# ON WEYL CORRESPONDENCE 

# \& PHASE SPACE QUANTUM MECHANICS 

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#### Abstract

There has been a resurgence of interest in the methods of "phase space quantum mechanics" consequent to the progress made in quantum computing and information processing because Gaussian quantum states, that are physically realizable in the laboratory, are particularly amenable to analysis in phase space. The premise of phase space quantum mechanics is the association of c-number functions in phase space that correspond to relevant operators in the Hilbert space of the quantum system. The formulation was pioneered by Weyl and hence, is referred to as the "Weyl correspondence". In this paper, we present a systematic and comprehensive assimilation of the mathematical features of the "Weyl correspondence" that are relevant to the physicist.


## 1. INTRODUCTION

The phase space formulation of quantum mechanics has gained considerable importance in the last decade with the progress achieved in quantum computing and quantum information processing employing continuous variable methods, particularly so, when processing is contemplated in the Gaussian/coherent state representation. The phase space formalism relies on associating a correspondence between the operator representation of quantum mechanics in Hilbert space with a $c$ number function/functional in phase space. Such a correspondence was initially identified by Weyl [1] through the Weyl transform and was soon thereafter applied by Wigner [2] to obtain corrections in classical statistical mechanics for quantum systems. While being useful in quantum information processing, the phase space formulation is extremely versatile, finding applications in statistical mechanics, nuclear physics, atomic and molecular physics, quantum optics, relativistic bound state problems and localized probability distributions.

In Section 2 of this article, we reproduce some well known results on the Weyl correspondence to facilitate ease of referencc and continuity. Section 3 looks at the physical implications of the Weyl correspondence whereas Section 4 examines the properties of the Weyl basis set. Section 5 sets up a comparative evaluation and studies the relationship between the Weyl basis and the Wigner basis. We shall be using the natural system of units in which $\hbar=c=1$ throughout this exposition.

## 2. THE WEYL CORRESPONDENCE [3-9]

For simplicity of exposition and to avoid proliferation of symbols, we shall consider the dynamics of a one-particle quantum system whose position and momentum eigenvectors satisfy the eigenvalue eqs.

$$
\begin{equation*}
\hat{q}|\mathbf{q}\rangle=\mathbf{q}|\mathbf{q}\rangle ; \hat{p}|\mathbf{p}\rangle=\mathbf{p}|\mathbf{p}\rangle \tag{1}
\end{equation*}
$$

The eigenvectors $|\mathbf{q}\rangle,|\mathbf{p}\rangle$ constitute a complete set and hence satisfy the closure relations

$$
\begin{equation*}
\int d \mathbf{q}|\mathbf{q}\rangle\langle\mathbf{q}|=\mathbf{I} ; \int d \mathbf{q}|\mathbf{p}\rangle\langle\mathbf{p}|=\mathbf{I} \tag{2}
\end{equation*}
$$

The orthogonality of the eigenvectors mandates

$$
\begin{equation*}
\left\langle\mathbf{q} \mid \mathbf{q}^{\prime}\right\rangle=\delta\left(\mathbf{q}-\mathbf{q}^{\prime}\right) ;\left\langle\mathbf{p} \mid \mathbf{p}^{\prime}\right\rangle=\delta\left(\mathbf{p}-\mathbf{p}^{\prime}\right) \tag{3}
\end{equation*}
$$

Furthermore, the operators $\hat{q}$ and $\hat{p}$ also satisfy the quantization conditions

$$
\begin{equation*}
\left[\hat{q}_{i}, \hat{q}_{j}\right]=\left[\hat{p}_{i}, \hat{p}_{j}\right]=0 ; \quad\left[\hat{q}_{i}, \hat{p}_{j}\right]=i \delta_{i j} \quad \forall i, j=1,2,3 . \tag{4}
\end{equation*}
$$

Using the resolution of identity motivated by the closure relations (2), we can expand an arbitrary operator $\hat{A}$ in the basis of the eigenvectors $|\mathbf{q}\rangle,|\mathbf{p}\rangle$ (that form a complete set) as

$$
\begin{align*}
& \hat{A}=\int d \mathbf{p} ' d \mathbf{p} " d \mathbf{q}^{\prime} d \mathbf{q}^{"}\left|\mathbf{q}^{\prime \prime}\right\rangle\langle\mathbf{q} " \mid \mathbf{p} "\rangle\langle\mathbf{p} "| \hat{A}\left|\mathbf{p}^{\prime}\right\rangle\left\langle\mathbf{p}^{\prime} \mid \mathbf{q}^{\prime}\right\rangle\left\langle\mathbf{q}^{\prime}\right| \\
& =(1 / 2 \pi)^{3} \int d \mathbf{p} d \mathbf{q} a(\mathbf{p}, \mathbf{q}) \Delta(\mathbf{p}, \mathbf{q}) \tag{5}
\end{align*}
$$

with

$$
\begin{equation*}
a(\mathbf{p}, \mathbf{q}) \equiv \int d \boldsymbol{\sigma} e^{i q . \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \hat{A}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle ; \Delta(\mathbf{p}, \mathbf{q}) \equiv \int d \boldsymbol{\tau} e^{i \mathbf{p} . \boldsymbol{\tau}}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau}| \tag{6}
\end{equation*}
$$

and the integration variables have been changed to

$$
\begin{equation*}
\mathbf{p}^{\prime}=\mathbf{p}-1 / 2 \boldsymbol{\sigma} ; \mathbf{p}{ }^{\prime \prime}=\mathbf{p}+1 / 2 \boldsymbol{\sigma} ; \mathbf{q}^{\prime}=\mathbf{q}-1 / 2 \boldsymbol{\tau} ; \mathbf{q}{ }^{\prime \prime}=\mathbf{q}+1 / 2 \boldsymbol{\tau} \tag{7}
\end{equation*}
$$

The function $a(\mathbf{p}, \mathbf{q})$ constitutes the Weyl transform of the operator $\hat{A}$ with respect to the operators $\hat{q}, \hat{p}$ and associates a $c$-number function $a(\mathbf{p}, \mathbf{q})$ with an arbitrary operator $\hat{A}$.

Defining the trace of an operator $\hat{A}$ in the usual way as
$\operatorname{Tr} \hat{A}=\int d \mathbf{p}\langle\mathbf{p}| \hat{A}|\mathbf{p}\rangle$
and using the completeness property of the eigenvectors $|\mathbf{p}\rangle$, we immediately obtain the identity $\langle\mathbf{p}| \hat{A}\left|\mathbf{p}^{\prime \prime}\right\rangle=\operatorname{Tr}\left(\hat{A}\left|\mathbf{p}^{\prime \prime}\right\rangle\left\langle\mathbf{p}^{\prime}\right|\right)$ since
$\operatorname{Tr}\left(\hat{A}\left|\mathbf{p}^{\prime \prime}\right\rangle\left\langle\mathbf{p}^{\prime}\right|\right)=\int d \mathbf{p}\langle\mathbf{p}|\left(\hat{A}\left|\mathbf{p}^{\prime \prime}\right\rangle\left\langle\mathbf{p}^{\prime}\right|\right)|\mathbf{p}\rangle=\left\langle\mathbf{p}^{\prime}\right| \hat{A}\left|\mathbf{p}{ }^{\prime \prime}\right\rangle$

We also have

$$
\begin{align*}
& \operatorname{Tr}[\hat{A} \Delta(\mathbf{p}, \mathbf{q})]=\int d \mathbf{p}\langle\mathbf{p}| \hat{A} \Delta(\mathbf{p}, \mathbf{q})|\mathbf{p}\rangle=\int d \mathbf{p} d \boldsymbol{\tau} e^{i p . \tau}\langle\mathbf{p}| \hat{A}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau} \mid \mathbf{p}\rangle \\
& =\int d \mathbf{p} d \boldsymbol{\tau} e^{i(\mathbf{p}-\hat{p}) \cdot \tau}\langle\mathbf{p}| \hat{A}|\mathbf{q}-1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau} \mid \mathbf{p}\rangle \\
& =(1 / 2 \pi)^{3} \int d \mathbf{p} d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}} e^{i(\mathbf{q}-1 / 2 \boldsymbol{\tau}-\hat{q}) \cdot \boldsymbol{\sigma}}\langle\mathbf{p}| \hat{A}|\mathbf{p}\rangle  \tag{10}\\
& =(1 / 2 \pi)^{3} \int d \mathbf{p} d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i[(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}+i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}]}\langle\mathbf{p}| \hat{A}|\mathbf{p}\rangle \tag{11}
\end{align*}
$$

which gives us an expression for $\Delta(\mathbf{p}, \mathbf{q})$ that is symmetric in $\mathbf{q}, \mathbf{p}$ as

$$
\begin{equation*}
\Delta(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i[(\mathbf{p}-\hat{p}) \cdot \tau+i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}]} \tag{12}
\end{equation*}
$$

We have used the following identities (for a plane wavefunction):

$$
\langle\mathbf{q} \mid \mathbf{p}\rangle=(1 / 2 \pi)^{3} e^{i \mathbf{p} . \mathbf{q}} ;|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle=e^{-i \tau \cdot \hat{p}}|\mathbf{q}-1 / 2 \boldsymbol{\tau}\rangle ; e^{A} e^{B}=e^{A+B+1 / 2[A, B]}
$$

in the respective steps leading to eq. (12).
From eq. (10), we also obtain

$$
\begin{align*}
& \operatorname{Tr}[\hat{A} \Delta(\mathbf{p}, \mathbf{q})]=(1 / 2 \pi)^{3} \int d \mathbf{p} d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i(\mathbf{p}-1 / 2 \boldsymbol{\sigma}-\hat{p}) \cdot \tau} e^{i(\mathbf{q}-\hat{q}) \cdot \sigma}\langle\mathbf{p}| \hat{A}|\mathbf{p}\rangle \\
& =\int d \mathbf{p} d \boldsymbol{\sigma} e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}\langle\mathbf{p}| \hat{A}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma} \mid \mathbf{p}\rangle \\
& =\int d \mathbf{p} d \boldsymbol{\sigma} e^{i \mathbf{q} \cdot \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \hat{A}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma} \mid \mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle \\
& =\int d \boldsymbol{\sigma} e^{i q \cdot \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \hat{A}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle=a(\mathbf{p}, \mathbf{q}) \tag{13}
\end{align*}
$$

Furthermore, using the equivalent expression for the trace, $\operatorname{Tr} \hat{A}=\int d \mathbf{q}\langle\mathbf{q}| \hat{A}|\mathbf{q}\rangle$, we get

$$
\begin{align*}
& a(\mathbf{p}, \mathbf{q})=\operatorname{Tr}[\hat{A} \Delta(\mathbf{p}, \mathbf{q})]=\int d \mathbf{q} d \boldsymbol{\tau} e^{i \mathbf{p} . \tau}\langle\mathbf{q}| \hat{A}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau} \mid \mathbf{q}\rangle \\
& =\int d \mathbf{q} d \boldsymbol{\tau} e^{i \mathbf{p} . \tau}\langle\mathbf{q}-1 / 2 \boldsymbol{\tau}| \hat{A}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau} \mid \mathbf{q}-1 / 2 \boldsymbol{\tau}\rangle \\
& =\int d \boldsymbol{\tau} e^{i \mathbf{p} \cdot \tau}\langle\mathbf{q}-1 / 2 \boldsymbol{\tau}| \hat{A}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle \tag{14}
\end{align*}
$$

An alternative expression for $\Delta(\mathbf{p}, \mathbf{q})$ may be directly obtained from eq. (10) as

$$
\begin{align*}
& \Delta(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}} e^{i(\mathbf{q}-1 / 2 \boldsymbol{\tau}-\hat{q}) \cdot \boldsymbol{\sigma}}=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i(\mathbf{p}-1 / 2 \boldsymbol{\sigma}-\hat{p}) \cdot \boldsymbol{\tau}} e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}} \\
& =\int d \boldsymbol{\sigma} e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma}|=\int d \boldsymbol{\sigma} e^{i \mathbf{q} \cdot \boldsymbol{\sigma}}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \tag{15}
\end{align*}
$$

Eq. (14) establishes the existence of a mapping that maps the set of operators in a Hilbert space upon a set of $c$-numbers, that are their respective Weyl transforms. The converse also holds, since, by virtue of eq. (5) and eq. (12), we obtain

$$
\begin{equation*}
\hat{A}=(1 / 2 \pi)^{6} \int d \mathbf{p} d \mathbf{q} d \boldsymbol{\sigma} d \boldsymbol{\tau} a(\mathbf{p}, \mathbf{q}) e^{i[(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}+i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}]} \tag{16}
\end{equation*}
$$

An alternative expression for $\hat{A}$ may also be written as:

$$
\begin{equation*}
\hat{A}=\int d \boldsymbol{\sigma} d \boldsymbol{\tau} \tilde{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) e^{-i(\hat{p} \cdot \boldsymbol{\tau}+\hat{q} \cdot \boldsymbol{\sigma})} \tag{17}
\end{equation*}
$$

where $\tilde{a}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ is the Fourier transform of $a(\mathbf{p}, \mathbf{q})$ given by:

$$
\begin{equation*}
\tilde{a}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{6} \int d \mathbf{p} d \mathbf{q} a(\mathbf{p}, \mathbf{q}) e^{i(\mathbf{q} \cdot \boldsymbol{\sigma}+\mathbf{p} \cdot \tau)} \tag{18}
\end{equation*}
$$

with the Fourier inverse

$$
\begin{equation*}
a(\mathbf{p}, \mathbf{q})=\int d \boldsymbol{\sigma} d \boldsymbol{\tau} \tilde{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) e^{-i(\mathbf{q} \cdot \boldsymbol{\sigma}+\mathbf{p} \cdot \boldsymbol{\tau})} \tag{19}
\end{equation*}
$$

We also obtain, from eq. (16)

$$
\begin{align*}
& \operatorname{Tr} \hat{A}=\operatorname{Tr}\left[(1 / 2 \pi)^{6} \int d \mathbf{p} d \mathbf{q} d \boldsymbol{\sigma} d \boldsymbol{\tau} a(\mathbf{p}, \mathbf{q}) e^{i[(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}+i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}]}\right] \\
& =\operatorname{Tr}\left[(1 / 2 \pi)^{6} \int d \mathbf{p} d \mathbf{q} d \boldsymbol{\sigma} d \boldsymbol{\tau} a(\mathbf{p}, \mathbf{q}) e^{i(\mathbf{p}-1 / 2 \boldsymbol{\sigma}-\hat{p}) \cdot \boldsymbol{\tau}} e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}\right] \\
& =\operatorname{Tr}\left[(1 / 2 \pi)^{3} \int d \mathbf{p} d \mathbf{q} d \boldsymbol{\sigma} a(\mathbf{p}, \mathbf{q}) e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma}|\right]=\int d \mathbf{p} d \mathbf{q} a(\mathbf{p}, \mathbf{q}) \tag{20}
\end{align*}
$$

It immediately follows from eq. (20) that
$\operatorname{Tr}[\Delta(\mathbf{p}, \mathbf{q})]=1$
Further, the Weyl transform of $\Delta(\mathbf{p}, \mathbf{q})$ is obtained from eq. (6) as:

$$
\begin{align*}
& \delta_{W}(\mathbf{p}, \mathbf{q}) \equiv \int d \boldsymbol{\sigma} e^{i \mathbf{q} \cdot \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right)|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle \\
& =\int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i \mathbf{q} \cdot \boldsymbol{\sigma}} e^{i \mathbf{p}^{\prime} \tau}\left\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma} \mid \mathbf{q}^{\prime}+1 / 2 \boldsymbol{\tau}\right\rangle\left\langle\mathbf{q}^{\prime}-1 / 2 \boldsymbol{\tau} \mid \mathbf{p}-1 / 2 \boldsymbol{\sigma}\right\rangle \\
& =\int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i\left(\mathbf{q}-\mathbf{q}^{\prime}\right) \cdot \boldsymbol{\sigma}} e^{i\left(\mathbf{p}^{\prime}-\mathbf{p}\right) \cdot \tau}=(2 \pi)^{3} \delta\left(\mathbf{q}-\mathbf{q}^{\prime}\right) \delta\left(\mathbf{p}-\mathbf{p}^{\prime}\right) \tag{22}
\end{align*}
$$

We also have

$$
\begin{aligned}
& \operatorname{Tr}\left[\Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right) \cdot \Delta(\mathbf{p}, \mathbf{q})\right]=(1 / 2 \pi)^{3} \int d \mathbf{p} d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i(\mathbf{p}-1 / 2 \boldsymbol{\sigma}-\hat{p}) \cdot \boldsymbol{\tau}} e^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}\langle\mathbf{p}| \Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right)|\mathbf{p}\rangle \\
& =\int d \mathbf{p} d \boldsymbol{\sigma} \boldsymbol{e}^{i(\mathbf{q}-\hat{q}) \cdot \boldsymbol{\sigma}}\langle\mathbf{p}| \Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right)|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma} \mid \mathbf{p}\rangle
\end{aligned}
$$

$$
\begin{align*}
& =\int d \mathbf{p} d \boldsymbol{\sigma} e^{i \mathbf{q} \cdot \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right)|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle\langle\mathbf{p}-1 / 2 \boldsymbol{\sigma} \mid \mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle \\
& =\int d \boldsymbol{\sigma} e^{i \mathbf{q} \cdot \sigma}\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma}| \Delta\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right)|\mathbf{p}-1 / 2 \boldsymbol{\sigma}\rangle \\
& =\int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i \mathbf{q} \cdot \sigma} e^{i \mathbf{p}^{\prime} \cdot \tau}\left\langle\mathbf{p}+1 / 2 \boldsymbol{\sigma} \mid \mathbf{q}^{\prime}+1 / 2 \boldsymbol{\tau}\right\rangle\left\langle\mathbf{q}^{\prime}-1 / 2 \boldsymbol{\tau} \mid \mathbf{p}-1 / 2 \boldsymbol{\sigma}\right\rangle \\
& =\int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i \mathbf{q} \cdot \sigma} e^{i \mathbf{p}^{\prime} \cdot \tau} e^{-i \mathbf{q}^{\prime} \cdot \boldsymbol{\sigma}} e^{-i \mathbf{p} \cdot \boldsymbol{\tau}} \\
& =\int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{i\left(q-\mathbf{q}^{\prime}\right) \cdot \boldsymbol{\sigma}} e^{i\left(\mathbf{p}^{\prime}-\mathbf{p}\right) \cdot \tau}=(1 / 2 \pi)^{3} \delta\left(\mathbf{p}-\mathbf{p}^{\prime}\right) \delta\left(\mathbf{q}-\mathbf{q}^{\prime}\right) \tag{23}
\end{align*}
$$

Furthermore,

$$
\begin{align*}
& \int d \mathbf{p} d \mathbf{q} \Delta(\mathbf{p}, \mathbf{q})=\int d \mathbf{p} d \mathbf{q} d \boldsymbol{\tau} e^{i \mathbf{p} \cdot \tau}|\mathbf{q}+1 / 2 \boldsymbol{\tau}\rangle\langle\mathbf{q}-1 / 2 \boldsymbol{\tau}| \\
& =(1 / 2 \pi)^{3} \int d \mathbf{p} d \boldsymbol{\tau} e^{i(\mathbf{p}-\hat{p}) \cdot \boldsymbol{\tau}}=\int d \mathbf{p}|\mathbf{p}\rangle\langle\mathbf{p}|=1  \tag{24}\\
& \text { । } \\
& \text { and }
\end{align*}
$$

$$
\begin{align*}
& \int d \mathbf{p} d \mathbf{q} a(\mathbf{p}, \mathbf{q}) b(\mathbf{p}, \mathbf{q}) \\
& =\int d \mathbf{p}_{A} d \mathbf{q}_{A} d \mathbf{p}_{B} d \mathbf{q}_{B} \operatorname{Tr}\left[\hat{A} \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right)\right] \operatorname{Tr}\left[\hat{B} \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right)\right] \delta\left(\mathbf{p}_{A}-\mathbf{p}_{B}\right) \delta\left(\mathbf{q}_{A}-\mathbf{q}_{B}\right) \\
& =\int d \mathbf{p}_{A} d \mathbf{q}_{A} \operatorname{Tr}\left[\hat{A} \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right)\right] \operatorname{Tr}\left[\hat{B} \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right)\right]=\operatorname{Tr} \hat{A} \hat{B} \tag{25}
\end{align*}
$$

The composition law for the Weyl transform of a product of operators is obtained by considering a product of operators $\hat{C}=\hat{A} \hat{B}$ and letting $c(\mathbf{p}, \mathbf{q})$ be the corresponding Weyl transform. Then, we have

$$
\begin{align*}
& c(\mathbf{p}, \mathbf{q})=\operatorname{Tr}[\hat{C} \Delta]=\operatorname{Tr}[\hat{A} \hat{B} \Delta]=\operatorname{Tr}\left[\int \begin{array}{l}
d \mathbf{p}_{A} d \mathbf{q}_{A} d \mathbf{p}_{B} d \mathbf{q}_{B} a\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \\
b\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta(\mathbf{p}, \mathbf{q})
\end{array}\right] \\
& =\int d \mathbf{p}_{A} d \mathbf{q}_{A} d \mathbf{p}_{B} d \mathbf{q}_{B} a\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) b\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \operatorname{Tr}\left[\Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta(\mathbf{p}, \mathbf{q})\right] \tag{26}
\end{align*}
$$

Now,

$$
\begin{aligned}
& \operatorname{Tr}\left[\Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta(\mathbf{p}, \mathbf{q})\right] \\
& =\int d \mathbf{q}_{1}\left\langle\mathbf{q}_{1}\right| \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta(\mathbf{p}, \mathbf{q})\left|\mathbf{q}_{1}\right\rangle \\
& =\int d \mathbf{q}_{1} d \mathbf{q}_{2} d \mathbf{q}_{3}\left\langle\mathbf{q}_{1}\right| \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right)\left|\mathbf{q}_{2}\right\rangle\left\langle\mathbf{q}_{2}\right| \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right)\left|\mathbf{q}_{3}\right\rangle\left\langle\mathbf{q}_{3}\right| \Delta(\mathbf{p}, \mathbf{q})\left|\mathbf{q}_{1}\right\rangle
\end{aligned}
$$

Also $\left\langle\mathbf{q}_{1}\right| \Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right)\left|\mathbf{q}_{2}\right\rangle=\int d \boldsymbol{\tau} e^{i \mathbf{p} \cdot \boldsymbol{\tau}}\left\langle\mathbf{q}_{1} \mid \mathbf{q}_{A}+1 / 2 \boldsymbol{\tau}\right\rangle\left\langle\mathbf{q}_{A}-1 / 2 \boldsymbol{\tau} \mid \mathbf{q}_{2}\right\rangle$
$=\int d \boldsymbol{\sigma} e^{i \boldsymbol{q} \cdot \boldsymbol{\sigma}}\left\langle\mathbf{q}_{1} \mid \mathbf{p}_{A}-1 / 2 \boldsymbol{\sigma}\right\rangle\left\langle\mathbf{p}_{A}+1 / 2 \boldsymbol{\sigma} \mid \mathbf{q}_{2}\right\rangle=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} e^{i \mathbf{q}_{A} \cdot \boldsymbol{\sigma}} e^{i\left(\mathbf{p}_{A}-1 / 2 \boldsymbol{\sigma}\right) \cdot \mathbf{q}_{1}} e^{i\left(\mathbf{p}_{A}+1 / 2 \boldsymbol{\sigma}\right) \cdot \mathbf{q}_{2}}$
$=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} e^{i\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right) \cdot \boldsymbol{p}} e^{i\left[\mathbf{q}_{A}-(1 / 2)\left(\mathbf{q}_{1}+\mathbf{q}_{2}\right)\right] \cdot \boldsymbol{\sigma}}=\delta\left[\mathbf{q}_{A}-(1 / 2)\left(\mathbf{q}_{1}+\mathbf{q}_{2}\right)\right] e^{i\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right) \cdot \mathbf{p}}$
Making use of (27) and similar expressions for $\left\langle\mathbf{q}_{2}\right| \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right)\left|\mathbf{q}_{3}\right\rangle$ and $\left\langle\mathbf{q}_{3}\right| \Delta(\mathbf{p}, \mathbf{q})\left|\mathbf{q}_{1}\right\rangle$ we obtain
$\operatorname{Tr}\left[\Delta\left(\mathbf{p}_{A}, \mathbf{q}_{A}\right) \Delta\left(\mathbf{p}_{B}, \mathbf{q}_{B}\right) \Delta(\mathbf{p}, \mathbf{q})\right]=\left\{\begin{array}{l}d \mathbf{q}_{1} d \mathbf{q}_{2} d \mathbf{q}_{3} \delta\left[\mathbf{q}_{A}-(1 / 2)\left(\mathbf{q}_{1}+\mathbf{q}_{2}\right)\right] \delta\left[\mathbf{q}_{B}-(1 / 2)\left(\mathbf{q}_{2}+\mathbf{q}_{3}\right)\right] \\ \delta\left[\mathbf{q}-(1 / 2)\left(\mathbf{q}_{3}+\mathbf{q}_{1}\right)\right] e^{i\left[\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right) \cdot \mathbf{p}_{A}+\left(\mathbf{q}_{2}-\mathbf{q}_{3}\right) \cdot \mathbf{p}_{B}+\left(\mathbf{q}_{3}-\mathbf{q}\right) \cdot \mathbf{p}\right]}\end{array}\right.$
$=2^{6} \exp \left\{2 i\left[\mathbf{p}_{A} \cdot\left(\mathbf{q}-\mathbf{q}_{B}\right)+\mathbf{p}_{B} \cdot\left(\mathbf{q}_{A}-\mathbf{q}\right)+\mathbf{p} \cdot\left(\mathbf{q}_{B}-\mathbf{q}_{A}\right)\right]\right\}$
where we have integrated over the $\delta$ functions. Making use of eqs. (26), (28) and (29), we obtain

$$
\begin{align*}
& c(\mathbf{p}, \mathbf{q})=\left(\frac{1}{\pi}\right)^{6} \int \begin{array}{l}
d \mathbf{p}^{\prime} d \mathbf{q}^{\prime} d \mathbf{p}^{\prime} d \mathbf{q} " a\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right) b\left(\mathbf{p}^{\prime \prime}, \mathbf{q}^{\prime \prime}\right) \times \\
e^{\left\{2 i\left[\left(\mathbf{q}^{\prime}-\mathbf{q}\right) \cdot\left(\mathbf{p}^{\prime \prime}-\mathbf{p}\right)-\left(\mathbf{p}^{\prime} \mathbf{p}\right) \cdot\left(\mathbf{q}^{\prime \prime} \mathbf{q}\right)\right]\right\}}
\end{array}  \tag{30}\\
& =\left(\frac{1}{\pi}\right)^{6} \int \begin{array}{l}
d \mathbf{p}^{\prime} d \mathbf{q}^{\prime} d \hat{p} d \hat{q} a\left(\mathbf{p}^{\prime}, \mathbf{q}^{\prime}\right) e^{\left\{2\left[\left[\left(\mathbf{q}^{\prime}-\mathbf{q}\right) \cdot \hat{p}-\left(\mathbf{p}^{\prime}-\mathbf{p}\right) \cdot \hat{q}\right]\right\}\right.} \times \\
e^{\left(\hat{p} \cdot \frac{\partial}{\partial \mathbf{p}}+\hat{q} \cdot \frac{\partial}{\partial \mathbf{q}}\right)} b(\mathbf{p}, \mathbf{q})
\end{array} \tag{31}
\end{align*}
$$

In arriving at eq. (31), we have introduced new variables $\hat{p}=\mathbf{p} "-\mathbf{p}$ and $\hat{q}=\mathbf{q} "-\mathbf{q}$ and expanded $b(\mathbf{p}+\hat{p}, \mathbf{q}+\hat{q})$ into a Taylor series around $b(\mathbf{p}, \mathbf{q})$. Eq. (31) may be simplified further by making the replacements $\hat{p} \equiv \frac{i}{2} \frac{\partial^{\leftarrow}}{\partial \mathbf{q}}, \hat{q} \equiv-\frac{i}{2} \frac{\partial^{\leftarrow}}{\partial \mathbf{p}}$ acting on the left. If we now integrate over $\hat{p}$ and $\hat{q}$, we obtain $\delta$ functions, Following this by integrations with respect to $\mathbf{p}^{\prime}$ and $\mathbf{q}^{\prime}$, we finally obtain

$$
\begin{equation*}
\left.c(\mathbf{p}, \mathbf{q})=e^{\left[\frac{1}{2 i}\left(\frac{\partial^{(a)}}{\partial \mathbf{p}} \cdot \frac{\partial^{(b)}}{\partial \boldsymbol{q}}-\frac{\hat{\partial}^{(a)}}{\partial \mathbf{q}} \cdot \frac{\partial^{(b)}}{\partial \mathbf{p}}\right)\right.}\right] a(\mathbf{p}, \mathbf{q}) b(\mathbf{p}, \mathbf{q}) \tag{32}
\end{equation*}
$$

We obtain a representation of $\Delta(\mathbf{p}, \mathbf{q})$ in coordinate space from eq. (27) as

$$
\begin{equation*}
\left\langle\mathbf{q}_{1}\right| \Delta(\mathbf{p}, \mathbf{q})\left|\mathbf{q}_{2}\right\rangle=\delta\left[\mathbf{q}-(1 / 2)\left(\mathbf{q}_{1}+\mathbf{q}_{2}\right)\right] e^{i\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right) \cdot \mathbf{p}} \tag{33}
\end{equation*}
$$

A similar line of reasoning leads us to the following representation of $\Delta(\mathbf{p}, \mathbf{q})$ in momentum space

$$
\begin{equation*}
\left\langle\mathbf{p}_{1}\right| \Delta(\mathbf{p}, \mathbf{q})\left|\mathbf{p}_{2}\right\rangle=\delta\left[\mathbf{p}-(1 / 2)\left(\mathbf{p}_{1}+\mathbf{p}_{2}\right)\right] e^{i \boldsymbol{q}\left(\mathbf{p}_{2}-\mathbf{p}_{1}\right)} \tag{34}
\end{equation*}
$$

The following identities relating to the Weyl transform of products of $\Delta(\mathbf{p}, \mathbf{q})$ follow trivially from the composition law of eq. (30)

$$
\begin{align*}
& \hat{\Delta}(\mathbf{p}, \mathbf{q}) \hat{\Delta}\left(\mathbf{p},, \mathbf{q}^{\prime}\right) \square \quad 2^{6} \int d \mathbf{p} " d \mathbf{q}^{\prime \prime} \hat{\Delta}\left(\mathbf{p} ", \mathbf{q}^{\prime \prime}\right) e^{\left\{2 i\left[\left(\mathbf{q}^{\prime}-\mathbf{q}\right) \cdot\left(\mathbf{p}^{\prime \prime}-\mathbf{p}\right)-\left(\mathbf{p}^{\prime}-\mathbf{p}\right) \cdot\left(\mathbf{q}^{\prime \prime}-\mathbf{q}\right)\right]\right\}}  \tag{35}\\
& \hat{\Delta}(\mathbf{p}, \mathbf{q}) \hat{\Delta}(\mathbf{p}, \mathbf{q}) \square \quad 2^{6} ; \hat{\Delta}(\mathbf{p}, \mathbf{q})^{-1} \square \quad[\hat{\Delta}(\mathbf{p}, \mathbf{q})] / 2^{6} \tag{36}
\end{align*}
$$

## 3. THE PHYSICAL IMPLICATIONS OF WEYL CORRESPONDENCE [10-16]

At this point, it is pertinent to examine the physical implications of the Weyl correspondence. The conventional picture of quantum mechanics (that emerged through the work of Schrodinger, Dirac, von Neumann \& Jordan) visualizes quantum dynamics as manifesting itself on a Hilbert space, H, on which linear operators, some of which represent physically relevant and measurable quantities act. Each such operator (that must necessarily be self adjoint) is referred to as an "observable" and the action of such an observable on the state vector representing the given system in H constitutes the act of "measurement" of the physical quantity represented by the observable. This process of "measurement" of the property represented by the observable through an appropriate measuring experiment returns one of the eigenvalues of the observable as the outcome of the measuring process. The spectrum of the observable, therefore, represents the set of possible values of the measured property that the system can take. Importantly, the dynamics of the quantum system is not affected by the choice of the Hilbert space or, in fact, the basis therein. If $\mathrm{H}^{(1)}$ and $\mathrm{H}^{(2)}$ are two differently described Hilbert spaces underlying the same quantum system then the dynamics reported by the two formulations are equivalent if the operators $\mathrm{A}^{(1)}$ and $\mathrm{A}^{(2)}$ corresponding to the same observable in $\mathrm{H}^{(1)}$ and $\mathrm{H}^{(2)}$ respectively are related as
$\mathrm{A}^{(2)}=\mathrm{UA}^{(1)} \mathrm{U}^{-1}$
where U is a unitary map from $\mathrm{H}^{(1)}$ to $\mathrm{H}^{(2)}$ [10].
The process of quantizing a classical system, conventionally, consists of (a) identifying and characterizing the relevant observables, usually, by reference to the canonical formalism of classical mechanics that comprises of a $2 n$-dimensional configuration space wherein points are labeled in terms of the position coordinates $q_{i}(i=1,2,3, \ldots, n)$ and the respective canonically conjugate momenta $p_{i}$ ( $i=1,2,3, \ldots, n$ ) and (b) the process of quantizing these classical variables by replacing them with the corresponding self adjoint operators that satisfy the quantization conditions of eq. (4) [10].

The commutation relations of eq. (4) fix the representation of the canonical operators $\hat{q}, \hat{p}$ in Hilbert space upto unitary equivalence for quantum systems with finite degrees of freedom. However, for systems with infinite degrees of freedom, this no longer holds and we have an infinite number of inequivalent irreducible representations of these commutation relations [10-16]. This aspect is of cardinal importance to our analysis and we look at it in greater detail. We illustrate this feature for the case of bosons i.e. particles with integral spin. The case of fermions can be exemplified on similar lines. For this purpose, we write the canonical commutation relations in terms of the creation and annihilation operators as:

$$
\begin{equation*}
\left[\hat{a}_{i}, \hat{a}_{j}\right]=\left[\hat{a}_{i}^{\dagger}, \hat{a}_{j}^{\dagger}\right]=0 ; \quad\left[\hat{a}_{i}, \hat{a}_{j}^{\dagger}\right]=\delta_{i j} \tag{38}
\end{equation*}
$$

In terms of these operators, the particle number operator $\hat{N}_{k}=\hat{a}_{k}^{\dagger} \hat{a}_{k}$ is a positive operator, whose eigenvalues must necessarily be non-negative. It gives the occupation number of the relevant quantum state and satisfies the commutation relations

$$
\begin{equation*}
\left[\hat{N}_{k}, \hat{a}_{k}\right]=-\hat{a}_{k} ;\left[\hat{N}_{k}, \hat{a}_{k}^{\dagger}\right]=\hat{a}_{k}^{\dagger} \tag{39}
\end{equation*}
$$

A kinematical description of the system encompassing all its quantum states can, then, be obtained in the system's Fock space $\mathrm{H}_{\mathrm{F}}$. This Fock space is the direct sum of all the $n$-particle spaces with $0 \leq n \leq \infty, n \in \square$ i.e.

$$
\begin{equation*}
\mathrm{H}_{\mathrm{F}}=\sum_{n=0}^{\infty}{ }^{\oplus}\left(\mathrm{H}_{1}^{\otimes n}\right)_{S, A} \tag{40}
\end{equation*}
$$

where the suffixes $S, A$ indicate symmetrized (for bosons) and anti-symmetrized (for fermions) $n$-fold tensor product of single particle Hilbert space $\mathrm{H}_{1}$ as the representative space of $n$-indistinguishable bosons or fermions respectively. A state vector in such a space is, then, an infinite hierarchy of symmetized/antisymmetrized wavefunctions, viz.
$\Psi=\left(\begin{array}{c}c \\ \psi_{1}(\xi) \\ \psi_{2}\left(\xi_{1}, \xi_{2}\right) \\ \cdots\end{array}\right)$
In the above expression for the wavefunction, the argument $\xi$ denotes both the position (or momentum) and spin component, $\psi_{n}\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$ is the probability amplitude of finding $n$ particles in the configuration $\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$ and $c \in \square$ is the vacuum state in the Fock space spanned by the basis vectors $\left\{\left|\xi_{1}, \xi_{2}, \ldots, \xi_{j}\right\rangle, j=1,2, \ldots, n\right\}$. The ket $\left|\xi_{1}, \xi_{2}, \ldots, \xi_{j}\right\rangle, j=1,2, \ldots, n$ represents the state wherein $j$ particles exist with the configuration $\xi_{1}$ for the particle $1, \xi_{2}$ for the particle $2, \ldots,, \xi_{j}$ for the $j^{\text {th }}$ particle at the given instant of time $x^{0}$.

An equivalent description of the kinematics can be represented in "occupation number" space. For this purpose, instead of using basis vectors $\left\{\left|\xi_{1}, \xi_{2}, \ldots, \xi_{j}\right\rangle, j=1,2, \ldots, n\right\}$ where the ket $\left|\xi_{1}, \xi_{2}, \ldots, \xi_{j}\right\rangle$ represents the quantum state of finding $j$ particles with configuration $\xi_{1}, \xi_{2}, \ldots, \xi_{j}$ respectively at an instant of time $x^{0}$, we introduce the "occupation number" basis. We identify a complete and orthonormal basis of single-particle state vectors and label them by an index $k$ ( $k=1,2, \ldots$ ). Let $(n)=n_{1}, n_{2}, \ldots$ denote an occupation number distribution so that $n_{k}$ is the number of particles in the $k^{\text {th }}$ state. ( $n$ ) is an infinite sequence of occupation numbers since there are infinitely many orthogonal quantum states for one particle. However, only such sequences are admitted for which the total particle number is finite, no matter how large i.e. for which $\sum n_{k}<\infty$. In the case of fermions each $n_{k}$ can, of course, only assume the values 0 or 1 . Let $\Psi_{(n)}$ denote the normalized state vector corresponding to the occupation number distribution $(n)$. These vectors form a complete orthonormal basis in $\mathrm{H}_{F}$ referred to as the "occupation number" basis. The kinematics of the system can, then, be represented in the Fock space $\mathrm{H}_{F}$ by introducing a system of annihilation and creation operators in this basis.

Now, the number operator $\hat{N}_{k}=\hat{a}_{k}^{\dagger} \hat{a}_{k}$ has, as its eigenvalue, an occupation number $n_{k}$ which is mandated to be a non-negative integer by the commutation relations. An occupation number distribution $(n)$ is an infinite sequence of such integers. It is possible to divide the set of such sequences into classes, by defining an equivalence relation as $\left(n^{(1)}\right)$ and $\left(n^{(2)}\right)$ are in the same class if the sequences differ only in a finite number of places. Operation by a creation or annihilation operator will change ( $n$ ) only in one place. Therefore, such an operation will not result in a change in the sequence class. It follows that the basis vectors $\Psi_{(n)}$ with $(n)$ restricted to one class, already span a representation space of the $\hat{a}_{k}, \hat{a}_{k}^{\dagger}$. Further, representations belonging to different classes cannot be unitarily equivalent. The set of representations obtained in this way does, however, not exhaust all possibilities [10]. Let us also introduce the operator $\hat{H}$ that satisfies the commutation relations [11-16]

$$
\begin{equation*}
\left[\hat{a}_{k}, \hat{a}_{k}^{\diamond}\right]=2 \hat{H}_{k} ;\left[\hat{H}_{k}, \hat{a}_{k}\right]=\left[\hat{H}_{k}, \hat{a}_{k}^{\dagger}\right]=\left[\hat{H}_{k}, \hat{N}_{k}\right]=0 \tag{42}
\end{equation*}
$$

$\hat{H}_{k}$ is a central operator that remains constant in each representation A Casimir operator for labeling the representations can be obtained as

$$
\begin{equation*}
\hat{C}_{k}=2 \hat{N}_{k} \hat{H}_{k}-\hat{a}_{k}^{\dagger} \hat{a}_{k} \tag{43}
\end{equation*}
$$

with the ground state corresponding to $\hat{C}_{k}=0$.
Now, the different unitarily inequivalent representations are related by the Bogolubov transformations. For the operators $\hat{a}_{k}^{\dagger}, \hat{a}_{k}$ these transformations are generated by [11-16]

$$
\begin{equation*}
\hat{G}=-i\left(\hat{a}_{1}^{\dagger} \hat{a}_{2}^{\dagger}-\hat{a}_{1} \hat{a}_{2}\right) \tag{44}
\end{equation*}
$$

and are obtained as

$$
\begin{equation*}
\hat{A}(\theta)=e^{i \theta \hat{G}} \hat{a}_{1} e^{-i \theta \hat{G}} ; \hat{B}(\theta)=e^{i \theta \theta \hat{G}} \hat{a}_{2} e^{-i \theta \hat{G}} \tag{45}
\end{equation*}
$$

giving

$$
\begin{equation*}
\hat{A}(\theta)=\hat{a}_{1} \cosh \theta-\hat{a}_{2}^{\dagger} \sinh \theta ; \hat{B}(\theta)=\hat{a}_{2} \cosh \theta-\hat{a}_{1}^{\dagger} \sinh \theta \tag{46}
\end{equation*}
$$

together with the respective hermitian conjugates [11-16]. It may be noted that $\hat{A}(\theta)$ and $\hat{B}(\theta)$ obey the commutators $\left[\hat{A}(\theta), \hat{A}^{\dagger}(\theta)\right]=\left[\hat{B}(\theta), \hat{B}^{\dagger}(\theta)\right]=1$ and $[\hat{A}(\theta), \hat{B}(\theta)]=\left[\hat{A}^{\dagger}(\theta), \hat{B}^{\dagger}(\theta)\right]=0$.

Now, it can be shown that the operators
$\hat{J}_{1}=1 / 2\left(\hat{a}_{1}^{\dagger} \hat{a}_{2}^{\dagger}+\hat{a}_{1} \hat{a}_{2}\right) ; \quad \hat{J}_{2}=-i / 2\left(\hat{a}_{1}^{\dagger} \hat{a}_{2}^{\dagger}-\hat{a}_{1} \hat{a}_{2}\right) ; \hat{J}_{3}=1 / 2\left(\hat{a}_{1}^{\dagger} \hat{a}_{1}+\hat{a}_{2}^{\dagger} \hat{a}_{2}+I\right)$
constitute the generators of an $\operatorname{su}(1,1)$ algebra. Furthermore, the corresponding generators $\hat{J}_{1,2,3}$ with the replacements $\hat{a}_{1} \rightarrow \hat{A}(\theta), \hat{a}_{2} \rightarrow \hat{B}(\theta)$ also generate an su(1,1) algebra. It follows, therefore, that $\hat{A}(\theta), \hat{B}(\theta)$ also constitute a irreducible representation of the canonical commutation relations for each value of the parameter $\theta$ [11-16]. What remains to be shown is that these representations are unitarily inequivalent in the case of systems with infinite degrees of freedom. Let $|0,0\rangle$ be the vacuum state (that certainly exists for systems with finite degrees of freedom) for the operators $\hat{A}(\theta), \hat{B}(\theta)$ so that $\hat{A}(\theta)|0,0\rangle=\hat{B}(\theta)|0,0\rangle=0$. The functional dependence of $|0,0\rangle$ on $\theta$ is obtained as

$$
\begin{equation*}
|0(\theta), 0(\theta)\rangle=e^{i \theta \hat{G}}|0,0\rangle=\frac{1}{\cosh \theta}\left\{e^{\left[\tanh \theta \hat{A}^{\dagger}(\theta) \hat{B}^{\dagger}(\theta)\right]}\right\}|0,0\rangle=\sum_{n} c_{n}|n, n\rangle \tag{48}
\end{equation*}
$$

Obviously, the above states stay normalized for all values of the parameter $\theta$.

The above results trivially generalize to $k$ ( $k$ finite) degrees of freedom as

$$
\begin{equation*}
|0(\theta), 0(\theta)\rangle=\prod_{k} e^{i \theta \hat{G}_{k}}|0,0\rangle=\prod_{k} \frac{1}{\cosh \theta}\left\{e^{\left[\tanh \theta \hat{A}_{k}^{t}(\theta) \hat{B}_{k}^{\hat{e}}(\theta)\right]}\right\}|0,0\rangle \tag{49}
\end{equation*}
$$

The infinite degrees of freedom generalization, $k \rightarrow \infty$, is implemented by the usual substitution of the summation operator by the integration operator as $\sum_{k} \rightarrow \frac{V}{(2 \pi)^{3}} \int d^{3} k$ whence, we have

$$
\begin{align*}
& \langle 0,0 \mid 0(\theta), 0(\theta)\rangle=\langle 0,0| \operatorname{Lim}_{k \rightarrow \infty} \prod_{k} e^{i \theta \hat{G}_{k}}|0,0\rangle \\
& =\langle 0,0| \operatorname{Lim}_{k \rightarrow \infty} \prod_{k} \frac{1}{\cosh \theta}\left\{e^{\left[\tanh \theta \hat{A}_{k}^{t}(\theta) \hat{B}_{k}^{\dagger}(\theta)\right]}\right\}|0,0\rangle=0 \forall \theta \neq 0 \tag{50}
\end{align*}
$$

It follows from this that in the case of systems with infinite degrees of freedom, the states $|0(\theta), 0(\theta)\rangle$ and $|0,0\rangle$ are mutually orthogonal $\forall \theta \neq 0$. Further, the representations of the canonical commutation relations so obtained are unitarily inequivalent in this infinite degrees limit [11-16].

It is customary to identify quantum states as "pure" states, for which optimal information is available and a statistical mixture of such pure states, for which optimal information is not available. Pure states are represented by vectors of unit length in Hilbert space. Mixed states cannot be so represented. However, a formalism that can describe both pure and mixed states is the "density" operator. The density operator, $\hat{\rho}$, is a positive self adjoint operator with unit trace. For pure states $\hat{\rho}$ degenerates into a projection operator on a one dimensional subspace of the parent Hilbert space. Therefore, the
description of a pure state in terms of the density operator and in terms of the unit vector that spans the one dimensional subspace are equivalent.

To elaborate on the association of the density operator with the process of measurement we consider a discrete observable represented by a self adjoint operator $\hat{A}$ that has a discrete set of eigenvalues $\left\{a_{k}\right\}$ corresponding to the respective eigenfunctions $\left\{E_{k}\right\}$ (that form a complete set of mutually orthogonal vectors in the Hilbert space of the system). The act of measurement shall return one of the eigenvalues $\left\{a_{k}\right\}$ as the outcome of measurement. We can, then, write the following expressions forthwith [10]:

$$
\begin{equation*}
E_{i} E_{j}=\delta_{i j} E_{j} ; \sum_{j} E_{j}=I ; \hat{A}=\sum_{j} a_{j} E_{j} \tag{51}
\end{equation*}
$$

The probability of getting a result $a_{i}$ is obtained in terms of the density operator as $\operatorname{Tr}\left(E_{i} \hat{\rho}\right)$ and the expectation value of the operator $\hat{A}$ in the quantum state $\hat{\rho}$ is given by [10]

$$
\begin{equation*}
\langle\hat{A}\rangle_{\hat{\rho}}=\sum_{j} a_{j} \operatorname{Tr}\left(E_{j} \hat{\rho}\right)=\operatorname{Tr}(\hat{A} \hat{\rho}) \tag{52}
\end{equation*}
$$

In the case of an operator with a continuous spectrum the probability that in an observation the operator $\hat{A}$ will take the value $a_{i}$ is given by $\operatorname{Pr}\left(a_{j}\right)=\operatorname{Tr}\left[\delta\left(\hat{A}-a_{j}\right) \hat{\rho}\right]$.

Further,

$$
\begin{equation*}
\hat{A}^{2}=\sum_{i} \sum_{j} a_{i} a_{j} E_{i \backslash} E_{j}=\sum_{i} \sum_{j} a_{i} a_{j} \delta_{i j} E_{j}=\sum_{j} a_{j}^{2} E_{j} \tag{53}
\end{equation*}
$$

If $f(\hat{A})$ is any real valued function of the operator $\hat{A}$ then

$$
\begin{equation*}
f(\hat{A})=\sum_{j} f\left(a_{j}\right) E_{j} \tag{54}
\end{equation*}
$$

It is worth emphasizing that the use of the operator valued function $f(\hat{A})$ instead of the operator $\hat{A}$ as an observable simply amounts to relabeling or recalibrating the results of the measurement i.e. if the measured outcome using the operator $\hat{A}$ returns the result of an observation as its eigenvalue $a_{k}$ then the observable $f(\hat{A})$ will return the result $f\left(a_{k}\right)$. It follows that any element of the abelian algebra generated by $\hat{A}$ could serve as the representation of the physical quantity being measured [10].

This cardinal fact leads us to the $C^{*}$ formalism of quantum mechanics wherein the system's dynamics are analyzed in terms of the algebra of its bounded operators equipped with the norm topology. The Hilbert space of the system plays the secondary role of being a representation space for this algebra.
$C^{*}$ algebras are complex algebras $\mathbf{A}$ that are complete in a norm $\|\|$ satisfying $\| a . b\|\leq\| a\| \| b \| \forall a, b \in \mathbf{A}$ and possess an involution $a \rightarrow a^{*}$ that satisfies $\|a * a\|=\|a\|^{2}$. The $C^{*}$ algebra picture of quantum mechanics visualizes a quantum system as a $C^{*}$ algebra whose self adjoint elements form the set of observables of the system. The quantum state in such a case is defined as a linear functional $\rho: \mathbf{A} \rightarrow \square$ that is positive in that $\rho\left(a^{*} a\right) \geq 0 \forall a \in \mathbf{A}$ and normalized $\rho(I)=1$ where $I$ is the unit element of $\mathbf{A}$. Further, $\rho(\hat{A})$ is interpreted as the expectation value of the observable $\hat{A}$ in the state $\rho$. Each state $\rho$ of a $C^{*}$ algebra $\mathbf{A}$ also defines a representation $\pi_{\rho}$ of the algebra $\mathbf{A}$ on the Hilbert space $H_{\rho}$ by means of the GNS construction. Firstly, let $\rho$ be faithful so that $\rho(a * a)>0 \forall a \neq 0, a \in \mathbf{A}$. Then, the map $\rho\left(a^{*} b\right)$ defines a positive definite sesquilinear form on $\mathbf{A}$. The completion of $\mathbf{A}$ in the corresponding norm is a Hilbert space $\mathrm{H}_{\rho}$. By construction, $\mathrm{H}_{\rho}$ contains $\mathbf{A}$ as a dense subspace. Let us now define for each $a \in \mathbf{A}$ an operator $\pi_{\rho}(a)$ on $\mathbf{A}$ by $\pi_{\rho}(a) b=a b$ for $b \in \mathbf{A}$. Then, since $\pi_{\rho}(a)$ is bounded, it may be extended by continuity to all of $H_{\rho}$. Further, $\pi_{\rho}: \mathbf{A} \rightarrow \mathrm{B}\left(\mathrm{H}_{\rho}\right)$ is linear and satisfies $\pi_{\rho}\left(a_{1} a_{2}\right)=\pi_{\rho}\left(a_{1}\right) \pi_{\rho}\left(a_{2}\right)$ and $\pi_{\rho}\left(a^{*}\right)=\pi_{\rho}(a) *$ showing thereby that $\pi_{\rho}$ defines a representation of $\mathbf{A}$ on $H_{\rho}$. In the event that $\rho$ is not faithful, the sesquilinear form defined as above is not positive definite but positive semidefinite. We need to take the quotient of $\mathbf{A}$ by the kernel $N_{\rho}$ of the form i.e. the collection of all $a \in \mathbf{A}$ for which $\rho\left(a^{*} a\right)=0$. The Hilbert space $\mathrm{H}_{\rho}$ is then constructed as the completion of $\mathbf{A} / N_{\rho}$ [10].

It is worth mentioning here that the closure of the $C^{*}$ algebra (i) under involution enables the identification of physically relevant observables through self-adjoint elements of $\mathbf{A}$ (ii) under linear combinations facilitates defining of mixed states in terms of pure states (iii) under multiplication enables the defining of pure states. The algebra $\mathbf{A}$ also contains the additive and multiplicative unities and is endowed with a scalar product that satisfies

$$
\begin{equation*}
\langle\hat{A} \mid \hat{B}\rangle=\langle\hat{B} \mid \hat{A}\rangle^{*}=\left\langle\hat{B}^{*} \mid \hat{A}^{*}\right\rangle=\operatorname{Tr}\left(\hat{A}^{*} \hat{B}\right) \tag{55}
\end{equation*}
$$

Hence, we now have the following situation: The algebra $\mathbf{A}$ is composed of all operator valued functions of the canonical variables $\hat{p}, \hat{q}$. This set of operator valued functions of $\hat{p}, \hat{q}$ forms a linear vector space whose points are operators. This vector space is spanned by a complete orthonormal basis. Given such a basis, we can write any operator as a linear combination of the basis elements with the coefficients being $c$-numbers. Obviously, given the set of these $c$-numbers, we can reconstruct the operator and vice versa so that these $c$-numbers facilitate a representation of the operator in the given basis.

## 4. THE WEYL BASIS $[3-5,8]$

The properties of the $\Delta$ function elaborated in the preceding section show that they constitute such a basis in the $\hat{p}, \hat{q}$ space. This basis set is usually termed as the Wigner basis and is extensively used in the phase space representation of quantum systems. Another equivalent basis referred to as the Weyl basis is related to the Wigner basis by the transformation

$$
\begin{equation*}
\left.\hat{\Delta}(\hat{\mathbf{p}}, \hat{\mathbf{q}})=(1 / 2 \pi)^{3} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} e^{[i(\sigma \cdot \mathbf{p}+\boldsymbol{\tau})}\right] \hat{T}(-\boldsymbol{\sigma},-\boldsymbol{\tau}) \tag{56}
\end{equation*}
$$

and constitutes a basis set in the $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ space that is dual to the $(\mathbf{p}, \mathbf{q})$ space. Eq. (56) follows from the equivalent representations of an operator $\hat{A}$ in the $\hat{\Delta}$ and the $\hat{T}$ bases as:

$$
\begin{equation*}
\hat{A}(\hat{\mathbf{p}}, \hat{\mathbf{q}})=\int d \boldsymbol{\sigma} d \boldsymbol{\tau} \hat{A}_{t}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})=\int d \mathbf{p} d \mathbf{q} \hat{A}_{\Delta}(\mathbf{p}, \mathbf{q}) \hat{\Delta}(\mathbf{p}, \mathbf{q}) \tag{57}
\end{equation*}
$$

From eqs. (12) and (56), we also obtain

$$
\begin{equation*}
\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})=e^{[i(\sigma \cdot \hat{p}+\tau \cdot \hat{\boldsymbol{q}})]} \tag{58}
\end{equation*}
$$

From eq. (8), (58) and the definition of the $\delta$ function as $\delta(\mathbf{x})=\int_{-\infty}^{+\infty} \frac{d \mathbf{p}}{(2 \pi)^{3}} e^{i \mathbf{p} \cdot \mathbf{x}}$, we immediately obtain

$$
\begin{equation*}
\operatorname{Tr}[\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})]=(2 \pi)^{3} \delta(\boldsymbol{\sigma}) \delta(\boldsymbol{\tau}) \tag{59}
\end{equation*}
$$

and
$\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \hat{T}\left(\boldsymbol{\sigma}^{\prime}, \boldsymbol{\tau}^{\prime}\right)=e^{[i(\sigma \cdot \hat{p}+\tau \cdot \hat{\boldsymbol{q}})]} e^{\left[i\left(\boldsymbol{\sigma}^{\prime} \cdot \hat{\mathbf{p}}+\tau^{\prime} \cdot \hat{\mathbf{q}}\right)\right]}=e^{i\left\{\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^{\prime}\right) \hat{\mathbf{p}}+\left(\tau+\tau^{\prime}\right) \cdot \hat{\mathbf{q}}-1 /\left[\left[(\sigma \cdot \hat{\mathbf{p}}+\tau \cdot \hat{\boldsymbol{q}}),\left(\boldsymbol{\sigma}^{\prime} \cdot \hat{\mathbf{p}}+\tau^{\prime} \cdot \hat{\mathbf{q}}\right)\right]\right\}\right.}$
$=e^{i / 2\left(\sigma \tau^{\prime}-\tau \boldsymbol{\sigma}^{\prime}\right)} e^{i\left[\left(\sigma+\boldsymbol{\sigma}^{\prime} \cdot \hat{\boldsymbol{p}}+\left(\boldsymbol{\tau}+\tau^{\prime}\right) \cdot \hat{\boldsymbol{q}}\right]\right.}=e^{i / 2\left(\sigma \tau^{\prime}-\tau \boldsymbol{\sigma}^{\prime}\right)} \hat{T}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^{\prime}, \boldsymbol{\tau}+\boldsymbol{\tau}^{\prime}\right)$
We also have
1
$\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})=e^{[i(\sigma \cdot \hat{\mathbf{p}}+\tau \cdot \hat{\mathbf{q}})]}=e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} e^{i \boldsymbol{\sigma} \cdot \hat{\mathrm{p}}} e^{i \tau . \hat{\mathbf{q}}}=e^{i / 2 \boldsymbol{\sigma} \tau \boldsymbol{\tau}} e^{i \tau \cdot \hat{\mathbf{q}}} e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}}$
The representation of $\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ in position and momentum space are respectively obtained as:

$$
\begin{align*}
& \left\langle\mathbf{q}^{\prime}\right| \hat{T}(\boldsymbol{\sigma} . \boldsymbol{\tau})\left|\mathbf{q}^{\prime \prime}\right\rangle=\left\langle\mathbf{q}^{\prime}\right| e^{i(\sigma \cdot \hat{p}+\tau \cdot \hat{\mathbf{q}})}\left|\mathbf{q}^{\prime \prime}\right\rangle=\left\langle\mathbf{q}^{\prime}\right| e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} \cdot e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}} e^{i \tau . \hat{\mathbf{q}}}\left|\mathbf{q}^{\prime \prime}\right\rangle \\
& =e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} \int\left\langle\mathbf{q}^{\prime}\right| e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}}|\mathbf{p}\rangle d \mathbf{p}\langle\mathbf{p}| e^{i \tau \cdot \hat{\mathbf{q}}}\left|\mathbf{q}^{\prime \prime}\right\rangle=e^{-i / 2 \boldsymbol{\sigma} \cdot \tau .} \int d \mathbf{p} e^{i \boldsymbol{\sigma} \cdot \mathbf{p}}\left\langle\mathbf{q}^{\prime} \mid \mathbf{p}\right\rangle\left\langle\mathbf{p} \mid \mathbf{q}^{\prime \prime}\right\rangle e^{i \tau \cdot \mathbf{q}^{\prime \prime}}=\delta\left(\mathbf{q}^{\prime}-\mathbf{q}^{\prime \prime}+\boldsymbol{\sigma}\right) e^{i \tau /\left(\mathbf{q}^{\prime}+\mathbf{q}^{\prime \prime}\right)} \tag{62}
\end{align*}
$$

and

$$
\begin{align*}
& \left\langle\mathbf{p}^{\prime}\right| \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})\left|\mathbf{p}^{\prime \prime}\right\rangle=\delta\left(\mathbf{p}^{\prime}-\mathbf{p} \mathbf{p}^{\prime \prime}+\boldsymbol{\tau}\right) e^{i \boldsymbol{\sigma} / 2\left(\mathbf{p}^{\prime}+\mathbf{p}^{\prime \prime}\right)}  \tag{63}\\
& \left\langle\mathbf{p}^{\prime}\right| \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})\left|\mathbf{q}^{\prime}\right\rangle=\frac{1}{(2 \pi)^{3 / 2}} e^{-i\left[\boldsymbol{\sigma} \boldsymbol{\tau} / 2-\sigma \cdot \mathbf{p}^{\prime}-\mathbf{q}^{\prime}\left(\tau-\mathbf{q}^{\prime}\right)\right]} \tag{64}
\end{align*}
$$

Eq. (59) also follows directly from the representations (62) or (63) by setting $\mathbf{q}^{\prime}=\mathbf{q}^{\prime \prime}$ and $\mathbf{p}^{\prime}=\mathbf{p}^{\prime \prime}$ respectively. Expression for $\operatorname{Tr}\left[\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \hat{T}\left(\boldsymbol{\sigma}^{\prime}, \boldsymbol{\tau}^{\prime}\right)\right]$ can be obtained directly from eqs. (59),(60) or from the representation (62). We have

$$
\begin{align*}
& \operatorname{Tr}\left[\hat{T}(\boldsymbol{\sigma} . \boldsymbol{\tau}) \hat{T}\left(\boldsymbol{\sigma}^{\prime} . \boldsymbol{\tau}^{\prime}\right)\right]=\langle\mathbf{q}| \hat{T}(\boldsymbol{\sigma} . \boldsymbol{\tau}) \hat{T}\left(\boldsymbol{\sigma}^{\prime} . \boldsymbol{\tau} \boldsymbol{\tau}^{\prime}\right)|\mathbf{q}\rangle=\langle\mathbf{q}| e^{i / 2\left(\boldsymbol{\sigma} \boldsymbol{\tau}^{\prime}-\boldsymbol{\tau} \boldsymbol{\sigma}^{\prime}\right)} \hat{T}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^{\prime}, \boldsymbol{\tau}+\boldsymbol{\tau}^{\prime}\right)|\mathbf{q}\rangle \\
& =(2 \pi)^{3} \delta\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^{\prime}\right) \delta\left(\boldsymbol{\tau}+\boldsymbol{\tau}^{\prime}\right) \tag{65}
\end{align*}
$$

## 5. THE ASSOCIATION BETWEEN WEYL \& WIGNER BASIS [8,17-21]

The expansion of an arbitrary operator $\hat{A}$ in the Weyl basis is given by [8]

$$
\begin{equation*}
\hat{A}=\int d \boldsymbol{\sigma} d \boldsymbol{\tau} \tilde{A}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \text { where } \tilde{A}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{3} \operatorname{Tr}[\hat{T}(-\boldsymbol{\sigma},-\boldsymbol{\tau}) \hat{A}(\hat{\mathbf{p}}, \hat{\mathbf{q}})] \tag{66}
\end{equation*}
$$

and its expectation value in a quantum state $\psi$ is [8]

$$
\begin{equation*}
\langle\hat{A}\rangle_{\psi}=(2 \pi)^{3} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} \tilde{A} \chi_{\psi}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \tag{67}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi_{\psi}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{3 / 2}\langle\psi \mid \hat{T} \psi\rangle \tag{68}
\end{equation*}
$$

is the characteristic function of the corresponding Wigner distribution given by

$$
\begin{equation*}
W_{\psi}(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}(\mathbf{q}-1 / 2 \mathbf{x}) e^{-i \mathbf{p} \cdot \mathbf{x}} \psi(\mathbf{q}+1 / 2 \mathbf{x}) \tag{69}
\end{equation*}
$$

It is instructive at this point to explore the effect of operating by $\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ on a quantum state $|\mathbf{q}\rangle$. We have

$$
\begin{equation*}
\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})|\mathbf{q}\rangle=e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} \cdot e^{i \boldsymbol{\sigma} \cdot \hat{p}} e^{i \tau \cdot \hat{\mathbf{q}}}|\mathbf{q}\rangle=e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} \cdot e^{i \tau \cdot \boldsymbol{q}} e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}}|\mathbf{q}\rangle=e^{-i / 2 \boldsymbol{\sigma} \cdot \tau} e^{i \tau . \boldsymbol{q}}|\mathbf{q}-\boldsymbol{\sigma}\rangle \tag{70}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau})|\mathbf{q}\rangle=e^{i / 2 \boldsymbol{\sigma} \cdot \tau} e^{i \tau \cdot \hat{\mathbf{q}}} e^{i \sigma \cdot \hat{\mathbf{p}}}|\mathbf{q}\rangle=e^{i / 2 \boldsymbol{\sigma} \cdot \tau} e^{i \tau \cdot \hat{\mathbf{q}}}|\mathbf{q}-\boldsymbol{\sigma}\rangle=e^{i / 2 \boldsymbol{\sigma} \cdot \tau} e^{i \tau \cdot(\mathbf{q}-\boldsymbol{\sigma})}|\mathbf{q}-\boldsymbol{\sigma}\rangle \tag{71}
\end{equation*}
$$

The maps $\boldsymbol{\sigma} \rightarrow \hat{T}_{\boldsymbol{\sigma}}(\boldsymbol{\sigma})=e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}}$ and $\boldsymbol{\tau} \rightarrow \hat{T}_{\tau}(\boldsymbol{\tau})=e^{i \tau . \hat{\boldsymbol{q}}}$ constitute unitary representations, (respectively generated by the infinitesimal generators $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ ) of the additive group of reals. It, further, follows from eqs. (60) and (61) that the map $\boldsymbol{\rho} \rightarrow \hat{T}_{\boldsymbol{\rho}}(\boldsymbol{\sigma}, \boldsymbol{\tau})=e^{i[(\boldsymbol{\rho \sigma}) \cdot \hat{\mathrm{p}}+(\boldsymbol{\rho} \tau) \cdot \hat{\boldsymbol{q}}]}$ also constitutes a representation of the additive group of reals for each pair $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ [17].

Consider the Wigner quasiprobability distribution function defined by:
$W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}_{2}(\mathbf{q}-1 / 2 \mathbf{x}) e^{-i \mathbf{p} \cdot \mathbf{x}} \psi_{1}(\mathbf{q}+1 / 2 \mathbf{x})$
The characteristic function of this distribution is given by the inverse Fourier transform $F_{\psi_{2}}^{-1} F_{\psi_{1}}^{-1} W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})$ which gives [17]
$\chi_{\psi_{1}, \psi_{2}}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}_{2}(\mathbf{x}-1 / 2 \boldsymbol{\sigma}) e^{i \tau . x} \psi_{1}(\mathbf{x}+1 / 2 \boldsymbol{\sigma})$
Introducing a change in variables $\mathbf{x} \rightarrow \mathbf{x}^{\prime}=\mathbf{x}-1 / 2 \boldsymbol{\sigma}$, we have,

$$
\begin{equation*}
\chi_{\psi_{1}, \psi_{2}}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x}^{\prime} \bar{\psi}_{2}\left(\mathbf{x}^{\prime}\right) e^{i \tau .\left(\mathbf{x}^{\prime}+1 / 2 \boldsymbol{\sigma}\right)} \psi_{1}\left(\mathbf{x}^{\prime}+\boldsymbol{\sigma}\right) \tag{74}
\end{equation*}
$$

We also have from eqs. (61), (70) and (71) that

$$
\begin{equation*}
\left[\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \psi_{1}\right]\left(\mathbf{x}^{\prime}\right)=e^{i \boldsymbol{\tau}\left(\mathbf{x}^{\prime}+\boldsymbol{\sigma}\right)} \psi_{1}\left(\mathbf{x}^{\prime}+\boldsymbol{\sigma}\right) \tag{75}
\end{equation*}
$$

whence, we obtain [17]

$$
\begin{equation*}
\chi_{\psi_{1}, \psi_{2}}(\boldsymbol{\sigma}, \boldsymbol{\tau})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x}^{\prime} \bar{\psi}_{2}\left(\mathbf{x}^{\prime}\right)\left[\hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \psi_{1}\right]\left(\mathbf{x}^{\prime}\right)=(1 / 2 \pi)^{3 / 2}\left\langle\psi_{2} \mid \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \psi_{1}\right\rangle \tag{76}
\end{equation*}
$$

Let us, now, consider an arbitrary function $f$ in the phase space of the system. $f$ admits a Fourier representation

$$
\begin{equation*}
f(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3 / 2} \int d \boldsymbol{\sigma} d \hat{f} \hat{f}(\boldsymbol{\sigma}, \boldsymbol{\tau}) e^{i(\boldsymbol{\sigma} p+\tau)} \tag{77}
\end{equation*}
$$

Let $\hat{A}_{f}$ be the operator given by

$$
\begin{equation*}
\hat{A}_{f}(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3 / 2} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} \hat{f}(\boldsymbol{\sigma}, \boldsymbol{\tau}) e^{i((\hat{\boldsymbol{p}}+\tau \hat{\mathbf{q}})} \tag{78}
\end{equation*}
$$

We, then, have [17]

$$
\begin{equation*}
\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle=\left\langle\bar{f} \mid F_{\psi_{1}} F_{\psi_{2}} \chi_{\psi_{1}, \psi_{2}}\right\rangle=\left\langle F_{\psi_{2}}^{-1} F_{\psi_{1}}^{-1} \bar{f} \mid \chi_{\psi_{1}, \psi_{2}}\right\rangle=\left\langle\overline{F_{\psi_{1}} F_{\psi_{2}} f} \mid \chi_{\psi_{1}, \psi_{2}}\right\rangle \tag{79}
\end{equation*}
$$

where we have used the unitarity of the Fourier transform and the property of the Fourier transform that $F^{-1} \bar{\psi}=(\overline{F \psi})$. Writing out explicitly, the expression for the inner product, we have
$\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle=\int d \mathbf{p} d \mathbf{q} f(\mathbf{p}, \mathbf{q}) W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})$
whence

$$
\begin{align*}
& \left\langle\bar{f} \mid W_{\psi_{1}+\psi_{1}, \psi_{2}}\right\rangle=\int d \mathbf{p} d \mathbf{q} f(\mathbf{p}, \mathbf{q}) W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q}) \\
& +\int d \mathbf{p} d \mathbf{q} f(\mathbf{p}, \mathbf{q}) W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})=\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle+\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle \tag{81}
\end{align*}
$$

where we have used the fact that

$$
\begin{align*}
& W_{\psi_{1}+\psi_{1}^{\prime}, \psi_{2}}(\mathbf{p}, \mathbf{q})=(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}_{2}(\mathbf{q}-1 / 2 \mathbf{x}) e^{-i \mathbf{p} . \mathbf{x}}\left(\psi_{1}+\psi_{1}{ }^{\prime}\right)(\mathbf{q}+1 / 2 \mathbf{x}) \\
& =(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}_{2}(\mathbf{q}-1 / 2 \mathbf{x}) e^{-i \mathbf{p} . \mathbf{x}} \psi_{1}(\mathbf{q}+1 / 2 \mathbf{x})+(1 / 2 \pi)^{3 / 2} \int d \mathbf{x} \bar{\psi}_{2}(\mathbf{q}-1 / 2 \mathbf{x}) e^{-i \mathbf{p} . \mathbf{x}} \psi_{1}^{\prime}(\mathbf{q}+1 / 2 \mathbf{x}) \\
& =W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})+W_{\psi_{1}^{\prime}, \psi_{2}}(\mathbf{p}, \mathbf{q}) \tag{82}
\end{align*}
$$

Similarly, the following properties of the inner product result from the corresponding properties of $W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})$ [17]:

$$
\begin{align*}
& \left\langle\bar{f} \mid W_{\lambda \psi_{1}, \psi_{2}}\right\rangle=\lambda\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle  \tag{83}\\
& \left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}+\psi_{2}}\right\rangle=\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle+\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}{ }^{\prime}}\right\rangle  \tag{84}\\
& \left\langle\bar{f} \mid W_{\psi_{1}, \lambda \psi_{2}}\right\rangle=\bar{\lambda}\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle \tag{85}
\end{align*}
$$

and also

$$
\begin{align*}
& \left\|W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})\right\|=\left\langle W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q}) \mid W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})\right\rangle^{1 / 2} \\
& =\int d \mathbf{p} d \mathbf{q} \bar{W}_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q}) W_{\psi_{1}, \psi_{2}}(\mathbf{p}, \mathbf{q})=\left|\psi_{1}\right|\left|\psi_{2}\right| \tag{86}
\end{align*}
$$

where we have used the expression (72) for the Wigner function.
We also have, by the Cauchy Schwartz inequality

$$
\begin{equation*}
\left|\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle\right| \leq\left|\left|W_{\psi_{1}, \psi_{2}}\|\cdot\| \bar{f}\|\leq\| \bar{f} \| \cdot\right| \psi_{1}\right|\left|\psi_{2}\right| \tag{87}
\end{equation*}
$$

Putting all these results together, we find that the inner product is a bonded sesquilinear form and hence, by the Riesz representation theorem, there exists an operator such that [17]

$$
\begin{equation*}
\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle=\left\langle\psi_{2} \mid \hat{A}_{f} \psi_{1}\right\rangle \tag{88}
\end{equation*}
$$

whence it follows that

$$
\left\langle\psi_{2} \mid \hat{A}_{f} \psi_{1}\right\rangle=\left\langle\bar{f} \mid W_{\psi_{1}, \psi_{2}}\right\rangle=\left\langle\overline{F_{\psi_{1}} F_{\psi_{2}} f} \mid \chi_{\psi_{1}, \psi_{2}}\right\rangle=\int d \boldsymbol{\sigma} d \boldsymbol{\tau} F_{\psi_{1}} F_{\psi_{2}} f(\boldsymbol{\sigma}, \boldsymbol{\tau}) \chi_{\psi_{1}, \psi_{2}}(\boldsymbol{\sigma}, \boldsymbol{\tau})
$$

$$
\begin{equation*}
=(1 / 2 \pi)^{3 / 2} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} F_{\psi_{1}} F_{\psi_{2}} f(\boldsymbol{\sigma}, \boldsymbol{\tau})\left\langle\psi_{2} \mid \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \psi_{1}\right\rangle \tag{89}
\end{equation*}
$$

whence [17]

$$
\begin{equation*}
\hat{A}_{f}=(1 / 2 \pi)^{3 / 2} \int d \boldsymbol{\sigma} d \boldsymbol{\tau} F_{\psi_{1}} F_{\psi_{2}} f(\boldsymbol{\sigma}, \boldsymbol{\tau}) \hat{T}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \tag{90}
\end{equation*}
$$

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