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CALCULATION OF THE GRAVITOELECTROMAGNETIC FORCE FOR THE BIANCHI TYPE I SPACETIME

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abstract:
We have studied the motion of a relativistic test particle in the Bianchi type I spacetime with applying the Hamilton-Jacobi formalism. Also, in threading formalism, the gravitoelectromagnetic force in this spacetime is calculated.

Keywords: *Bianchi type I spacetime, particle trajectory, gravitoelectromagnetism force.*

1. Introduction

The slicing and threading points of view today are introduced respectively by Misner, Thorne and Wheeler [1] in 1973 and, Landau and Lifshitz [2] in 1975. Both points of view can be traced back when the Landau and Lifshitz [3] in 1941 introduced the threading point of view splitting of the spacetime metric. After them, Lichnerowicz [4] introduced the beginnings of slicing point of view. The slicing point of view is commonly referred as 3+1 or ADM formalism and also term 1+3 formalism has been suggested for the threading point of view. In this paper, we are going to work in threading formalism.
In threading point of view, splitting of spacetime is introduced by a family of timelike congruences with unit tangent vector field, may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in tangent space. Let[2] $(\mathrm{M}, \mathrm{g}_{\alpha\beta})$ be a 4-dimensional manifold of a stationary spacetime. We can construct a 3-dimensional orbit manifold as $\bar{\mathrm{M}} = \frac{\mathrm{M}}{\mathrm{G}}$ with projected metric tensor γ_{ij} by the smooth map $\Sigma : \mathrm{M} \to \bar{\mathrm{M}}$ where $\Sigma(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in \mathrm{M}$ and G is 1-dimensional group of transformations generated by timelike Killing vector of the spacetime under consideration, [5,6]. The threading decomposition leads to the following line element, [2,6,7]:

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta = h\left(dt - \mathrm{g}_i dx^i\right)^2 - \gamma_{ij} dx^i dx^j , \tag{1}$$

where the components of metric are

$$\mathrm{g}_{00} = h,\ \ \mathrm{g}_{0i} = -h\mathrm{g}_i ,\ \ \mathrm{g}_{ij} = -\gamma_{ij} + h\mathrm{g}_i \mathrm{g}_j , \tag{2}$$

and their inverse are

$$\mathrm{g}^{00} = -\mathrm{g}^2 + \frac{1}{h},\ \ \mathrm{g}^{0i} = -\mathrm{g}^i ,\ \ \mathrm{g}^{ij} = -\gamma^{ij} , \tag{3}$$

such that $\mathrm{g}^2 = \mathrm{g}^i \mathrm{g}_i = \gamma^{ij} \mathrm{g}_i \mathrm{g}_j$. In a spacetime with the time dependent metric (1), the gravitoelectromagnetic force acting on a relativistic test particle whose mass m due to time

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[2]The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3.

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dependent gravitoelectromagnetic[3] fields as measured by threading observers is described by the following equation[4], we use gravitational units with c=1, [10,11]:

$$ {}^*\mathbf{F} = \frac{{}^*d\,{}^*\mathbf{p}}{dt} - \frac{m}{\sqrt{1-{}^*v^2}}\left\{ {}^*\mathbf{E} + {}^*\mathbf{v} \times {}^*\mathbf{B} + {}^*\mathbf{M} \right\}, \tag{4} $$

where ${}^*\mathrm{p}^i = \dfrac{m\,{}^*v^i}{\sqrt{1-{}^*v^2}}$ such that ${}^*v^2 = \gamma_{ij}\,{}^*v^i\,{}^*v^j$ in which ${}^*v^i = \dfrac{v^i}{\sqrt{h}\left(1-\mathrm{g}_k v^k\right)}$ while $v^i = \dfrac{dx^i}{dt}$.

Also, the starry total derivative with respect to time is defined as $\dfrac{{}^*d}{dt} = \dfrac{{}^*\partial}{\partial t} + {}^*v^i\,{}^*\partial_i$ where $\dfrac{{}^*\partial}{\partial t} = \dfrac{1}{\sqrt{h}}\dfrac{\partial}{\partial t}$ and ${}_{*i} = {}^*\partial_i = \partial_i + \mathrm{g}_i\dfrac{\partial}{\partial t}$. In equation (4), the last term is defined as

$$ {}^*\mathrm{M}^i = -{}^*\lambda^i_{jk}\,{}^*v^j\,{}^*v^k - 2\mathrm{D}^i_k\,{}^*v^k, \tag{5} $$

where the 3-dimensional starry Christoffel symbols are defined with the following form

$$ {}^*\lambda^i_{jk} = \frac{1}{2}\gamma^{il}\left(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l}\right). \tag{6} $$

Also, deformation rates of the reference frame with respect to the observer are represented by tensors $\mathrm{D}_{ij} = \dfrac{1}{2}\dfrac{{}^*\partial\gamma_{ij}}{\partial t}$ and $\mathrm{D}^{ij} = -\dfrac{1}{2}\dfrac{{}^*\partial\gamma^{ij}}{\partial t}$. Finally, the time dependent gravitoelectromagnetism fields are defined in terms of the gravoelectric potential $\phi = \ln\sqrt{h}$ and the gravomagnetic vector potential $\mathbf{g} = \left(\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3\right)$ as follows

$$ {}^*\mathbf{E} = -{}^*\nabla\phi - \frac{\partial\mathbf{g}}{\partial t},\ {}^*\mathrm{E}_i = -\phi_{*i} - \frac{\partial\mathrm{g}_i}{\partial t}, \tag{7} $$

$$ \frac{{}^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla\times\mathbf{g},\ \frac{{}^*\mathrm{B}^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{g}_{[k*j]}, \tag{8} $$

here the curl of an arbitrary vector in a 3-space with metric γ_{ij} is defined by $\left({}^*\nabla\times\mathbf{A}\right)^i = \dfrac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{A}_{[k*j]}$ while the symbol [] represent the anticommutation over indices. For more details about applications of gravitoelectromagnetic fields, see references [12,13].

2. Motion of a test particle in the Bianchi type I spacetime

2.1. Calculation of the gravitoelectromagnetic force

As is well known, the Bianchi type cosmological models play a vital role in general relativity to discuss the early stages of evolution of universe, [14]. The Friedmann-Robertson-Walker metrics are isotropic, which are particular cases of types I, V, VII$_h$ and IX. The Bianchi type I models include the Kasner metric as a special case. Hence, we consider the Bianchi type I spacetime in Cartesian coordinates with the following line element

$$ ds^2 = dt^2 - t^2\left(dx^2 + dy^2\right) - dz^2. \tag{9} $$

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[3]The gravitoelectromagnetic refers to a set of analogies between Maxwell equations and a reformulation of the Einstein field equations in general relativity, [8,9].

[4]The vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ has components as $\mathrm{C}^i = \dfrac{\varepsilon^{ijk}}{\sqrt{\gamma}}\mathrm{A}_j\mathrm{B}_k$ in which $\gamma = \det(\gamma_{ij})$ and 3-dimensional Levi-Civita tensor ε_{ijk} is antisymmetric in any exchange of indices while $\varepsilon_{123} = \varepsilon^{123} = 1$, [2].

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We firstly determine the trajectory of a relativistic test particle of mass m moving in this spacetime by using the Hamilton-Jacobi equation, [15-17]. Therefore, we have

$$t^2\left(\frac{\partial S}{\partial t}\right)^2-\left(\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial y}\right)^2-t^2\left(\frac{\partial S}{\partial z}\right)^2-m^2t^2=0. \tag{10}$$

We now use the method of separation of variables for the Hamilton-Jacobi function for solving this equation as follows

$$S(x,y,z,t)=ax+by+cz+\psi(t), \tag{11}$$

where a, b and c are arbitrary constants and can be identified respectively as the angular momentum components of test particle along x, y and z-directions. Afterwards, with substituting the relation (11) into Hamilton-Jacobi equation, the unknown function ψ is given by

$$\psi=-L\sinh^{-1}\left(\frac{L}{\ell t}\right)+\lambda, \tag{12}$$

where $\lambda=\sqrt{\ell^2t^2+L^2}$ in which $L^2=a^2+b^2$ and $\ell^2=m^2+c^2$. Next, the equations for the trajectory can be obtained by considering the following conditions, [15-17]:

$$\frac{\partial S}{\partial a}=\text{constant},\ \frac{\partial S}{\partial b}=\text{constant},\ \frac{\partial S}{\partial c}=\text{constant}, \tag{13}$$

without loss of generality one can consider the above constants to be zero. Hence, the equations (13) respectively convert to the following equalities

$$x=\frac{a}{L}\sinh^{-1}\left(\frac{L}{\ell t}\right), \tag{14}$$

$$y=\frac{b}{L}\sinh^{-1}\left(\frac{L}{\ell t}\right), \tag{15}$$

$$z=-\frac{c}{\ell^2}\lambda. \tag{16}$$

Therefore, from equations (14-16), the trajectory of particle is obtained as

$$Z^2=1+\frac{2}{\sinh^2X+\sinh^2Y}, \tag{17}$$

where $X=\frac{L}{a}x$, $Y=\frac{L}{b}y$ and $Z=-\frac{\ell^2}{cL}z$. From the equations (14-16), we conclude

$$^{*}v^{i}=-\frac{1}{\lambda t}\begin{cases}a & i=1,\\ b & i=2,\\ ct^2 & i=3.\end{cases} \tag{18}$$

With applying the last relation and after simplifying, we lead to

$$\frac{m}{\sqrt{1-{}^{*}v^2}}=\frac{\lambda}{t}, \tag{19}$$

$$^{*}\mathbf{M}=\frac{2}{\lambda t^2}\left(a,b,0\right). \tag{20}$$

Finally, by considering equations (18-20) and using this fact that all components of gravitoelectromagnetic fields and starry Christoffel symbols are zero, we obtain

$$^{*}\mathbf{F}=0. \tag{21}$$

3. Conclusion

The behaviour of test particles in the Bianchi type I spacetime have been studied. We proved that the particles can be trapped by this gravitational field. Also, it was shown that the gravitoelectromagnetic force acting on particles in this spacetime is vanish.

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