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# MAXIMAL REDUCTION OF QUANTUM-MECHANICAL THREE PARTICLE EQUATIONS

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***Abstract:***

*It is known such formulation of integral equations for AGSK operators, when they obey 3- dimensional equations (instead of 6-dimensional), and effective potentials are to be found from the Faddeev-like equations. In this formulation various approximate methods are developed and have many advantages for studying some problems.*

*Below we use the spectral representations for Green functions and show that all positive properties of AGSK will remain and at the same time effective potentials also should be simplified.*

***Keywords:*** *Faddeev equations, Green functions.*

## 1. Introduction: kinematics and definitions

Let us consider three particles with masses $m_1, m_2, m_3$ and momentum vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$. Total momentum $\boldsymbol{\mathscr{P}} = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3$. Jacobi variables are introduced usually

$$\boldsymbol{p}_\alpha = \frac{1}{m_\beta + m_\gamma}\left(m_\beta \boldsymbol{k}_\gamma - m_\gamma \boldsymbol{k}_\beta\right)$$ - relative momentum of $(\beta,\gamma)$-subsystem,

$$\boldsymbol{q}_\alpha = \frac{1}{m_\alpha + m_\beta + m_\gamma}\left[m_\alpha\left(\boldsymbol{k}_\beta + \boldsymbol{k}_\gamma\right) - \left(m_\beta + m_\gamma\right)\boldsymbol{k}_\alpha\right]$$ - momentum of 3-d particle relative to mass-centrum of $(\beta,\gamma)$ subsystem.

The total energy of system is

$$E_0 = \frac{\mathscr{P}^2}{2M} + \frac{p_\alpha^2}{2\eta_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha},$$

where the following notations are used

$$M = m_1 + m_2 + m_3; \qquad \eta_\alpha = \frac{m_\beta m_\gamma}{m_\beta + m_\gamma}; \qquad \mu_\alpha = \frac{m_\alpha\left(m_\beta + m_\gamma\right)}{M}$$

It is useful also to introduce a shorthand definition for squares of moments

$$\tilde{p}_\alpha^2 = \frac{p_\alpha^2}{2\eta_\alpha}, \qquad \tilde{q}_\alpha^2 = \frac{q_\alpha^2}{2\mu_\alpha}$$

In the system of centrum of mass (MC system) $\boldsymbol{\mathscr{P}} = \boldsymbol{0}$ and only two independent moments $(\boldsymbol{p}_\alpha, \boldsymbol{q}_\alpha)$ remain, so we are working in $L_2(\boldsymbol{p},\boldsymbol{q})$ Hilbert space. Therefore state vectors of the Hilbert space are $|\boldsymbol{p}_\alpha, \boldsymbol{q}_\alpha\rangle$. We have a total Hamiltonian $H = H_0 + V_1 + V_2 + V_3$, and partial or subsystem Hamiltonians $H_\alpha = H_0 + V_\alpha$; $\alpha = 0,1,2.3.$ $V_0 = 0$. Correspondingly we have several Green functions (or resolvents)

Total $\qquad G(z) = (H - z)^{-1}$

Partial $\qquad G_\alpha(z) = (H_\alpha - z)^{-1}, \qquad \alpha = 0,1,2,3.$

In the 3-particle Hilbert space corresponding matrix elements are written in the following form

$$\left\langle \boldsymbol{p}'_\alpha \boldsymbol{q}'_\alpha \right| G_0(z) \left| \boldsymbol{p}_\alpha \boldsymbol{q}_\alpha \right\rangle = \left(\tilde{p}_\alpha^2 + \tilde{q}_\alpha^2 - z\right)^{-1} \delta\left(\boldsymbol{p}'_\alpha - \boldsymbol{p}_\alpha\right)\delta\left(\boldsymbol{q}'_\alpha - \boldsymbol{q}_\alpha\right)$$

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$$\langle \boldsymbol{p}'_\alpha \boldsymbol{q}'_\alpha | \hat{O}_2(z) | \boldsymbol{p}_\alpha \boldsymbol{q}_\alpha \rangle = \langle \boldsymbol{p}'_\alpha | \hat{O}_2 | \boldsymbol{p}_\alpha \rangle \delta(\boldsymbol{q}'_\alpha - \boldsymbol{q}_\alpha)$$

$$\langle \boldsymbol{p}'_\alpha \boldsymbol{q}'_\alpha | V_\alpha | \boldsymbol{p}_\alpha \boldsymbol{q}_\alpha \rangle = \langle \boldsymbol{p}'_\alpha | V_\alpha | \boldsymbol{p}_\alpha \rangle \delta(\boldsymbol{q}'_\alpha - \boldsymbol{q}_\alpha),$$

where $\hat{O}_2$ denotes 2-particle operator. Such operators we furnish by hats.

## 2. Lippmann-Schwinger and Faddeev Equations

The Lippmann-Schwinger equation for total Scattering matrix T has a form

$$T(z) = V + VG_0(z)T(z),$$

where $V = \sum_\gamma V_\gamma$ is the sum of pair potentials. Therefore the kernel of the Lippmann-Schwinger equation consists of disconnected terms and correspondingly, delta functions. Because of this fact the kernel and its any power is very singular and does not belong to the Hilbert-Schmidt class, so equations are not regular.

L.D. Faddeev introduced the following decomposition

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z)$$

It is clear that

$$T^{(\alpha)}(z) = V_\alpha + V_\alpha G_0(z)T(z),$$

From which it follows

$$\left[1 - V_\alpha G_0(z)\right]T^{(\alpha)}(z) = V_\alpha + V_\alpha G_0(z)\left[T^{(\beta)}(z) + T^{(\gamma)}(z)\right]$$

It is known from the 2-body problem that

$$\hat{T}_\alpha(z) = V_\alpha G_0(z)\hat{T}_\alpha(z) + V_\alpha$$

or

$$\left[1 - V_\alpha G_0(z)\right]\hat{T}_\alpha(z) = V_\alpha$$

from which we derive

$$\hat{T}_\alpha(z) = \left[1 - V_\alpha G_0(z)\right]^{-1} V_\alpha$$

Taking it into account we obtain

$$T^{(\alpha)}(z) = \hat{T}_\alpha(z) + \hat{T}_\alpha(z)G_0(z)\left[T^{(\beta)}(z) + T^{(\gamma)}(z)\right]; \qquad \alpha,\beta,\gamma = 0,1,2,3$$

They are the Faddeev equations. Their squared kernel (iterative kernel) does not contain delta functions anymore and therefore is a Hilbert-Schmidt like operator and equations are of Fredholm type.

The important property of Faddeev equations - they contain 2-body scattering matrices, $\hat{T}_\alpha(z)$, but not pair potentials.

But the Faddeev equations have one practical lack: Its components have a complicated relation to the observable physical amplitudes. For example, they do not contain physical amplitudes. The following quantities need be calculated

$$\int d\boldsymbol{p}'_\gamma \langle \boldsymbol{p}'_\gamma \boldsymbol{q}'_\gamma | T(z) | \boldsymbol{p}_\gamma \boldsymbol{q}_\gamma \rangle \Phi_{\gamma n}(\boldsymbol{p}_\gamma),$$

where $\Phi_{\gamma n}$ is a wave function of bound state in $\gamma$ channel.

Let us remember of Ekshtein (H.Ekshtein. Phys. Rev.101,880,1956) theorem: In the many channel system it is impossible to introduce a single scattering operator $S$ , which matrix elements $(\phi_\beta, S\phi_\alpha)$ would give scattering amplitudes for all channels, for example, for $\alpha \to \beta$ transition.

Here there are 4 partial Hamiltonians in 3 – particle system $H_\alpha\ (\alpha = 0,1,2,3)$. Therefore it is necessary to define at least 16 operators $S_{\alpha\beta}$ in order to describe all 3-particle processes.

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**3. Lovelace equations (C.Lovelace. Phys.Rev. 135,B1225, 1964)**

Basing on symmetry of total Green function let us rewrite its equation in two forms

$$G(z) = G_\beta(z) - G_\beta(z)\sum_{\gamma\neq\beta} V_\gamma G(z) =$$

$$= G_\alpha(z) - G(z)\sum_{\delta\neq\alpha} V_\delta G_\alpha(z)$$

Let substitute the second line into the first one

$$G(z) = G_\beta(z) - G_\beta(z)\left[\sum_{\gamma\neq\beta} V_\gamma - \sum_{\substack{\gamma\neq\beta\\ \delta\neq\alpha}} V_\gamma G(z) V_\delta\right] G_\alpha(z)$$

But if we substitute on the contrary the first line into the second one, one obtains

$$G(z) = G_\alpha(z) - G_\beta(z)\left[\sum_{\delta\neq\alpha} V_\delta - \sum_{\substack{\gamma\neq\beta\\ \delta\neq\alpha}} V_\gamma G(z) V_\delta\right] G_\alpha(z)$$

Denote the quantities in parenthesis by $U_{\beta\alpha}^{(\pm)}(z)$, we derive following equations for them

$$U_{\beta\alpha}^{(+)}(z) = \sum_\gamma \bar{\delta}_{\beta\gamma} V_\gamma - \sum_\delta \bar{\delta}_{\alpha\delta} U_{\beta\delta}^{(+)}(z) G_0(z) \hat{T}_\delta(z)$$

and

$$U_{\beta\alpha}^{(-)}(z) = \sum_\delta \bar{\delta}_{\alpha\delta} V_\delta - \sum_\gamma \bar{\delta}_{\beta\gamma} \hat{T}_\gamma(z) G_0(z) U_{\gamma\alpha}^{(-)}(z), \qquad \text{where } \bar{\delta}_{\beta\gamma} \equiv 1 - \delta_{\beta\gamma}$$

These are the Lovelace equations for $U_{\beta\alpha}^{(\pm)}(z)$ operators. They have the following properties:

Kernel has the same structure, as Faddeev's one. Therefore the equations are of Fredholm type.

The relations with the process amplitudes are simple. For example, let us consider process

$$(1,2)_n + 3 \;\rightarrow\; 1 + (2,3)_m$$

The amplitude of this channel is

$$\int d\boldsymbol{p}_3 d\boldsymbol{p}_1 \psi_{3n}^*(\boldsymbol{p}_3)\langle \boldsymbol{p}_3\boldsymbol{q}_3 | U_{31}^{(\pm)}(z) | \boldsymbol{p}_1'\boldsymbol{q}_1'\rangle \psi_{1m}(\boldsymbol{p}')$$

Analogously, for other channels. As the lack of Lovelace equations is considered the presence of potentials, as well as 2-body amplitudes, i.e. the symmetry of the Faddeev equations is discarded. Moreover, we need 2 sets of amplitudes $U_{\beta\alpha}^{(\pm)}(z)$, which is connected to the continuation behind the energetic surface.

**4. AGSK equations (Alt E.O., GrassbergerP.,Sandhas W.-Nucl.Phys. 82, 167, 1967;А.А.Хелашвли. ОИЯИ, P2-3371,Дубна,1967)**

Let introduce $A_{\alpha\beta}$ operators in the following symmetrical manner

$$G(z) = \delta_{\beta\alpha} G_\alpha(z) - G_\beta(z) A_{\beta\alpha}(z) G_\alpha(z),$$

(there is no summation in the repeated indices).

By this definition the channel amplitudes are related to the matrix elements of $A_{\alpha\beta}$ directly. For example

$$\left(\psi_{\beta n}^{(-)}, \psi_{\alpha m}^{(+)}\right) = \delta_{\beta\alpha}\delta_{nm} - 2\pi i \delta\left(E_{\beta n} - E_{\alpha m}\right)\left(\Phi_{\beta n}, A_{\beta\alpha}(E_{\alpha m} + i0)\Phi_{\alpha m}\right),$$

and so on, for all other amplitudes.

The introduced operators satisfy the following equations

$$A_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_\delta \bar{\delta}_{\alpha\delta} A_{\beta\delta}(z) G_0(z) \hat{T}_\delta(z)$$

and

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$$A_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha}G_0^{-1}(z) - \sum_{\gamma}\bar{\delta}_{\beta\gamma}\hat{T}_{\gamma}(z)G_0(z)A_{\gamma\delta}(z)$$

They are connected to the Lovelace operators by

$$A_{\beta\alpha}(z) = U_{\beta\alpha}^{(+)}(z) - \bar{\delta}_{\beta\alpha}G_{\alpha}^{-1}(z) =$$

$$= U_{\beta\alpha}^{(-)}(z) - \bar{\delta}_{\beta\alpha}G_{\beta}^{-1}(z)$$

The equations obtained have the following positive features

- The all advantages of the Faddeev equations are retained. The symmetry, among them, as they consist only 2-body amplitudes, $\hat{T}_{\alpha}(z)$.
- They are related to channel amplitudes by simple manner.
- Indices take values 0,1,2,3, i.e. these equations take into account all channels in 3-body system .
- There are 16 operators , but not 32, as in Lovelace case.

**5. Reduction of AGSK equations**

It is well known that 2-body scattering operator can be always decomposed into factorizable (separable) and remainder parts:

$$\hat{T}_{\gamma}(z) = \hat{T}_{\gamma}^{f}(z) + \hat{T}_{\gamma}^{R}(z)$$

Naturally this decomposition is not unique, but in any problem it may be achieved. After such a decomposition AGSK equations split into two systems :

$$A_{\beta\alpha}(z) = B_{\beta\alpha}(z) - \sum_{\gamma}B_{\beta\gamma}(z)G_0(z)\hat{T}_{\gamma}^{f}(z)G_0(z)A_{\gamma\alpha}(z)$$

and

$$B_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha}G_0^{-1}(z) - \sum_{\gamma}\bar{\delta}_{\beta\gamma}\hat{T}_{\gamma}^{R}(z)G_0(z)B_{\gamma\alpha}(z)$$

We see that the first one is a system of 3-dimensional equations, in which $B_{\gamma\alpha}(z)$ are operators of “effective potentials”. On the other hand, they must be constructed from the equation with the same difficulties.

These forms of equations dictate a rather simple scheme of approximate calculations: potential operators can be constructed from the second equations in needed approximation and then use them in the first equations, which are 3-dimensional.

This is the *first* step of reduction. The followed step is connected with the special choice of a separable term. We note that usually authors were restricted only by separable parts in their calculations.

Below we want to move further. In particular, Karlson and Zeiger ( B.R.Karlson,E.M. Zeiger. Phys. Rev. D9, 1761; D10,129,1974) used a special form for separable parts, basing on the spectral representation of 2-body Green function

$$G_{\gamma}(z) = \sum_{r}\int\frac{\left|\Phi_{\gamma r},\boldsymbol{q}_{\gamma}''\right\rangle d\boldsymbol{q}_{\gamma}''\left\langle\Phi_{\gamma r},\boldsymbol{q}_{\gamma}''\right|}{\tilde{q}_{\gamma}''^{2} - E_{\gamma r} - z} + \int\frac{\left|\psi_{\boldsymbol{p}_{\gamma}''}^{\gamma}\boldsymbol{q}_{\gamma}''\right\rangle d\boldsymbol{p}_{\gamma}''d\boldsymbol{q}_{\gamma}''\left\langle\psi_{\boldsymbol{p}_{\gamma}''}^{\gamma}\boldsymbol{q}''\right|}{\tilde{q}_{\gamma}''^{2} + \tilde{p}_{\gamma}''^{2} - z} \equiv G_{\gamma}^{B}(z) + G_{\gamma}^{C}(z)$$

Here $\Phi_{\gamma r}$ is a bound state wave function for the $\gamma$ channel with energy $E_{\gamma r}$ (discrete spectrum) and $\psi_{\boldsymbol{p}''}^{\gamma}$ is a wave function of the continuous spectrum in the same channel. Here only the completeness relation is explored in the 2-body sub-systems. Let us take this representation into account in the definition of the amplitude

$$\hat{T}_{\gamma}(z)G_0(z) = V_{\gamma}G_{\gamma}(z)$$

and take as separable contribution bound state parts

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$$\hat{T}_\gamma^f(z)G_0(z)=V_\gamma G_\gamma^B(z)$$

Continuous contribution consider by remainder relation

$$\hat{T}_\gamma^R(z)G_0(z)=V_\gamma G_\gamma^C(z)$$

If we use the relation

$$G_\gamma(z)=G_0(z)+G_0(z)\hat{T}_\gamma(z)G_0(z),$$

it follows

$$G_0(z)\hat{T}_\gamma^f(z)G_0(z)=G_\gamma^B(z)$$

This is a crucial moment, on validity of which our further consideration is dependent. In this respect our equations become

$$A_{\beta\alpha}(z)=B_{\beta\alpha}(z)+\sum_\gamma B_{\beta\gamma}(z)\tilde{G}_\gamma^B(z)A_{\gamma\alpha}(z), \qquad (*)$$

where

$$\tilde{G}_\gamma^B(z)=G_0(z)V_\gamma G_\gamma^B(z)=G_\gamma^B(z)V_\gamma G_0(z),$$

and

$$B_{\beta\alpha}(z)=-\bar{\delta}_{\beta\alpha}G_0^{-1}(z)-\sum_\gamma \bar{\delta}_{\beta\gamma}V_\gamma G_\gamma^C(z)B_{\gamma\alpha}(z)$$

From (*) it follows 3-dimensional equations for all physical channels.

Very non-trivial result follows for potential matrices. After substitution of the spectral representation we derive the intermediate relations:

- Lippmann-Schwinger like equation

$$V_\gamma\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle=\hat{T}_\gamma\left(\tilde{p}''^2+i0\right)\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle$$

and

$$B_{\beta\alpha}(z)=-\bar{\delta}_{\beta\alpha}G_0^{-1}(z)-\sum_\gamma\bar{\delta}_{\beta\gamma}\int\frac{\hat{T}_\gamma\left(\tilde{p}''^2+i0\right)\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle d\boldsymbol{p}_\gamma''d\boldsymbol{q}_\gamma''\left\langle\psi_{\boldsymbol{p}_\gamma''}^\gamma\boldsymbol{q}_\gamma''\right|B_{\gamma\alpha}(z)}{q_\gamma''^2+\tilde{p}_\gamma''^2-z}$$

Here $\hat{T}_\gamma$ entering kernel does not depend on $z$. Moreover one of moments $\boldsymbol{p}_\lambda''$ rests on the energy shell.

Therefore, we need from two particle problem the following things:

- the bound state wave functions and
- semi-energetic scattering amplitudes.

Let us rewrite derived equations in the explicit form

$$T_{\beta m,\alpha n}\left(\boldsymbol{q}_\beta',\boldsymbol{q}_\alpha;z\right)=V_{\beta m,\alpha n}\left(\boldsymbol{q}_\beta',\boldsymbol{q}_\alpha;z\right)+\sum_{\delta r}\int d\boldsymbol{q}_\delta''\frac{V_{\beta m,\delta r}\left(\boldsymbol{q}_\beta',\boldsymbol{q}_\delta'';z\right)}{E_{\delta r}+z-\tilde{\boldsymbol{q}}_\delta''^2}T_{\delta r,\alpha n}\left(\boldsymbol{q}_\delta'',\boldsymbol{q}_\alpha;z\right)$$

where the potentials are

$$V_{\beta m,\alpha n}\left(\boldsymbol{q}_\beta',\boldsymbol{q}_\alpha;z\right)=\int d\boldsymbol{p}_\beta'\Phi_{\beta m}^*\left(\boldsymbol{p}_\beta'\right)\left\langle\boldsymbol{p}_\beta'\boldsymbol{q}_\beta'\right|B_{\beta\alpha}(z)\left|\boldsymbol{p}_\alpha\boldsymbol{q}_\alpha\right\rangle\Phi_{\alpha n}\left(\boldsymbol{p}_\alpha\right)d\boldsymbol{p}_\alpha$$

Hence, problem requires knowledge of the following quantities:

$$\left\langle\Phi_{\beta m}\boldsymbol{q}_\beta'\right|\hat{T}_\gamma\left(\tilde{p}_\gamma''^2\right)\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle=\int d\boldsymbol{p}_\beta'\Phi_{\beta m}^*\left(\boldsymbol{p}_\beta'\right)\left\langle\boldsymbol{p}_\beta'\boldsymbol{q}_\beta'\right|\hat{T}_\gamma\left(\tilde{p}_\gamma''^2\right)\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle$$

and

$$\left\langle\Phi_{\beta m}\boldsymbol{q}_\beta'\right|\hat{T}_\gamma\left(\tilde{p}_\gamma''^2\right)\left|\boldsymbol{p}_\gamma''\boldsymbol{q}_\gamma''\right\rangle=\Phi_{\beta m}^*\left(\boldsymbol{q}_\gamma''+\frac{m_\gamma}{m_\alpha+m_\gamma}\boldsymbol{q}_\beta'\right)t_\gamma\left(-\boldsymbol{q}_\beta'-\frac{m_\beta}{m_\alpha+m_\beta}\boldsymbol{q}_\gamma'',\boldsymbol{p}_\gamma'';\tilde{p}_\gamma''^2+i0\right),$$

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$$\left\langle \psi^{\beta}_{\mathbf{p}'_\beta} \mathbf{q}'_\beta \left| \hat{T}_\gamma \left( \tilde{p}''^2_\gamma + i0 \right) \right| \mathbf{p}''_\gamma \mathbf{q}''_\gamma \right\rangle = \psi^{*\beta}_{\mathbf{p}'_\beta} \left( \mathbf{q}''_\gamma + \frac{m_\gamma}{m_\alpha + m_\gamma} \mathbf{q}'_\beta \right) t_\gamma \left( -\mathbf{q}'_\beta - \frac{m_\beta}{m_\alpha + m_\beta} \mathbf{q}''_\gamma , \mathbf{p}''_\gamma ; \tilde{p}''^2_\gamma + i0 \right).$$

Other needed matrix elements may be written analogously in the explicit form.

## 6. Conclusions

We considered possible maximal reduction of three-body problem. It was established above that in final equations we have the following quantities:

1. two-body physical (on- energy- shell) amplitudes,
2. potentials do not contain two-particle bound-state singularities in the 3-body space. therefore wave functions have the same phases as scattering amplitudes.
3. When we perform the angular decomposition two-particle phases will cancelled and effective potentials become real functions.

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