components of reinforcement learning. Reinforcement learning is not one specific algorithm but a class of various algorithms (e.g., Q learning is one of the most well known). If learning of a group structure is possible within this framework, it will spawn interdisciplinary research.

As an example, consider learning a reduced residue class group $(Z/nZ)^{\times}$ from a set of integers Z/nZ. More concretely, consider integers 0-7. This set is called Z/8Z. Operations in this set are basically equivalent to modular arithmetic: For example, just as $2\cdot 4\equiv 0 \pmod 8$ and $3\cdot 7\equiv 5 \pmod 8$ in modular arithmetic, $[2]\cdot[4]=[0]$ and $[3]\cdot[7]=[5]$ in Z/8Z. As not all the elements have the inverse

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element, Z/8Z is not a group. However, consider only elements that are relatively prime to 8: [1], [3], [5], and [7]. This set forms a group and called a reduced residue class group $(Z/8Z)^{\times}$. (See Table 1 for the multiplication table.) Naturally, this is an Abelian group.

Reinforcement learning can implicitly acquire $(Z/8Z)^{\times}$ from Z/8Z through trial and error. Assume a state to be an integer in Z/8Z, and assume an action to be multiplying it with an integer chosen from Z/8Z. So here we may multiply the current state (e.g., [7]) by some randomly chosen integer (e.g., [6]), and the state is transitioned to another state. As [7]·[6]=[2], the state is now transitioned to [2]. If the state becomes [1] (identity element), then a reinforce is delivered. So in this case no reinforce is delivered in [2] and learning cycles are continued, further taking actions and observing the next sates. Maybe actions and the resulting state transitions eventually lead to the goal state [1], or maybe they lead to a state not relatively prime to eight, in which case further actions never lead to the goal state. Intuitively, such states can be considered like a gutter from which one cannot escape.

In this framework, reinforcement learning can acquire the group structure. Set r (reward) to be 1, and γ (discount rate) to be .5 here. (These specific valued are not important, and the learning rate α can be arbitrary). Then, the optimal Q values are as in Table 2. Elements that are not relatively prime to eight are sieved out as zero. In addition, the remaining elements [1], [3], [5], [7] form the reduced residue class group $(Z/8Z)^{\times}$. A Q value of one indicates that this action directly leads to the goal state [1], thus showing the action to be the inverse element. A Q value of 0.5 means that the action is not optimal but a detour, which is nonetheless better than irreversibly dropping out from $(Z/8Z)^{\times}$

The fact that one can make the table of Q values means that this structure can be learned with some reinforcement-learning algorithm, regardless of its details. One can easily acquire this structure by, for example, Q learning:

$Q(s,a) = Q(s,a)+\alpha(r+\gamma\max_{a'}Q(s',a')-Q(s,a))$

Where a represents an action, s the state, and s' the next state brought by the action. Learning algorithms other than Q learning (e.g., Sarsa, actor critic) are similarly applicable, and specific action rules do not matter either (e.g., softmax, ε-greedy).

Incidentally, this group $(Z/8Z)^{\times}$ is nothing other than the Klein four-group. Therefore, this result is relevant to the works of Levi-Strauss and Piaget, who proposed the existence of this structure in our society or mind but was silent about the mechanism to acquire it.

Extending this to other group structures will be desirable. There are two types of groups of order 4, of which one is the Klein four-group. The other is the cyclic group, also an Abelian group. A cyclic group is generated by a single element, called a generator. As a concrete example, again number theory can be used. A generator in modular arithmetic is called a primitive root, and this concept has important applications in information sciences (e.g., cryptography). Primitive roots always exist in $(Z/pZ)^{\times}$, where p is a prime number. In addition, using Euler's phi function, there are $\phi(p-1)$ primitive roots in those cases. For instance, in $(Z/7Z)^{\times}$, there are $\phi(6)=2$ primitive roots, [3] and [5]. $[3]^n$ is never equal to [1] until $[3]^6=[1]$, at which point all the elements in $(Z/7Z)^{\times}$ have been visited.

Applying the present approach to the cyclic group is only straightforward. Here, the only possible action is multiplying one same element consecutively, and the agent receives reinforcement when it reaches [1]. For instance, in $(Z/7Z)^{\times}$, an element is randomly chosen as an initial state. This may be [2], and in this case not all states are visited; $[2]^2=[4]$, and next $[2]^3=[1]$, leaving other states ([3], [5] and [7]) with 0 values. However, if a primitive root [3] or [5] is chosen first, then all states

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are eventually assigned non-zero values. Thus, one can now distinguish two types of groups of order 4, the Klein four-group and the cyclic group. In contrast to the cyclic group, all states are never assigned non-zero values starting from any initial state for the Klein four-group. Without this additional mechanism, the two group structures are indistinguishable.

One may find this additional mechanism trivial, because only one action is allowed and merits of reinforcement learning may not be felt. However, it should be noted that this kind of simple mechanism has already attracted tremendous attention in neuroscience (see [3] that showed the relevance of temporal difference learning to the response of neurons in animal conditioning).

The author does not claim that this framework is the most elegant or efficient algorithm for learning the group structure. Without doubt, one can newly invent a learning algorithm that is specifically suited to learning a group. However, building a framework that addresses a broad range of problems is preferable to inventing a new mechanism for each of various problems. What is important is that an algorithm already studied extensively, and with some hints of biological plausibility, can provide the framework. The author hopes that this article spawns interdisciplinary investigations, and that works of past intellectual giants are fruitfully continued in the present day.

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Table 1. Reduced residue class group $(Z/8Z)^{\times}$

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Table 2. Optimal Q Values.

Action \ State	0	1	2	3	4	5	6	7
1	0	R	0	0.5	0	0.5	0	0.5
2	0	e	0	0	0	0	0	0
3	0	w	0	1	0	0.5	0	0.5
4	0	a	0	0	0	0	0	0
5	0	r	0	0.5	0	1	0	0.5
6	0	d	0	0	0	0	0	0
7	0		0	0.5	0	0.5	0	1

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