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THE EXACT SOLUTIONS OF THE RAYCHAUDHURI SPACETIME BASED ON LYRA MANIFOLD

Morteza Yavari
Islamic Azad University, Kashan Branch, Kashan, Iran

Abstract:
The exact solutions of the vacuum field equations for the Raychaudhuri spacetime in presence of a massless scalar field within the framework of Lyra manifold are obtained.

Keywords: *Raychaudhuri spacetime, Lyra manifold, massless scalar field.*

1. INTRODUCTION

Einstein provided a general theory of gravitation by geometry and this theory has been very successful in describing the gravitational phenomena. Einstein field equations without the cosmological constant admitted only nonstatic solutions and he introduced the cosmological constant in order to obtain the static models. The properties of the spacetime require the Riemannian geometry for their description. Several modifications of Riemannian geometry have been suggested to unify gravitation, electromagnetism and other effects in universe. One of the modified theories has been introduced by Lyra, [1]. He introduced an additional gauge function into the structureless manifold as a result of which a displacement vector field arises naturally from geometry. The Einstein field equations in normal gauge based on Lyra manifold defined by Sen [2] and Sen and Dunn [3] as[1]

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R+\frac{3}{2}\zeta_{\alpha}\zeta_{\beta}-\frac{3}{4}g_{\alpha\beta}\zeta_{\mu}\zeta^{\mu}=T_{\alpha\beta}, \tag{1}$$

where ζ_{α} is the Lyra displacement vector field and other symbols have their usual meaning as in Riemannian geometry. In Lyra formalism, the constant displacement vector field plays the same role as the cosmological constant in the standard general relativity, [4]. Also, the scalar-tensor treatment based on Lyra manifold predicts some effects, within the observational limit, as in Einstein theory, [4].

2. THE METRIC AND FIELD EQUATIONS

We assume that the metric of the spacetime is of the Raychaudhuri form, in the cylindrical coordinates, with the following line element, [5]:

$$ds^2=-\gamma^{-2}dt^2+\rho^{2m^2}\gamma^2(d\rho^2+dz^2)+\rho^2\gamma^2d\phi^2, \tag{2}$$

where γ is an unknown function of ρ and m is a constant. In this analysis, we assume that the energy-momentum tensor corresponding to the massless scalar field Φ is defined as follows[2]

$$T_{\mu\nu}=\Phi_{,\mu}\Phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\Phi_{,\sigma}\Phi^{,\sigma}, \tag{3}$$

such that Φ satisfies the Klein-Gordon equation as

[1] We choose the geometric units in which 8π G=c=1.

[2] The comma denotes partial derivatives with respect to the appropriate coordinates.

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$$\Box\Phi = 0, \tag{4}$$

where $\Box\Phi = \frac{1}{\sqrt{-g}}[\sqrt{-g}g^{\mu\nu}\Phi_{,\nu}]_{,\mu}$ while $g = \det(g_{\mu\nu})$. At first, by considering $\Phi = \Phi(\rho, t)$, the equation (4) changed to the following relation

$$\Phi'' + \frac{1}{\rho}\Phi' - \rho^{2m^2}\gamma^4\ddot{\Phi} = 0, \tag{5}$$

where the over head dot and prime indicate partial differentiation with respect to t and ρ respectively. To continue our analysis, we consider the Lyra displacement vector to be a time-like vector as

$$\zeta_{\varpi} = (\lambda, 0, 0, 0), \tag{6}$$

where λ is either a constant or a function. In the next step, the field equations (1) for the spacetime metric (2) lead to the following set of equations

$$2\frac{2\rho^2\gamma\gamma'' - \rho^2(\gamma')^2 + 2\rho\gamma\gamma' - m^2\gamma^2}{\rho^{2m^2+2}\gamma^6} + \frac{3}{2}\lambda^2 - (\dot{\Phi})^2 - \frac{(\Phi')^2}{\rho^{2m^2}\gamma^4} = 0, \tag{7}$$

$$2\frac{\rho^2(\gamma')^2 - m^2\gamma^2}{\rho^{2m^2+2}\gamma^6} + \frac{3}{2}\lambda^2 - (\dot{\Phi})^2 - \frac{(\Phi')^2}{\rho^{2m^2}\gamma^4} = 0, \tag{8}$$

$$2\frac{m^2\gamma^2 - \rho^2(\gamma')^2}{\rho^{2m^2+2}\gamma^6} + \frac{3}{2}\lambda^2 - (\dot{\Phi})^2 + \frac{(\Phi')^2}{\rho^{2m^2}\gamma^4} = 0. \tag{9}$$

By comparing equations (8) and (9), one finds

$$\lambda = \pm\sqrt{\frac{2}{3}}\,\dot{\Phi}. \tag{10}$$

From the equations (7) and (10), yields

$$\Phi' = \pm\sqrt{2}\sqrt{2\frac{\gamma''}{\gamma} - (\frac{\gamma'}{\gamma})^2 + \frac{2}{\rho}\frac{\gamma'}{\gamma} - \frac{m^2}{\rho^2}}. \tag{11}$$

In order to solve the field equations, we assume the separable form of the massless scalar field as follows

$$\Phi(\rho, t) = \Phi_1(t) + \Phi_2(\rho). \tag{12}$$

By substituting the equations (11) and (12) into equation (5), we can conclude

$$-\frac{\gamma'''}{\gamma} + \frac{2\gamma'\gamma''}{\gamma^2} - (\frac{\gamma'}{\gamma})^3 + \frac{2}{\rho}(\frac{\gamma'}{\gamma})^2 - \frac{3}{\rho}\frac{\gamma''}{\gamma} - \frac{1}{\rho^2}\frac{\gamma'}{\gamma} \pm \frac{c_1}{\sqrt{2}}\rho^{2m^2}\gamma^4\sqrt{2\frac{\gamma''}{\gamma} - (\frac{\gamma'}{\gamma})^2 + \frac{2}{\rho}\frac{\gamma'}{\gamma} - \frac{m^2}{\rho^2}} = 0, \tag{13}$$

and

$$\Phi_1(t) = \frac{1}{2}c_1t^2 + c_2t + c_3, \tag{14}$$

where c_1, c_2 and c_3 are constants of integration. In continuation, from the equations (10) and (14), we find that

$$\lambda = \pm\sqrt{\frac{2}{3}}\,(c_1t + c_2). \tag{15}$$

Unfortunately only an integral expression as $\rho = \rho(\gamma)$ can be obtained from the solution of the equation (13). However, this equation can be solved exactly only when $c_1 = 0$ and it can be shown that its solution is

$$\gamma = p \pm p\sin(a\ln(\rho) + b), \tag{16}$$

in which p, a and b are arbitrary constants. By applying this result, the integration of equation (11) with respect to ρ yields

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$$\Phi_2(\rho) = c\ln(\rho) + d, \tag{17}$$

where c and d are integration constants. Finally, we have

$$\Phi = c\ln(\rho) + \frac{1}{2}c_1 t^2 + c_2 t + \Phi_0. \tag{18}$$

Here Φ_0 is a constant.

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