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A SIMPLE APPROACH TO THE DESCRIPTION OF BARIONS ON THE BASIS OF SALPETER EQUATION

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Abstract:
The approach is developed to the solution of a problem of three bound constituent quarks (baryon) on the basis of Salpeter equation with two required 8-component spinors, having clear physical meaning in the spirit of a particle-antiparticle (baryon-antibaryon), without so-called relativization of full wave function. The doubtful character of consideration of two-particle interaction under quark confinement conditions is stressed. It is proposed to use expansion in terms of hyperspherical harmonics for calculations of compact bound systems with three-particle interactions. Two elementary types of the central three-particle interaction - linear and oscillatory potentials are considered. The approach proposed in this paper will be applied numerically to light baryon calculations.

Key words: *bound three-quark systems, baryons, Salpeter equation, 8-component spinors, three-particle linear and oscillatory potentials, hyperspherical harmonics.*

1. Introduction

Advances in nonrelativistic quark models for description of the bound states of qqq-systems are well-known [1]. However relatively large binding energies of light baryons (N, Σ, Λ, Ξ, Δ, Ω), consisting of constituent quarks (u, d, s), lead to the necessity of accounting for relativism. The relativistic-covariant approach is realized within the framework of Bethe-Salpeter (BS) equation [2] generalized to three particles [3]. The homogeneous integral BS equation for the bound states of quarks is derived from the first principles – the 4-dimensional six-point Green's function has a pole at energy equal to bound state (baryon) mass [4-6]. Therefore quark confinement (lack of the free spectator quark, or lack of asymptotic states of the inhomogeneous BS equation) does not interfere with the statement of the equation for three bound quarks, rather it specifies the basic role of three-particle interactions in baryons. However, problems related to probabilistic interpretation and normalization of a wave function arise. The absurdity of wave function dependence on relative time of particles is enough precisely and wittily described by the following expression: "Electron today and proton tomorrow do not form the bound state – a hydrogen atom" (see e.g. [7]). As with two particles [6], these problems are overcome at the instantaneous approximation excluding the BS equation kernel dependence on relative energy variables in momentum space. It is very important that the obtained 3-dimensional Salpeter equation remains relativistic-invariant [8] and the wave function gains usual probabilistic sense. At the same time, in a coordinate space the instantaneous approximation arranges all three quarks on a space-like hypersurface, interquarks interaction takes a potential character, i.e. interaction propagates with infinite velocity and effects of retardation are formally missing. The conventional two-particle forces (considered for either reasons in the kernel of 4-dimensional BS equation [8]) in Salpeter equation have a formal character - because of quark confinement they really are three-particle forces. At first it is necessary to make the instantaneous approximation in the kernel of 4-dimensional BS equation, and then to talk (or not to talk!) about multiparticleness of interquark interactions, rather than the reverse. The principal virtues of Salpeter equation – relativistic invariance and simultaneous affinity of potential reviewing to a nonrelativistic picture, make this equation especially attractive for the description of baryons. At the

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same time the present state of QCD does not allow a build-up of the kernel of BS equation and consequently we are forced to choose it phenomenological.

The basic role of three-particle forces in calculations of baryons together with phenomenological choice of Salpeter equation's kernel pushes us to use expansions (for required wave functions) in terms of hyperspherical harmonics (HH) most natural for this case [9, 19, 20]. In particular, using solutions of nonrelativistic Schrödinger equations with oscillatory three-particle potential along with simplification of analytical calculations we hope to achieve fast convergence in specific numerical calculations. In a sense it is possible to consider this paper as prolongation of early examinations for $q\bar{q}$ systems [10-11].

1. Bethe-Salpeter amplitudes
2.

BS amplitudes of three quarks (making a hadron) $\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}$ and their adjoint $\bar{\chi}^{Pn}_{\alpha_1\alpha_2\alpha_3}$ are defined as follows

$$\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) \equiv \langle 0 | T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\psi_{\alpha_3}(x_3)] | Pn\rangle , \quad (1)$$

$$\bar{\chi}^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) \equiv \langle Pn | T[\bar{\psi}_{\alpha_1}(x_1)\bar{\psi}_{\alpha_2}(x_2)\bar{\psi}_{\alpha_3}(x_3)] | 0\rangle . \quad (2)$$

Here: Heisenberg's field operators

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix} \quad (3)$$

(annihilation and production operators of quarks, respectively) satisfies to Dirac's equations:

$$(i\gamma^n\partial_n - m)\psi(x) = 0 , \quad \bar{\psi}(x)(i\overleftarrow{\partial}_n\gamma^n - m) = 0 ; \quad (4)$$

x_1, x_2, x_3 are usual space-time coordinates: $x^m = (x^0,\, \vec{x}),\ \ x_m = (x_0,\, -\vec{x})$; The index $\alpha_i \equiv q_i,\, f_i, c_i$ denotes the set of flavor (f_i) and color (c_i) of the Dirac quark field (q_i); $|0\rangle$ - vacuum state and $|Pn\rangle$ - state with total 4-momentum P and with a set of other quantum numbers „n”. Entering in Dirac's equations (4) γ-matrixes, satisfying to Clifford's commutation relations

$$\gamma^m\gamma^n + \gamma^n\gamma^m = 2g^{mn} \quad (5)$$

with metrics

$$g^{mn} = g_{mn} = diag\,(1,-1,-1,-1). \quad (6)$$

in Dirac's picture have the following (block) appearance:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}; \quad (\gamma^0)^\dagger = \gamma^0, \ \ \vec{\gamma}^\dagger = -\vec{\gamma} , \quad (7)$$

where the 3-dimensional vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ consists of Pauli matrices:

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$$\left.\begin{array}{l}\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\\ \sigma_i\sigma_j=i\varepsilon_{ijk}\sigma_k+\delta_{ij}.\end{array}\right\}\quad (8)$$

By means of bispinor space-time translation at shift ($x\to x+a$)

$$e^{iPa}\psi(x)e^{-iPa}=\psi(x+a) \quad (9)$$

We have

$$\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)=e^{iPa}\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1+a,x_2+a,x_3+a). \quad (10)$$

Choosing $a=-X$ and taking the determination of Jacobi coordinates into account (see. Appendix), we will obtain

$$\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)=e^{-iPX}\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(\eta,\xi), \quad (11)$$

$$\chi^{Pn}_{\alpha_1\alpha_2\alpha_3}(\eta,\xi)\equiv$$
$$\equiv\langle 0|T[\psi_{\alpha_1}\left(\sqrt{\tfrac{\mu_2}{\mu_1(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\psi_{\alpha_2}\left(-\sqrt{\tfrac{\mu_1}{\mu_2(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\psi_{\alpha_3}\left(-\sqrt{\tfrac{\mu_1+\mu_2}{\mu_3}}\xi\right)]|Pn\rangle\ . \quad (12)$$

Similarly for adjoint amplitudes we will obtain the corresponding relations:

$$\overline{\chi}^{Pn}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)=e^{iPX}\overline{\chi}^{Pn}_{\alpha_1\alpha_2\alpha_3}(\eta,\xi), \quad (13)$$

$$\overline{\chi}^{Pn}_{\alpha_1\alpha_2\alpha_3}(\eta,\xi)\equiv$$
$$\equiv\langle Pn|T[\overline{\psi}_{\alpha_1}\left(\sqrt{\tfrac{\mu_2}{\mu_1(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\overline{\psi}_{\alpha_2}\left(-\sqrt{\tfrac{\mu_1}{\mu_2(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\overline{\psi}_{\alpha_3}\left(-\sqrt{\tfrac{\mu_1+\mu_2}{\mu_3}}\xi\right)]|0\rangle \quad (14)$$

BS-amplitudes in momentum space (k -representation) we calculate using relations (11) and (12):

$$\begin{aligned}\chi^{Pn}(k_1,k_2,k_3)&=\int dx_1dx_2dx_3e^{ik_1x_1+ik_2x_2+ik_3x_3}\chi^{Pn}(x_1,x_2,x_3)=\\ &=I\int dX\,d\eta\,d\xi e^{iKX+iq\eta+ip\xi-iPX}\chi^{Pn}(\eta,\xi)=\\ &=I(2\pi)^4\delta(K-P)\chi^{Pn}(q,p)\equiv\chi^{Pn}(K;q,p),\end{aligned} \quad (15)$$

$$\chi^{Pn}(q,p)=\int d\eta\,d\xi e^{iq\eta+ip\xi}\chi^{Pn}(\eta,\xi)\ , \quad (16)$$

where $I=1/(\mu_1\mu_2\mu_3)^2$ represents Jacobian of transformation from Cartesian 4-momenta k_1, k_2 and k_3 to Jacobi 4-momenta K, q and p. In these relations for convenience the lower indexes of amplitudes are omitted. Inverse transformation:

$$\chi^{Pn}(\eta,\xi)=\int\frac{dq}{(2\pi)^4}\frac{dp}{(2\pi)^4}e^{-iq\eta-ip\xi}\chi^{Pn}(q,p)\ . \quad (17)$$

For adjoint BS amplitudes we have:

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$$\begin{gathered}\bar{\chi}^{Pn}(k_1,k_2,k_3)=I(2\pi)^4\delta(K-P)\bar{\chi}^{Pn}(q,p)\equiv\bar{\chi}^{Pn}(K;\,q,p),\\ \bar{\chi}^{Pn}(q,p)=\int d\eta d\xi e^{-iq\eta-ip\xi}\,\bar{\chi}^{Pn}(\eta,\xi),\\ \bar{\chi}^{Pn}(\eta,\xi)=\int\frac{dq}{(2\pi)^4}\frac{dp}{(2\pi)^4}e^{iq\eta+ip\xi}\,\bar{\chi}^{Pn}(q,p)\end{gathered}\tag{18}$$

3. The six-point Green's function for three quarks

The six-point Green's function is defined as follows

$$\begin{gathered}G_{\alpha_1\alpha_2\alpha_3\alpha_1'\alpha_2'\alpha_3'}(x_1,x_2,x_3;\,x_1',x_2',x_3')\equiv\\ \equiv\left\langle 0\right|T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\psi_{\alpha_3}(x_3)\bar{\psi}_{\alpha_1'}(x_1')\bar{\psi}_{\alpha_2'}(x_2')\bar{\psi}_{\alpha_3'}(x_3')]\left|0\right\rangle\end{gathered}\tag{19}$$

Using transformation (9), similar to relations (11) and (12), it is easy to get its translational-invariant expression:

$$\begin{aligned}&G_{\alpha_1\alpha_2\alpha_3\alpha_1'\alpha_2'\alpha_3'}(x_1,x_2,x_3;\,x_1',x_2',x_3')=G_{\alpha_1\alpha_2\alpha_3\alpha_1'\alpha_2'\alpha_3'}(X-X';\,\eta,\xi;\,\eta',\xi')\equiv\\ &\equiv\left\langle 0\right|T[\psi_{\alpha_1}\left(\sqrt{\tfrac{\mu_2}{\mu_1(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\psi_{\alpha_2}\left(-\sqrt{\tfrac{\mu_1}{\mu_2(\mu_1+\mu_2)}}\eta+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi\right)\psi_{\alpha_3}\left(-\sqrt{\tfrac{\mu_1+\mu_2}{\mu_3}}\xi\right)\times\\ &\times e^{-iP(X-X')}\bar{\psi}_{\alpha_1}\left(\sqrt{\tfrac{\mu_2}{\mu_1(\mu_1+\mu_2)}}\eta'+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi'\right)\bar{\psi}_{\alpha_2}\left(-\sqrt{\tfrac{\mu_1}{\mu_2(\mu_1+\mu_2)}}\eta'+\sqrt{\tfrac{\mu_3}{\mu_1+\mu_2}}\xi'\right)\bar{\psi}_{\alpha_3}\left(-\sqrt{\tfrac{\mu_1+\mu_2}{\mu_3}}\xi'\right)\left|0\right\rangle.\end{aligned}\tag{20}$$

In k -representation we have (the lower indices again are omitted)

$$\begin{aligned}&G(k_1,k_2,k_3;\,k_1',k_2',k_3')=\\ &=\int dx_1dx_2dx_3dx_1'dx_2'dx_3'e^{ik_1x_1+ik_2x_2+ik_3x_3-ik_1'x_1'-ik_2'x_2'-ik_3'x_3'}G(x_1,x_2,x_3;\,x_1',x_2',x_3')=\\ &=I^2(\pi)^4\delta(K-K')G(K;\,q,p;\,q',p'),\end{aligned}\tag{21}$$

where

$$G(K;\,q,p;\,q',p')=\int dXd\eta d\xi d\eta' d\xi' e^{iKX+iq\eta+ip\xi-iq'\eta'-ip'\xi_3'}G(X;\,\eta,\xi;\,\eta',\xi')\,.\tag{22}$$

Inverse transformation:

$$\begin{gathered}\int\frac{dK}{(2\pi)^4}G(X;\,\eta,\xi;\,\eta',\xi')=\\ \int\frac{dq}{(2\pi)^4}\frac{dp}{(2\pi)^4}\frac{dq'}{(2\pi)^4}\frac{dp'}{(2\pi)^4}e^{-iKX-iq\eta-ip\xi+iq'\eta'+ip'\xi_3'}G(K;\,q,p;\,q',p')\end{gathered}\tag{23}$$

To define contribution of qqq -system bound state to the Green's function, we will substitute a complete set of states $\sum_n\left|n\right\rangle\left\langle n\right|=1$ into expression (20):

$$\begin{aligned}G(x_1,x_2,x_3;\,x_1',x_2',x_3')=&\sum_n\left\langle 0\right|T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\psi_{\alpha_3}(x_3)]\left|n\right\rangle\\ &\times\left\langle n\right|T[\bar{\psi}_{\alpha_1'}(x_1')\bar{\psi}_{\alpha_2'}(x_2')\bar{\psi}_{\alpha_3'}(x_3')]\left|0\right\rangle\times\\ &\times\theta[\min(x_{10},x_{20},x_{30})-\max(x_{10}',x_{20}',x_{30}')]+\,...\end{aligned}\tag{24}$$

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Among all possible summands we select the term, in whom three quarks are the product in time-points $x'_{10}, x'_{20}, x'_{30}$ and space-points $\vec{x}_1{}', \vec{x}_2{}', \vec{x}_3{}'$ (production operators $\bar{\psi}$) and they annihilate in later time-points x_{10}, x_{20}, x_{30} in space-points $\vec{x}_1, \vec{x}_2, \vec{x}_3$ (annihilation operators ψ). Therein lies the essence of propagation of a hadron as a whole consisting of three quarks. This circumstance is provided by step-like θ-function $(\varepsilon > 0,\ \varepsilon \to 0)$:

$$\theta(z) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikz}}{k + i\varepsilon} = \begin{cases} 1, & z > 0\ , \\ 0, & z \le 0\ . \end{cases} \tag{25}$$

Further for calculation of a bound state contribution in this selected term we will make replacement:

$$\sum_n |n\rangle\langle n| = 1 \to \int |K_B\rangle \frac{dK}{(2\pi)^4} \theta(K_0)(2\pi)\delta(K^2 - M_B^2)\langle K_B| = 1\ , \tag{26}$$

where

$$K_B = (\Omega_B, \vec{K}), \quad \Omega_B \equiv \sqrt{M_B^2 + \vec{K}^2}\ . \tag{27}$$

Let's note that condition of completeness (26) is equivalent to the Lorentz-invariant relations:

$$\left.\begin{aligned} &\int |K_B\rangle \frac{d\vec{K}}{(2\pi)^3 \Omega_B} \langle K_B| = 1\ , \\ &\langle K_B | K'_B\rangle = 2\Omega_B (2\pi)^3 \delta(\vec{K} - \vec{K}')\ . \end{aligned}\right\} \tag{28}$$

So, the pure contribution of the bound state $|K_B\rangle$ to the Green's function is equal

$$\begin{aligned} B = \int \frac{dK}{(2\pi)^4} &\langle 0|T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\psi_{\alpha_3}(x_3)]|K_B\rangle\langle K_B|T[\bar{\psi}_{\alpha'_1}(x'_1)\bar{\psi}_{\alpha'_2}(x'_2)\bar{\psi}_{\alpha'_3}(x'_3)]|0\rangle \times \\ &\theta(K_0)(2\pi)\delta(K^2 - M_B^2)\theta[\min(x_{10}, x_{20}, x_{30}) - \max(x'_{10}, x'_{20}, x'_{30})]\ . \end{aligned} \tag{29}$$

At calculation of B we use the definition of BS-amplitudes (1) and (2) together with the definition of the step-like function (25) and formulae of transformation from x- representation to k -representation. In the limit $K \to K_B$ (more precisely, $K_0 \to \Omega_B$) we obtain the following expression of the six-point Green's function with a pole [4-6, 8]:

$$G(K; q, p; q', p') \xrightarrow[K \to K_B]{} \frac{i}{2\Omega_B} \frac{\chi^{K_B}(q,p)\bar{\chi}^{K_B}(q',p')}{K_0 - \Omega_B + i\varepsilon} = i\frac{\chi^{K_B}(q,p)\bar{\chi}^{K_B}(q',p')}{K^2 - M_B^2 + i\varepsilon}\ . \tag{30}$$

It is obvious that other members arising in expression (24) not only because of other time-orderings, but also other intermediate states $|n\rangle$, give a regular contribution.

4. BS equation for three bound quarks.

Based on works [2, 3] and according to the representation of the six-point Green's function in the form of decomposition in an infinite series with respect to interaction constant, it is possible to get

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generalization of non-uniform integral equation of BS in case of three fermions (see, for example, [8]). So, in x - representation we have:

$$G = G_0 - iG_0 \mathsf{K} \quad G = G_0 - iG\mathsf{K} \quad G_0 \ . \tag{31}$$

Here in symbolical record the following multiplication rule of operators is assumed:

$$\begin{aligned}&[AB]_{\alpha_1\alpha_2\alpha_3\alpha_1'\alpha_2'\alpha_3'}(x_1,x_2,x_3;\ x_1',x_2',x_3') = \\ &= \sum_{\beta_1\beta_2\beta_3}\int dy_1 dy_2 dy_3 A_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}(x_1,x_2,x_3;\ y_1,y_2,y_3) B_{\beta_1\beta_2\beta_3\alpha_1'\alpha_2'\alpha_3'}(y_1,y_2,y_3;\ x_1',x_2',x_3')\ .\end{aligned} \tag{32}$$

The Green's free function

$$G_{0\alpha_1\alpha_2\alpha_3\alpha_1'\alpha_2'\alpha_3'}(x_1,x_2,x_3;\ x_1',x_2',x_3') = S_{\alpha_1\alpha_1'}(x_1,x_1') S_{\alpha_2\alpha_2'}(x_2,x_2') S_{\alpha_3\alpha_3'}(x_3,x_3') \tag{33}$$

is expressed through product of full propagators of constituent quarks

$$\left.\begin{aligned}&S_{\alpha_i\alpha_i'}(x_i,x_i') \equiv \langle 0|T[\psi_{\alpha_i}(x_i)\bar{\psi}_{\alpha_i'}(x_i')]|0\rangle\ , \\ &S(x,x') = S(x-x') = i\int \frac{dk}{(2\pi)^4}\frac{\hat{k}+m}{k^2-m^2+i\varepsilon}e^{-ik(x-x')} \\ &S(k) = \int dx e^{ikx} S(x) = i\frac{\hat{k}+m}{k^2-m^2+i\varepsilon} = \frac{i}{\hat{k}-m+i\varepsilon}\ .\end{aligned}\right\}, \tag{34}$$

where the Feynman rule of poles bypass is chosen: $m \to m - i\varepsilon$ $(\varepsilon > 0,\ \varepsilon \to 0)$. The kernel K of the equation of BS, generally speaking, is the sum of all possible two- and three-body connected irreducible diagrams, which, in turn, are caused by interactions of two- and three-fermions. However in case of quarks, because of confinement, there is no particle-spectator (a free quark) and therefore it is reasonable to consider only irreducible kernel of three-particle interaction. Moreover, due to the lack of asymptotic states with free quarks, there is no sense in considering the non-uniform equation of BS. But before, using the relations (15)-(18) and (19), (21)-(23), we will write down BS equation (31) in k -representation:

$$G = G_0 + \beta G_0 \mathsf{K} \quad G = G_0 + \beta G\mathsf{K} \quad G_0\ , \quad \beta = -iI^2\ . \tag{35}$$

Taking into account a multiplication rule of operators

$$\begin{aligned}&[A(K)B(K)](q,p;\ q',p') = \\ &= \int \frac{dq''}{(2\pi)^4}\frac{dp''}{(2\pi)^4} A(K;\ q,p;\ q'',p'') B(K;\ q'',p'';\ q',p')\end{aligned} \tag{36}$$

in expanded form this equation registers as follows:

$$\begin{aligned}&G(K;q,p;q',p') = (2\pi)^4\delta(q-q')(2\pi)^4\delta(p-p')G_0(K;q,p) + \\ &+ \beta G_0(K;q,p)\int \frac{dq''}{(2\pi)^4}\frac{dp''}{(2\pi)^4}\mathsf{K}\ \ (K;\ q,p;\ q'',p'')G(K;\ q'',p'';\ q',p')\ ,\end{aligned} \tag{37}$$

where

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$$G_0(K;q,p) = S(k_1)S(k_2)S(k_3) \ , \tag{38}$$

and k_1, k_2 , k_3 variables also are defined by relations (A.2). Let's note that in the case of a choice of standard coordinates of Jacobi $I = 1$ and $\beta = -i$. Now we have all necessary to write down the BS equation for bound three quarks. Let's multiply the equation (39) on $(K^2 - M_B^2 + i\varepsilon)$ and then pass to the limit $K \to K_B$:

$$\begin{aligned}\chi^{K_B}(q,p) = \beta S\left(\mu_1 K_B + \sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}\, q + \mu_1\sqrt{\frac{\mu_3}{\mu_1+\mu_2}}\, p \right)\times \\ \times S\left(\mu_2 K_B - \sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}\, q + \mu_2\sqrt{\frac{\mu_3}{\mu_1+\mu_2}}\, p \right)\times \\ \times S\left(\mu_3 K_B - \sqrt{\mu_3(\mu_1+\mu_2)}\, p\right)\int \frac{dq'}{(2\pi)^4}\frac{dp'}{(2\pi)^4} \mathsf{K}\ \ (K_B;\, q,p;\, q',p')\chi^{K_B}(q',p') \ .\end{aligned} \tag{39}$$

New equation from the first equation (35) follows

$$G_0^{-1} G = 1 + \beta \mathsf{K}\ \ G \ , \tag{40}$$

where

$$G_0^{-1}(K;q,p;q',p') = (2\pi)^4 \delta(q-q')(2\pi)^4 \delta(p-p')[G_0(K;q,p)]^{-1} \ , \tag{41}$$

$$[1(K)](q,p;\, q',p') = (2\pi)^4 \delta(q-q')(2\pi)^4 \delta(p-p') \ . \tag{42}$$

It is easy to check that expression (42) is the “solution” of relation (36) in case of A=B=1. From the equation (40), in turn, relations follow:

$$\left.\begin{aligned} (G_0^{-1} - \beta \mathsf{K}\ \ )G = 1 \ , \\ G_0^{-1} - \beta \mathsf{K}\ \ = G^{-1} \ . \end{aligned}\right\} \tag{43}$$

If $G^{-1}(K)$ exists, then differentiating the equation (40) with respect to K_0 and using relations (43), we come to the expression:

$$\frac{\partial G}{\partial K_0} = -G\left(\frac{\partial G_0^{-1}}{\partial K_0} - \beta \frac{\partial \mathsf{K}}{\partial K_0} \right) G = -G \frac{\partial G^{-1}}{\partial K_0} G \ . \tag{44}$$

Now we will pass to the limit $K \to K_B$ and use pole-expression (30) for the Green’s function. As a result so-called conditions of normalization turn out:

$$\begin{cases} i\bar{\chi}^{K_B}\left(\dfrac{\partial}{\partial K_0} G^{-1} \right)_{K_0=\Omega_B} \chi^{K_B} = 2\Omega_B, \\ i\displaystyle\int \frac{dq}{(2\pi)^4}\frac{dp}{(2\pi)^4}\frac{dq'}{(2\pi)^4}\frac{dp'}{(2\pi)^4} \bar{\chi}^{K_B}(q,p) \left[\frac{\partial}{\partial K_0} G^{-1}(K;\, q,p;\, q',p') \right]_{K_0=\Omega_B} \chi^{K_B}(q',p') = 2\Omega_B \ . \end{cases} \tag{45}$$

5. **Three dimensional relativistic equations.**

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As stated above dependence of BS amplitude on relative time of particles in x-representation (or on relative energy of particles in k- representation) creates difficulties with the probabilistic interpretation of the wave function. One of the ways to overcome these difficulties consists in averaging with respect to energy variables [12]. Below by analogy to work [11] at first we make the averaging of the BS equation with respect to "energy" variables which leads to quasipotential equation. Next we take the kernel of this equation in instantaneous approximation and obtain Salpeter equation. Natural generalization in the case of three particles is the concept "double tilde operation" for arbitrary $\tilde{\tilde{A}}(K;q,p;q',p') \equiv \tilde{\tilde{A}}(K;\vec{q},\vec{p};\vec{q}',\vec{p}')$:

$$\tilde{\tilde{A}}(K;\vec{q},\vec{p};\vec{q}',\vec{p}') \equiv \tilde{\tilde{A}}(K;\vec{\kappa},\vec{\kappa}') = \int \frac{dq_0}{2\pi}\frac{dp_0}{2\pi}\frac{dq_0'}{2\pi}\frac{dp_0'}{2\pi} A(K;q,p;q',p') \ . \tag{46}$$

(Here and below we use contracted notations from Appendix A). For the free Green's function we have:

$$\tilde{\tilde{G}}_0(K;\vec{\kappa},\vec{\kappa}') = (2\pi)^6 \delta(\kappa-\vec{\kappa}')\int \frac{dq_0}{2\pi}\frac{dp_0}{2\pi} G_0(K;q,p) \ . \tag{47}$$

Integrations taking into account the Feynman rule of pole bypass $m \to m - i\varepsilon \quad (\varepsilon \to 0)$ give the following result:

$$\begin{aligned} &\tilde{\tilde{G}}_0(K;\vec{\kappa},\vec{\kappa}') = iI^{1/4}(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}') \times \\ &\times \left\{ \frac{\Lambda_1^+(\vec{k}_1)\Lambda_2^+(\vec{k}_2)\Lambda_3^+(\vec{k}_3)}{K^0 + i\varepsilon - \omega_1(\vec{k}_1) - \omega_2(\vec{k}_2) - \omega_3(\vec{k}_3)} + \frac{\Lambda_1^-(\vec{k}_1)\Lambda_2^-(\vec{k}_2)\Lambda_3^-(\vec{k}_3)}{K^0 + \omega_1(\vec{k}_1) + \omega_2(\vec{k}_2) + \omega_3(\vec{k}_3)} \right\} \gamma_0^{(1)} \cdot \gamma_0^{(2)} \cdot \gamma_0^{(3)} = \\ &= iI^{1/4}(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\left[K^0 + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right]^{-1} \hat{\Pi}(\vec{K};\vec{\kappa})\gamma_0^{(1)} \cdot \gamma_0^{(2)} \cdot \gamma_0^{(3)} \ . \end{aligned} \tag{48}$$

Here the projector operator

$$\hat{\Pi}(\vec{K};\vec{\kappa}) = \Lambda_1^+(\vec{k}_1)\Lambda_2^+(\vec{k}_2)\Lambda_3^+(\vec{k}_3) + \Lambda_1^-(\vec{k}_1)\Lambda_2^-(\vec{k}_2)\Lambda_3^-(\vec{k}_3) \tag{49}$$

is expressed through usual one-particle projection operators

$$\Lambda^{\pm}(\vec{k}) = \frac{\omega(\vec{k}) \pm h(\vec{k})}{2\omega(\vec{k})} \ , \tag{50}$$

where $h(\vec{k}) = \gamma_0\vec{\gamma}\cdot\vec{k} + \gamma_0 m$ - is the Dirac Hamiltonian of a free particle and $\omega(\vec{k}) = \sqrt{m^2 + \vec{k}^2}$. When obtaining expression (48) we use the following relations

$$\begin{cases} \gamma_0 h(\vec{k}) = h(-\vec{k})\gamma_0 , \\ \gamma_0 \Lambda^{\pm}(-\vec{k}) = \Lambda^{\pm}(\vec{k})\gamma_0 , \\ h^2(\vec{k}) = \omega^2 , \\ h(\vec{k})\Lambda^{\pm}(\vec{k}) = \pm\omega\Lambda^{\pm}(\vec{k}). \end{cases} \tag{51}$$

In center-of-mass system (CMS) $K = (M, \vec{0})$:

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$$\tilde{\tilde{G}}_0(M;\vec{\kappa},\vec{\kappa}') = F_M(\vec{\kappa},\vec{\kappa}')\hat{\Pi}(-\vec{\kappa}) , \tag{52}$$

$$F_M(\vec{\kappa},\vec{\kappa}') = iI^{1/4}(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\left[M+i\varepsilon-h_1(k_1)-h_2(k_2)-h_3(k_3)\right]^{-1}\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)} . \tag{53}$$

On the grounds of the projection operator properties, it is possible to show that for the free Green's function $\tilde{\tilde{G}}_0$ there is no inverse operator $\tilde{\tilde{G}}_0^{-1}$. Let's perform the double tilde-operation (46) over the BS equation in CMS:

$$\tilde{\tilde{G}} = \tilde{\tilde{G}}_0 + \beta G_0 \mathbf{K} \quad G . \tag{54}$$

Inverse operator of $\tilde{\tilde{G}}$ has the following appearance:

$$\tilde{\tilde{G}}^{-1} = \tilde{\tilde{G}}_0^{-1} - \beta\tilde{\tilde{G}}_0^{-1} G_0 \mathbf{K} \quad G\tilde{\tilde{G}}_0^{-1} + \beta^2\tilde{\tilde{G}}_0^{-1} G_0 \mathbf{K} \quad G\tilde{\tilde{G}}_0^{-1} G_0 \mathbf{K} \quad G\tilde{\tilde{G}}_0^{-1} - \ldots . \tag{55}$$

Direct check gives $\tilde{\tilde{G}}^{-1}\tilde{\tilde{G}} = 1$. However, since $\tilde{\tilde{G}}_0^{-1}$ doesn't exist, from equation (55) it is clear that $\tilde{\tilde{G}}^{-1}$ doesn't exist either. Let's introduce such auxiliary operator

$$\tilde{\tilde{G}}' = F + \beta G_0 \mathbf{K} \quad G , \tag{56}$$

that for it there is an inverse operator $\tilde{\tilde{G}}'^{-1}$. From equation (54) and relations (52), (53) we obtain:

$$\tilde{\tilde{G}}' = \tilde{\tilde{G}} + F_M(\vec{\kappa},\vec{\kappa}')[1-\hat{\Pi}(-\vec{\kappa})] . \tag{57}$$

The inverse operator:

$$\begin{aligned}\tilde{\tilde{G}}'^{-1} &= [1+\beta F^{-1} G_0 \mathbf{K} \quad G]^{-1}F^{-1} = \\ &= F^{-1} - \beta F^{-1} G_0 \mathbf{K} \quad GF^{-1} + \beta^2 F^{-1} G_0 \mathbf{K} \quad GF^{-1} G_0 \mathbf{K} \quad GF^{-1} - \ldots = F^{-1} - U .\end{aligned} \tag{58}$$

Here the quasipotential U is defined by the following expression:

$$U = \beta F^{-1} G_0 \mathbf{K} \quad GF^{-1} - \beta^2 F^{-1} G_0 \mathbf{K} \quad GF^{-1} G_0 \mathbf{K} \quad GF^{-1} + \ldots . \tag{59}$$

From (53) we obtain:

$$F_M^{-1}(\vec{\kappa},\vec{\kappa}') = -iI^{-1/4}(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\gamma_0\cdot\gamma_0\cdot\gamma_0\left[M+i\varepsilon-h_1(k_1)-h_2(k_2)-h_3(k_3)\right] . \tag{60}$$

Thus, we have relations:

$$\tilde{\tilde{G}}'^{-1}\tilde{\tilde{G}}' = \tilde{\tilde{G}}'\tilde{\tilde{G}}'^{-1} = 1 , \tag{61}$$

which are confirmed by direct multiplication of expressions (56) and (58). From the pole- type Green's function (30) and definition of "double tilde operation" (46) we obtain:

$$\tilde{\tilde{G}}' \approx \tilde{\tilde{G}} \xrightarrow[M\to M_B]{} i\frac{\Phi_M(\vec{\kappa})\bar{\Phi}_M(\vec{\kappa}')}{2M_B(M-M_B+i\varepsilon)} = i\frac{\Phi_M(\vec{\kappa})\bar{\Phi}_M(\vec{\kappa}')}{M^2-M_B^2+i\varepsilon} , \tag{62}$$

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where

$$\Phi_M(\vec{\kappa}) = \int \frac{dq_0}{2\pi}\frac{dp_0}{2\pi}\chi^M(q,p) \,. \tag{63}$$

Substitution (62) into (61) leads to the equations:

$$\tilde{\tilde{G}}'^{-1}\Phi_M = 0; \quad \bar{\Phi}_M \tilde{\tilde{G}}'^{-1} = 0 \,, \tag{64}$$

which can be presented in expanded form:

$$\begin{cases} \int \tilde{\tilde{G}}'^{-1}(M;\vec{\kappa},\vec{\kappa}')\Phi_M(\vec{\kappa}')d\vec{\kappa}' = 0, \\ \int \bar{\Phi}_M(\vec{\kappa}')\tilde{\tilde{G}}'^{-1}(M;\vec{\kappa},\vec{\kappa}')d\vec{\kappa}' = 0. \end{cases} \tag{65}$$

Substituting expression (58) for $\tilde{\tilde{G}}'^{-1}$ together with expression (60) for F_M^{-1} in equations (65), we obtain the ***q u a s i p o t e n t i a l*** equations for three bound particles:

$$\left[M + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right]\Phi_M(\vec{\kappa}) = \\ = iI^{1/4}\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int U(M;\vec{\kappa},\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi_M(\vec{\kappa}') \,; \tag{66}$$

$$\tilde{\Phi}_M(\vec{\kappa})\left[M + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right] = \\ = iI^{1/4}\int \tilde{\Phi}_M(\vec{\kappa}')\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\frac{d\vec{\kappa}'}{(2\pi)^6}U(M;\vec{\kappa}',\vec{\kappa}) \,, \tag{67}$$

$$\tilde{\Phi}_M(\vec{\kappa}) = \bar{\Phi}_M(\vec{\kappa})\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)} = \Phi_M^{\dagger}(\vec{\kappa}). \tag{68}$$

Now we will deduce some useful relations. We multiply the relation (58) on the right on $\tilde{\tilde{G}}'$, and then on the left on F :

$$\tilde{\tilde{G}}' = F + FU\tilde{\tilde{G}}' \,. \tag{69}$$

Comparison of relations (56) and (69) gives compact expression for quasipotential:

$$U = \beta F^{-1}\bar{G}_0 K \; G\tilde{\tilde{G}}'^{-1} \,. \tag{70}$$

In a kernel of the BS equation (33) we will make so-called ***i n s t a n t a n e o u s a p p r o x i m a t i o n***:

$$K \;(K;q,p;\; q',p') \to V(M;\vec{q},\vec{p};\; \vec{q}',\vec{p}') \,. \tag{71}$$

Direct calculations give the following result:

$$\bar{G}_0 K \; G = \tilde{\tilde{G}}_0 V\,\tilde{\tilde{G}} = F\hat{\Pi}(-\vec{\kappa})V\,\tilde{\tilde{G}} \,. \tag{72}$$

At first we will insert expression (72) into relation (70), and then the obtained expression in relation (58):

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$$\tilde{\tilde{G}}'^{-1} = F^{-1} - \beta\hat{\Pi}(-\vec{\kappa})V\tilde{\tilde{G}}\tilde{\tilde{G}}'^{-1} \ . \tag{73}$$

After that the equations (64) assume the following form:

$$\begin{cases} F^{-1}\Phi_M = \beta\hat{\Pi}(-\vec{\kappa})V\tilde{\tilde{G}}\tilde{\tilde{G}}'^{-1}\Phi_M \ , \\ \bar{\Phi}_M F^{-1} = \beta\bar{\Phi}_M\hat{\Pi}(-\vec{\kappa})V\tilde{\tilde{G}}\tilde{\tilde{G}}'^{-1}. \end{cases} \tag{74}$$

From relations (57) and (62) it is visible that in a limit $M \to M_B$, the equalities $\tilde{\tilde{G}}' \approx \tilde{\tilde{G}}$ and therefore $\tilde{\tilde{G}}\tilde{\tilde{G}}'^{-1} = 1$ takes place. In this case the equations (74) become simpler:

$$\begin{cases} F^{-1}\Phi_M = \beta\hat{\Pi}(-\vec{\kappa})V\Phi_M \ , \\ \bar{\Phi}_M F^{-1} = \beta\bar{\Phi}_M\hat{\Pi}(-\vec{\kappa})V. \end{cases} \tag{75}$$

From here, using relations (60) and (49), we obtain ***S a l p e t e r*** equations:

$$\begin{aligned} &\left[M + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right]\Phi_M(\vec{\kappa}) = \\ &= I^{9/4}\hat{\Pi}(\vec{\kappa})\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi_M(\vec{\kappa}') \ ; \end{aligned} \tag{76}$$

$$\begin{aligned} &\tilde{\Phi}_M(\vec{\kappa})\left[M + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right] = \\ &= I^{9/4}\int\tilde{\Phi}_M(\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}\hat{\Pi}(\vec{\kappa})\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}V(M;\vec{\kappa}',\vec{\kappa}) \ . \end{aligned} \tag{77}$$

These equations can also be written in another form. So, for example, acting on equation (76) with operator $\left[M + i\varepsilon - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3)\right]^{-1}$ at the left and using relations (51) we shall obtain Salpeter equation in the form:

$$\begin{aligned} \Phi_M(\vec{\kappa}) = I^{9/4}&\left\{\frac{\Lambda_1^+(k_1)\Lambda_2^+(k_2)\Lambda_3^+(k_3)}{M + i\varepsilon - \omega_1 - \omega_2 - \omega_3} + \frac{\Lambda_1^-(k_1)\Lambda_2^-(k_2)\Lambda_3^-(k_3)}{M + \omega_1 + \omega_2 + \omega_3}\right\}\times \\ &\times\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi_M(\vec{\kappa}') \ . \end{aligned} \tag{78}$$

From here, taking $\Lambda^\pm(\vec{k})$ projector operators properties into account, it is easy to obtain very useful additional equations:

$$\Phi_M^{++-}(\vec{\kappa}) = \Phi_M^{+-+}(\vec{\kappa}) = \Phi_M^{-++}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = \Phi_M^{-+-}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = 0 \ , \tag{79}$$

where

$$\Phi_M^{\pm\pm\pm}(\vec{\kappa}) = \Lambda_1^\pm(\vec{k}_1)\Lambda_2^\pm(\vec{k}_2)\Lambda_3^\pm(\vec{k}_3)\Phi_M(\vec{\kappa}) \ . \tag{80}$$

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Further, representing 64-component spinor $\Phi_M(\vec{\kappa})$ in the form of a block column involving Φ_1, Φ_2 ,..., Φ_8 eight 8-component spinors

$$\Phi_M(\vec{\kappa}) = \begin{pmatrix} \Phi_1(\vec{\kappa}) \\ \Phi_2(\vec{\kappa}) \\ \Phi_3(\vec{\kappa}) \\ \Phi_4(\vec{\kappa}) \\ \Phi_5(\vec{\kappa}) \\ \Phi_6(\vec{\kappa}) \\ \Phi_7(\vec{\kappa}) \\ \Phi_8(\vec{\kappa}) \end{pmatrix}, \tag{81}$$

and solving six (block) equations (84), it is possible to express the spinors Φ_i (i=2,3,…7) through two spinors Φ_1 and Φ_8 :

$$\begin{cases} \Phi_2 = \{(1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_1 + (1+\varepsilon_3)\lambda_1\lambda_2\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \\ \Phi_3 = \{(1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_1 + (1+\varepsilon_2)\lambda_1\lambda_3\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \\ \Phi_4 = \{(1+\varepsilon_1)\lambda_2\lambda_3\Phi_1 - (1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \\ \Phi_5 = \{(1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_1 + (1+\varepsilon_1)\lambda_2\lambda_3\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \\ \Phi_6 = \{(1+\varepsilon_2)\lambda_1\lambda_3\Phi_1 - (1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \\ \Phi_7 = \{(1+\varepsilon_3)\lambda_1\lambda_2\Phi_1 - (1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_8\} / (1+\varepsilon_1\varepsilon_2\varepsilon_3), \end{cases} \tag{82}$$

where

$$\begin{gathered} \lambda_i(\vec{k}_i) \equiv \vec{\sigma}_i \cdot \vec{k}_i \,/\, \omega_i(\vec{k}_i) + m_i, \quad \lambda_i\lambda_i = \varepsilon_i, \quad \varepsilon_i = (\omega_i - m_i)/(\omega_i + m_i), \\ (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot[\vec{a}\times\vec{b}] \end{gathered} \tag{83}$$

Thus, the determination of 64 components of the spinor Φ_M is reduced to the determination of 16 components of two 8-component spinors Φ_1 and Φ_8. Let's note here that in calculations with direct multiplication of matrices, it is important to adhere to the one chosen multiplication rule everywhere.

The Feynman rule of poles bypass $m \to m - i\varepsilon$ $(\varepsilon > 0,\ \varepsilon \to 0)$ in the full propagator of the quark (34), as it is known [13], leads to the possibility of the motion of particles both forward and backward in time. The particle moving backward in time is equivalent to an antiparticle moving forward. This rule maintains the covariance of the theory as infinitesimal imaginary addition $-i\varepsilon$ is entered into invariant mass m. However, another possibility of poles bypass exists: $k_0 \to k_0 + i\varepsilon$. It violates the covariance of the theory and corresponds to the propagation of particles only forward in time (there are no antiparticles), i.e. in this case we have only the retarded Green's function: $S(x) = 0$, when $x_0 < 0$. If in the calculations mentioned above we replace the Feynman propagator $i(\hat{k}+m)\big/(k^2 - m^2 + i\varepsilon)$ on the retarded Green's function $i(m+\hat{k})\big/[(k_0 + i\varepsilon)^2 - \omega^2]$, instead of the relation (48) for the averaged free Green's function we will get an expression in which the projective operator $\hat{\Pi}(\vec{K};\vec{\kappa})$ is replaced on "1":

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$$\tilde{\tilde{G}}_0(K;\vec{\kappa},\vec{\kappa}') = iI^{1/4}(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')$$
$$\left[K^0 + i\varepsilon - h_1(k_1) - h_2(k_2) - h_3(k_3)\right]^{-1}\cdot\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\ . \tag{84}$$

It is clear that because of the absence of projector operators in the expression of $\tilde{\tilde{G}}_0$ there is the inverse operator $\tilde{\tilde{G}}_0^{-1}$, and also $\tilde{\tilde{G}}^{-1}$. As a result we will obtain equations of the form (66) and (67) with a modified quasipotential (see expression (70)):

$$U = \beta\tilde{\tilde{G}}_0^{-1}\, G_0 \mathbf{K} \quad G\tilde{\tilde{G}}^{-1}\ . \tag{85}$$

Performing here transition to instantaneous approximation (74), we get a simple expression for "quasipotential" U :

$$U = \beta V\ . \tag{86}$$

Further substituting this expression in the equation of form (66), we come to a generalization of the known equation of Breit [14] for a case of three bound particles:

$$\left[M + i\varepsilon - h_1(k_1) - h_2(k_2) - h_3(k_3)\right]\Phi_M(\vec{\kappa}) =$$
$$= I^{9/4}\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi_M(\vec{\kappa}')\ . \tag{87}$$

Externally this equation differs from Salpeter equation (76) only on factor $\hat{\Pi}(\vec{\kappa})$. Behind this seemingly simple difference there is sufficiently deep physical sense (see, for example, [15, 8]). The matter is that BS amplitude dependence on relative time is caused by the existence of antiparticles. Existence of antiparticles gives the possibility to particles move in time back. Motion forward-back in time makes essential configurations for which individual routes in time are different for the bound particles and so relative time is large. The projector operator $\hat{\Pi}(\vec{\kappa})$ in Salpeter equation is that "relict" of averaging of BS equation which corresponds to the contribution of these configurations in bound state amplitude. At the same time, as well as has to be, relative time in Breit equation completely disappears. In it the contribution of antiparticles absolutely is absent, but the relativistic kinematics is considered. If the covariant choice of rule of pole bypass, and also covariant form of instantaneous approximation (71) at the expense of introduction of variables of the Wallace-Mandelzweig's [16], results the Lorentz- invariant Salpeter equation (see, for example, [8]), then in case of Breit equation the Lorentz-invariance is violated from the beginning because of the not covariant rule of pole bypass.
As it was noted above, finding of full Salpeter amplitude is reduced to finding of two 8-component spinors. This circumstance favorable distinguishes Salpeter equation from all other 3-dimensional relativistic equations, and further we effectively use it at construction of amplitude and solution of equation.

6. The normalizing condition for amplitudes of quasipotential and Salpeter equations

Let's multiply equation (58) on the right on $\tilde{\tilde{G}}'$ and then differentiate the obtained relation with respect to M:

$$\frac{\partial\tilde{\tilde{G}}'}{\partial M} = -\tilde{\tilde{G}}'\left(\frac{\partial F^{-1}}{\partial M} - \frac{\partial U}{\partial M}\right)\tilde{\tilde{G}}'\ . \tag{88}$$

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Taking into account relation (62) this expression in point $M = M_B$ will register as the following normalization condition:

$$\frac{1}{I^{1/4}}\int\frac{d\vec{\kappa}}{(2\pi)^6}\tilde{\Phi}_M(\vec{\kappa})\Phi_M(\vec{\kappa}) - i\int\frac{d\vec{\kappa}}{(2\pi)^6}\frac{d\vec{\kappa}'}{(2\pi)^6}\tilde{\Phi}_M(\vec{\kappa})\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\times$$
$$\times\left[\frac{\partial}{\partial M}U(M;\vec{\kappa},\vec{\kappa}')\right]_{M=M_B}\Phi_M(\vec{\kappa}') = 2M_B\ . \tag{89}$$

In instantaneous approximation from (71) and (73), it follows that quasipotential $U = \beta\hat{\Pi}(-\vec{\kappa})\mathbf{K}$. Substituting this expression in condition (88), we will obtain the general normalization condition for Salpeter amplitude:

$$\frac{1}{I^{1/4}}\int\frac{d\vec{\kappa}}{(2\pi)^6}\Phi^{\dagger}{}_M(\vec{\kappa})\Phi_M(\vec{\kappa}) -$$
$$-I^2\int\frac{d\vec{\kappa}}{(2\pi)^6}\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi^{\dagger}{}_M(\vec{\kappa})\hat{\Pi}(-\vec{\kappa})\left[\frac{\partial}{\partial M}\mathbf{K}\ (M;\vec{\kappa},\vec{\kappa}')\right]_{M=M_B}\Phi_M(\vec{\kappa}') = 2M_B\ . \tag{90}$$

If the kernel $\mathbf{K}\ (M;\vec{\kappa},\vec{\kappa}')$ doesn't depend on the mass M, the second term in (90) vanishes and the normalization condition becomes simpler:

$$\frac{1}{I^{1/4}}\int\frac{d\vec{\kappa}}{(2\pi)^6}\Phi^{\dagger}{}_M(\vec{\kappa})\Phi_M(\vec{\kappa}) = 2M_B\ . \tag{91}$$

Taking into account relations (81)-(83), we shall obtain an even more simplified normalization condition:

$$\int\frac{d\vec{\kappa}}{(2\pi)^6}\mathbf{N}\ (\vec{\kappa})\left\{\left|\Phi_1(\vec{\kappa})\right|^2 + \left|\Phi_8(\vec{\kappa})\right|^2\right\} = 2M\ , \tag{92}$$

where

$$\mathbf{N}\ (\vec{\kappa}) = \frac{1}{I^{1/4}}\frac{(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)}{1+\varepsilon_1\varepsilon_2\varepsilon_3} = \frac{1}{I^{1/4}}\frac{4\omega_1\omega_2\omega_3}{A}\ ,$$
$$A = \frac{1}{2}(\omega_1+m_1)(\omega_2+m_2)(\omega_3+m_3)(1+\varepsilon_1\varepsilon_2\varepsilon_3). \tag{93}$$

It should be noted that only two 8-component spinors enter into a normalization condition (92) instead of one 64-component spinor.

7. **Symmetry of the Salpeter equation and construction of Salpeter amplitude**

Charge conjugation $\mathbf{C}$, space parity $\mathbf{P}$ and time-reversal $\mathbf{T}$ operators for baryons are obtained by direct multiplication of corresponding one-particle quark operators and in $\vec{\kappa}$-representation operate as follows:

$$\mathbf{C}\ \Phi_M(\vec{\kappa}) = \gamma_0\gamma_2\cdot\gamma_0\gamma_2\cdot\gamma_0\gamma_2\Phi^*_M(\vec{\kappa})\ ,$$
$$\mathbf{P}\ \Phi_M(\vec{\kappa}) = \gamma_0\cdot\gamma_0\cdot\gamma_0\Phi_M(-\vec{\kappa})\ , \tag{94}$$
$$\mathbf{T}\ \Phi_M(\vec{\kappa}) = -i\gamma_1\gamma_3\cdot\gamma_1\gamma_3\cdot\gamma_1\gamma_3\Phi^*_M(-\vec{\kappa})\ .$$

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From here we obtain action of operator **C P T** on $\Phi_M(\vec{\kappa})$:

$$\mathbf{C\ P\ T}\ \Phi_M(\vec{\kappa}) = -\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi_M(\vec{\kappa}) , \tag{95}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Therefore **C P T** -symmetry of a strong interaction will be expressed by the following commutation relation (see eq. (76)):

$$\left[\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5, V(M;\vec{\kappa},\vec{\kappa}')\right] = 0 . \tag{96}$$

Besides, $\Phi_M(\vec{\kappa})$ solutions of Salpeter equation (76) have also a certain value of parity π since parity operator **P** kommutes with interaction V:

$$\mathbf{P}\ \Phi_{M,\pi}(\vec{\kappa}) = \pi\,\Phi_{M,\pi}(\vec{\kappa}) . \tag{97}$$

Further, in Salpeter equation (76) with a negative value of mass $-M < 0$ we shall insert expressions $1 = (\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5)(\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5)$ before amplitudes $\Phi_{-M,\pi}(\vec{\kappa})$ and $\Phi_{-M,\pi}(\vec{\kappa}')$. Then the left expressions $\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5$ we shall carry up to the end to the left, using a kommutator (96) and an anticommutator

$$\{\gamma_0\gamma_5, h(\vec{k})\} = 0 . \tag{98}$$

The "new" Salpeter equation, which is obtained as a result of these manipulations, already describes a baryon with the positive mass $M > 0$ and opposite parity $-\pi$:

$$\Phi'_{M,-\pi}(\vec{\kappa}) = \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi_{-M,\pi}(\vec{\kappa}) . \tag{99}$$

Really, by means of relation $\gamma_0\gamma_5 = -\gamma_5\gamma_0$ and by direct calculations we receive:

$$\mathbf{P}\ \Phi'_M(\vec{\kappa}) = -\pi\Phi'_M(\vec{\kappa}) . \tag{100}$$

Thus, the solution $\Phi'_M(\vec{\kappa})$ describes an antibaryon. Also taking into account zero anticommutator (98), from relation (99) we obtain the following connections:

$$\begin{cases} \Phi^{+++}_{M,-\pi}(\vec{\kappa}) = \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi^{---}_{-M,\pi}(\vec{\kappa}), \\ \Phi^{---}_{M,-\pi}(\vec{\kappa}) = \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi^{+++}_{-M,\pi}(\vec{\kappa}), \end{cases} \tag{101}$$

where the summands of positive and negative energy ($\Phi^{+++}_M(\vec{\kappa})$ and $\Phi^{---}_M(\vec{\kappa})$) of the total Salpeter amplitude

$$\Phi_M(\vec{\kappa}) = \Phi^{+++}_M(\vec{\kappa}) + \Phi^{---}_M(\vec{\kappa}) \tag{102}$$

are defined according to relation (80). At the same time the component-wise structure of Salpeter amplitude (76) and relation (99) (with an explicit form of matrix $\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5$) allow to explain the physical sense of 8-component spinors in the spirit of "particle-antiparticle":

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$$\left.\begin{aligned}\Phi'_{M1,-\pi}(\vec{\kappa}) &= -\Phi_{-M8,\pi}(\vec{\kappa})\ ,\\ \Phi'_{M8,-\pi}(\vec{\kappa}) &= \Phi_{-M1,\pi}(\vec{\kappa})\ .\end{aligned}\right\} \tag{103}$$

Baryon consisting of three constituent quarks is described by the total Salpeter amplitude $\Phi_M^{J^\pi M_J TM_T \mathbf{S}^*}$ (81) having a quite definite quantum number set: JM_J – the total angular momentum and its projection, π – space parity, TM_T – the total isospin and its projection, $\mathbf{S}^*$ – strangeness. It is possible to be convinced that two 8-component spinors $\Phi_{M1,\pi}^{JM_J TM_T \mathbf{S}^*}$ and $\Phi_{M8,-\pi}^{JM_J TM_T \mathbf{S}^*}$ (which are components of the total amplitude) have almost the same quantum numbers set. Elimination is orbital parity. Here the lower indexes «π» and «$-\pi$» designate numerical values of orbital parity π'. In the former case numerically it is equal to total space parity, and it is opposite to it in the letter case. All this can be checked up acting with a parity operator on $\Phi_M(\vec{\kappa})$ (using an explicit form of matrix $\gamma_0 \cdot \gamma_0 \cdot \gamma_0$)

$$\mathbf{P}\ \ \Phi_M(\vec{\kappa}) = \gamma_0 \cdot \gamma_0 \cdot \gamma_0 \Phi_M(-\vec{\kappa}) = \begin{pmatrix} \Phi_1(-\vec{\kappa}) \\ -\Phi_2(-\vec{\kappa}) \\ -\Phi_3(-\vec{\kappa}) \\ \Phi_4(-\vec{\kappa}) \\ -\Phi_5(-\vec{\kappa}) \\ \Phi_6(-\vec{\kappa}) \\ \Phi_7(-\vec{\kappa}) \\ -\Phi_8(-\vec{\kappa}) \end{pmatrix} \tag{104}$$

and then line by line comparing the obtained expression with $\pi \cdot \Phi_M(\vec{\kappa})$.

Further we shall construct the required totally antisymmetric 8-component spinor $\Phi_{M1,\pi}^{JM_J TM_T \mathbf{S}^*}$ ($\Phi_{M8,-\pi}^{JM_J TM_T \mathbf{S}^*}$ can be constructed similarly; labels are borrowed from paper [8]):

$$\Phi_{M1,\pi}^{JM_J TM_T \mathbf{S}^*}(\vec{\kappa}) = \sum_{\mathbf{R}_L \mathbf{R}_S \mathbf{R}_F} \left\{ \left\{ \left\{ \left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathbf{R}_L} \cdot \left[\chi^S\right]_{\mathbf{R}_S} \right\}^{JM_J} \cdot \left[\varphi^{TM_T \mathbf{S}^*}\right]_{\mathbf{R}_F} \right\}_{\mathbf{S}} \cdot c_{\mathbf{A}} \right\}_{\mathbf{A}} . \tag{105}$$

Here: $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathbf{R}_L}$ is space wave function (WF) in $\vec{\kappa}$-representation with total orbital momentum L, orbital parity π (coinciding with baryon total parity) and quarks permutation symmetry $\mathbf{R}_L \in \{\mathbf{S}, \mathbf{M_S}, \mathbf{M_A}\}$ [17]; $\left[\chi^S\right]_{\mathbf{R}_S}$ is 8-component spin WF (spinor) with total spin S and symmetry $\mathbf{R}_S \in \{\mathbf{S}, \mathbf{M_S}, \mathbf{M_A}\}$; $\left[\varphi^{TM_T \mathbf{S}^*}\right]_{\mathbf{R}_F}$ is flavour WF with total isospin T and its projection M_T, strangeness $\mathbf{S}^*$ and symmetry $\mathbf{R}_F \in \{\mathbf{S}, \mathbf{M_S}, \mathbf{M_A}, \mathbf{A}\}$ (see Appendix B) and $c_{\mathbf{A}}$ is the function describing totally antisymmetric color (colourless) singlet.

Let's pay attention to one circumstance. In the proposed approach there is no need for the so-called relativization of the Salpeter amplitude $\Phi_M^{J^\pi M_J TM_T \mathbf{S}^*}$ [8], as its spinor (and, therefore, relativistic) structure is ensured with required spinors $\Phi_{M1,\pi}^{JM_J TM_T \mathbf{S}^*}(\vec{\kappa})$ and $\Phi_{M8,-\pi}^{JM_J TM_T \mathbf{S}^*}(\vec{\kappa})$, whose 8-component structure in turn is set by spin WF $\left[\chi^S\right]_{\mathbf{R}_S}$. The remainder gives us the solution of the Salpeter equation with required unknown coefficients in expansion of space WF with given symmetry $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathbf{R}_L}$. The most laborious is the construction of this space WF. At their construction we use expansion on functions $\Psi_{NK}^{l_q l_p L M_L}(\vec{\kappa})$ (see.(A30)) which are solutions of the nonrelativistic equation with an oscillatory potential (see below the nonrelativistic limit of the Salpeter equation). They contain the hyperspherical harmonics (HH) (or K-harmonics) $\phi_K^{l_q l_p}(\Omega_{\vec{\kappa}})$. Per se HH $\phi_K^{l_q l_p}(\Omega_{\vec{\kappa}})$ have no certain symmetry with respect to permutations group S_3, i.e. they are not basis functions of the irreducible representation of this group. By the action of the so-called Young operators [9] on them we reach certain symmetry. These operators, in turn, comprise quark permutations operators. As a result in expansion appear the HH with different sets of the coordinates obtained from the set $1 \equiv (123)$ by cyclic permutation of particles: $2 \equiv (231)$, $3 \equiv (312)$ (see. Appendix A). Therefore it is necessary to have a relation between them. This relation with coefficients of Reynal-Revai [24, 9] is provided:

$$\phi_K^{l_{q_j} l_{p_j} L M_L}(\Omega_j) = \sum_{l_{q_i} l_{p_i}} \left\langle l_{q_i} l_{p_i} \middle| l_{q_j} l_{p_j} \right\rangle_{KL} \phi_K^{l_{q_i} l_{p_i} L M_L}(\Omega_i) . \tag{106}$$

Because of invariance of the 6-dimensional operator $\Delta_{\vec{\kappa}}$ concerning rotations and permutations of particles to Reynal-Revai coefficients conserved quantum numbers K and L are assigned. By means of relation (106) it is possible to express the result of Young operator actions through HH $\phi_{KL}^{l_1 l_2}$ functions of one type that considerably simplifies calculations. The problem of the construction of full orthonormalized basic function $\phi_{KL}^{\{f\}\mu\sigma}$ of group S_3 is solved by the calculation of so-called symmetrization coefficients which are expressed also by coefficients of Reynal-Revai [9]. Here $\{f\}$ is Young's diagram: $\{3\}$ – symmetric and $\{1^3\}$ – antisymmetric 1-dimensional representations, $\{21\}$ – 2-dimensional mixed symmetry representations; μ – Yamanouchi symbol for $\{21\}$ diagram (representations $8_{\mathbf{S}}$ and $8_{\mathbf{A}}$ for $\mu = 1$ and $\mu = 2$, respectively); σ – the number of identical representations of S_3 group. After that it is necessary to determine unknown coefficients in expansion of space wave function $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathbf{R}_L}$ already by the solution of the Salpeter equation.

8. The Salpeter equation with the three-particle central potentials (oscillator and linear). Nonrelativistic limit.

The most popular (local) potentials used in calculations of quark-antiquark bound systems [21] are easily generalized in the case of three-quark bound systems ($\vec{\rho}$-representation, see. Appendix A):

$$\langle \vec{\rho} | V_n | \vec{\rho}' \rangle = \hat{\Pi}_0 \mathbf{V}_n(\vec{\rho}, \vec{\rho}') = \hat{\Pi}_0 \delta(\vec{\rho} - \vec{\rho}')[\mathbf{V}_0 + \eta_n \rho^n e^{-\varepsilon\rho}] \ , \tag{107}$$

$$\hat{\Pi}_0 = \lim_{\vec{\kappa}\to 0} \hat{\Pi}(\vec{\kappa}) = \frac{1}{4}\{1\cdot 1\cdot 1 + \gamma_0\cdot\gamma_0\cdot 1 + \gamma_0\cdot 1\cdot\gamma_0 + 1\cdot\gamma_0\cdot\gamma_0\} \ , \tag{108}$$

where $\mathbf{V}_0$ and η_n - parameters of three-quark interaction. Values $n = -1, 1, 2$ correspond to Coulomb, linear and oscillator potentials, respectively. The multiplier $e^{-\varepsilon}$ ($\varepsilon \to 0$) is introduced for regularization of potentials in $\vec{\kappa}$- representation. Here at γ_0 - matrices we omit quarks indexes.
In $\vec{\kappa}$-representation we have

$$\langle \vec{\kappa} | V_n | \vec{\kappa}' \rangle = \hat{\Pi}_0 \left\{ (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}') \mathbf{V}_0 + \eta_n \int e^{i(\vec{\kappa}-\vec{\kappa}')\vec{\rho} - \varepsilon\rho} \rho^n d\vec{\rho} \right\} . \tag{109}$$

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For the calculation of integral

$$I_n(\vec{\kappa},\vec{\kappa}';\varepsilon)=\eta_n\int e^{i(\vec{\kappa}-\vec{\kappa}')\vec{\rho}-\varepsilon\rho}\rho^n d\vec{\rho} \tag{110}$$

at first we will carry out expansion of 6-dimensional plane waves separately $e^{i\vec{\kappa}\vec{\rho}}$ and $e^{-i\vec{\kappa}'\vec{\rho}}$ on HH $\phi_K(\Omega)$ (see. The appendix, here for brevity through K all quantum numbers are denoted), and then produce integration with angular variables. As a result we will obtain:

$$I_n(\vec{\kappa},\vec{\kappa}';\varepsilon)=\frac{(2\pi)^6\eta_n}{(\kappa\kappa')^2}\sum_K v_K^n(\kappa,\kappa';\varepsilon)\phi_K^*(\Omega_{\vec{\kappa}})\phi_K(\Omega_{\vec{\kappa}'})\ , \tag{111}$$

where

$$v_K^n(\kappa,\kappa';\varepsilon)\equiv\int_0^\infty e^{-\varepsilon\rho}\rho^{n+1}J_{K+2}(\kappa\rho)J_{K+2}(\kappa'\rho)d\rho\ . \tag{112}$$

The relation follows from definition (112)

$$v_K^n(\kappa,\kappa';\varepsilon)=(-1)^{\,n+1}\frac{\partial^{n+1}}{\partial\varepsilon^{n+1}}v_K^{-1}(\kappa,\kappa';\varepsilon). \tag{113}$$

The integral (112) for $n=-1$ is equal [22]:

$$v_K^{-1}(\kappa,\kappa';\varepsilon)=\int_0^\infty e^{-\varepsilon\rho}J_{K+2}(\kappa\rho)J_{K+2}(\kappa'\rho)d\rho=\frac{1}{\pi\sqrt{\kappa\kappa'}}Q_{K+3/2}\left(\frac{\kappa^2+\kappa'^2+\varepsilon^2}{2\kappa\kappa'}\right)\ . \tag{114}$$

Here $J_\nu(z)$ is a Bessel cylinder function of the first kind, $Q_\nu(z)$ is a Legendre function of the second kind. Using known expressions for Legendre generating function, from relation (114) with double differentiation with respect to ε we shall obtain the following expression for linear potential:

$$\begin{aligned}v_K^1(\kappa,\kappa';\varepsilon)=\frac{2K+3}{\pi(\kappa\kappa')^{3/2}}\frac{1}{z^2-1}\Bigg\{&(z-z_0)\left[\left(K+5/2-\frac{2z^2}{z^2-1}\right)Q_{K+3/2}(z)+\frac{2z}{z^2-1}Q_{K+1/2}(z)\right]+\\&+\frac{1}{2}\left[zQ_{K+3/2}(z)-Q_{K+1/2}(z)\right]\Bigg\}\ ,\end{aligned} \tag{115}$$

where

$$z=z_0+\frac{\varepsilon^2}{2\kappa\kappa'}\ ,\quad z_0=\frac{1}{2}\left(\frac{\kappa}{\kappa'}+\frac{\kappa'}{\kappa}\right). \tag{116}$$

From expression (116) when $\kappa\neq\kappa'$ we obtain finite values ($z_0\neq 1$):

$$v_K^1(\kappa,\kappa';\varepsilon)\xrightarrow[\varepsilon\to 0]{}\frac{2K+3}{2\pi(\kappa\kappa')^{3/2}}\frac{1}{z_0^2-1}\left[z_0Q_{K+3/2}(z_0)-Q_{K+1/2}(z_0)\right]. \tag{117}$$

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When $\kappa = \kappa'$ we have singularity ($z_0 = 1$): При $\kappa = \kappa'$ имеем особенность ($z_0 = 1$):

$$v_K^1(\kappa,\kappa';\varepsilon) \approx -\frac{2K+3}{4\pi\kappa^3}\frac{1}{z-1}\left[Q_{K+3/2}(1) - Q_{K+1/2}(1)\right] \approx \frac{1}{\kappa}\frac{1}{\pi\varepsilon}\xrightarrow[\varepsilon\to 0]{}\infty. \tag{118}$$

Formally there is also a second way of calculating the integral (110) which consists of the expansion of the 6-dimensional plane wave $e^{i(\vec{\kappa}-\vec{\kappa}')\vec{\rho}}$ (as whole) on HH functions and the subsequent integration first with respect to angular variables, and then to a collective variable ρ [22]:

$$\begin{aligned} I_n(\vec{\kappa},\vec{\kappa}';\varepsilon) &= \frac{(2\pi)^3\eta_n}{|\vec{\kappa}-\vec{\kappa}'|^2}\int_0^8 e^{-\varepsilon\rho}\rho^{n+3}J_2(|\vec{\kappa}-\vec{\kappa}'|\rho)\,d\rho = \\ &= \frac{(|\vec{\kappa}-\vec{\kappa}'|/\varepsilon)^2(n+5)!}{8\varepsilon^{n+4}}\frac{1}{(1+|\vec{\kappa}-\vec{\kappa}'|^2/\varepsilon^2)^{\frac{2n+7}{2}}}F\left(-\frac{n+1}{2},-\frac{n}{2};3;-|\vec{\kappa}-\vec{\kappa}'|^2/\varepsilon^2\right). \end{aligned} \tag{119}$$

Here $F(a,b;c;z)$ is the hypergeometrical function. However, this way coincides with the first if at comparison of integrals (112) and (119) we take into account relation [23]

$$\rho^{n+3}J_2(|\vec{\kappa}-\vec{\kappa}'|\rho) = (2\pi)^3\left(\frac{|\vec{\kappa}-\vec{\kappa}'|}{\kappa\kappa'}\right)^2\sum_K \rho^{n+1}J_{K+2}(\kappa\rho)J_{K+2}(\kappa'\rho)\phi_K^*(\Omega_{\vec{\kappa}})\phi_K(\Omega_{\vec{\kappa}'}). \tag{120}$$

The outlined approach for the determination of a potential form in $\vec{\kappa}$ -representation we demonstrated for Coulomb and linear potentials: $n = -1, 1$. However oscillatory potential in this representation can be obtained in another, simpler way. Fourier-form of potential (107) with $n = 2$ and $\eta_2 = M_0\Omega_0^2/2$ in $\vec{\kappa}$ -representation has the following form (see (A28) and (A33)):

$$\begin{aligned} \langle\vec{\kappa}|V|\vec{\kappa}'\rangle &= \hat{\Pi}_0\mathbf{V}(\vec{\kappa},\vec{\kappa}') = \hat{\Pi}_0\left\{(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\mathbf{V}_0 - (2\pi)^6\frac{M_0\Omega_0^2}{2}\left[\Delta_{\vec{\kappa}'}\delta(\vec{\kappa}-\vec{\kappa}')\right]\right\} = \\ &= \hat{\Pi}_0\left\{(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\left[\mathbf{V}_0 - \frac{\vec{\kappa}^2}{2M_0}\right] + \sum_\nu \Omega_0(2N+K+3)\langle\nu|\vec{\kappa}'\rangle\langle\vec{\kappa}|\nu\rangle\right\}. \end{aligned} \tag{121}$$

For simplification of further calculations we will write down the Salpeter equation (76) in a different form:

$$D_M(\vec{\kappa})\Phi_M(\vec{\kappa}) = I^{9/4}\hat{\Pi}(\vec{\kappa})\gamma_0\cdot\gamma_0\cdot\gamma_0\int\langle\vec{\kappa}|V|\vec{\kappa}'\rangle\frac{d\vec{\kappa}'}{(2\pi)^6}\Phi_M(\vec{\kappa}'), \tag{122}$$

$$D_M(\vec{\kappa}) \equiv M - h_1(\vec{k}_1) - h_2(\vec{k}_2) - h_3(\vec{k}_3). \tag{123}$$

At substitution of potential (107) in equation (122) a simple matrix product appears:

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$$\gamma_0\cdot\gamma_0\cdot\gamma_0\hat{\Pi}_0\Phi_M=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\\ \Phi_5\\ \Phi_6\\ \Phi_7\\ \Phi_8\end{pmatrix}=\begin{pmatrix}\Phi_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ -\Phi_8\end{pmatrix}, \tag{124}$$

where points represent zero: $\cdot=0$. Regardless of the type of potential according to relations (82) for the solution of the Salpeter equation (122) it is enough to write out the equations for 8×1 dimensional two components Φ_1 and Φ_8. In our concrete case we will obtain the following system of the equations (the multiplier $I^{9/4}$ "is introduced" in $\mathbf{V}(\vec{\kappa},\vec{\kappa}')$):

$$\left.\begin{aligned}(D_M\Phi)_1(\vec{\kappa})&=\Pi_{11}(\vec{\kappa})\int\mathbf{V}(\vec{\kappa},\vec{\kappa}')\Phi_1(\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}-\Pi_{18}(\vec{\kappa})\int\mathbf{V}(\vec{\kappa},\vec{\kappa}')\Phi_8(\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6},\\ (D_M\Phi)_8(\vec{\kappa})&=\Pi_{81}(\vec{\kappa})\int\mathbf{V}(\vec{\kappa},\vec{\kappa}')\Phi_1(\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}-\Pi_{88}(\vec{\kappa})\int\mathbf{V}(\vec{\kappa},\vec{\kappa}')\Phi_1(\vec{\kappa}')\frac{d\vec{\kappa}'}{(2\pi)^6}.\end{aligned}\right\} \tag{125}$$

For a solution of this system of equations it is necessary to calculate columns $(D_M\Phi)_1(\vec{\kappa})$ and $(D_M\Phi)_8(\vec{\kappa})$ every 8×1-dimensional, and also matrices $\Pi_{11}(\vec{\kappa})$, $\Pi_{18}(\vec{\kappa})$, $\Pi_{81}(\vec{\kappa})$ and $\Pi_{88}(\vec{\kappa})$ every 8×8-dimensional. After enough difficult calculations we obtain the following expressions:

$$\begin{cases}(D_M\Phi)_1(\vec{\kappa})=\left(M-\Omega\dfrac{B}{A}\right)\Phi_1(\vec{\kappa})-\Omega\dfrac{\hat{C}}{A}\Phi_8(\kappa)\ ,\\ (D_M\Phi)_8(\vec{\kappa})=-\Omega\dfrac{\hat{C}}{A}\Phi_1(\vec{\kappa})+\left(M+\Omega\dfrac{B}{A}\right)\Phi_8(\vec{\kappa})\ ,\end{cases} \tag{126}$$

$$\begin{cases}\hat{C}\equiv(\vec{\sigma}_1\cdot\vec{k}_1)(\vec{\sigma}_2\cdot\vec{k}_2)(\vec{\sigma}_3\cdot\vec{k}_3)\ ,\\ \Pi_{11}(\vec{\kappa})=\Pi_{88}(\vec{\kappa})=\Pi(\vec{\kappa})=\dfrac{A}{4\omega_1\omega_2\omega_3}=\dfrac{1}{I^{1/4}\mathbf{N}(\vec{\kappa})}\ ,\quad \Pi_{18}(\vec{\kappa})=\Pi_{81}(\vec{\kappa})=0,\\ \Omega=\omega_1+\omega_2+\omega_3,\\ B=\dfrac{1}{2}(\omega_1+m_1)(\omega_2+m_2)(\omega_3+m_3)(1-\varepsilon_1\varepsilon_2\varepsilon_3).\end{cases} \tag{127}$$

Other terms entering in expressions (126), (127) are already determined above. Taking into account these definitions and formula (84) it is possible to check the correctness of the following auxiliary relations:

$$B=A(m_i\leftrightarrow\omega_i)\ ,\quad \frac{B}{A}=\frac{1+\varepsilon_1\varepsilon_2\varepsilon_3}{1-\varepsilon_1\varepsilon_2\varepsilon_3}\ ,\quad \hat{C}^2=\frac{4\varepsilon_1\varepsilon_2\varepsilon_3}{(1+\varepsilon_1\varepsilon_2\varepsilon_3)^2}A^2\ ,\quad A^2=B^2+\hat{C}^2\ . \tag{128}$$

Now we will substitute expressions (126) and (127) in the system of equations (125):

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$$\begin{cases} \left(M - \Omega \dfrac{B}{A} \right) \Phi_1(\vec{\kappa}) - \dfrac{\Omega}{A} \hat{C} \Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa}) \int \mathbf{V}(\vec{\kappa}, \vec{\kappa}') \Phi_1(\vec{\kappa}') \dfrac{d\vec{\kappa}'}{(2\pi)^6} , \\ \dfrac{\Omega}{A} \hat{C} \Phi_1(\vec{\kappa}) - \left(M + \Omega \dfrac{B}{A} \right) \Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa}) \int \mathbf{V}(\vec{\kappa}, \vec{\kappa}') \Phi_8(\vec{\kappa}') \dfrac{d\vec{\kappa}'}{(2\pi)^6} . \end{cases} \tag{129}$$

In the case of free movement $\mathbf{V}(\vec{\kappa}, \vec{\kappa}') = 0$ for existence of solutions it is necessary that the determinant of system equaled to zero:

$$\begin{vmatrix} M - \Omega \dfrac{B}{A} & -\dfrac{\Omega}{A} \hat{C} \\ -\dfrac{\Omega}{A} \hat{C} & M + \Omega \dfrac{B}{A} \end{vmatrix} = 0 , \tag{130}$$

or, as well as has to be

$$M = \pm \Omega . \tag{131}$$

Returning expression (131) into the system of the equations with $\mathbf{V}(\vec{\kappa}, \vec{\kappa}') = 0$, we obtain 8-component spinors of free movement:

$$\begin{cases} \Phi_1^{(\pm)} = \pm \dfrac{\hat{C}}{A \mp B} \Phi_8^{(\pm)} , \\ \Phi_8^{(\pm)} = \pm \dfrac{\hat{C}}{A \pm B} \Phi_1^{(\pm)} , \end{cases} \tag{132}$$

where according to definitions (93), (127) and (83):

$$A \pm B = (\omega_1 \pm m_1)(\omega_2 \pm m_2)(\omega_3 \pm m_3) . \tag{133}$$

On the left side of relations (132) signs ($\pm$) designate "big" and "small" components (see below), respectively. On the right side we have the opposite picture – signs ($\pm$) designate "small" and "big" components. Signs $\pm$ before fractions on the right side correspond to solutions $M = \pm\Omega$. Substituting expressions from (127) in relations (132) we obtain:

$$\begin{cases} \Phi_1^{(\pm)} = \pm \dfrac{\vec{\sigma}_1 \cdot \vec{k}_1}{\omega_1 \mp m_1} \dfrac{\vec{\sigma}_2 \cdot \vec{k}_2}{\omega_2 \mp m_2} \dfrac{\vec{\sigma}_3 \cdot \vec{k}_3}{\omega_3 \mp m_3} \Phi_8^{(\pm)} , \\ \Phi_8^{(\pm)} = \pm \dfrac{\vec{\sigma}_1 \cdot \vec{k}_1}{\omega_1 \pm m_1} \dfrac{\vec{\sigma}_2 \cdot \vec{k}_2}{\omega_2 \pm m_2} \dfrac{\vec{\sigma}_3 \cdot \vec{k}_3}{\omega_3 \pm m_3} \Phi_1^{(\pm)} . \end{cases} \tag{134}$$

These 8×1 dimensional columns (or spinors) we will express through a direct product of three 2×1 dimensional columns:

$$\begin{cases} \Phi_1^{(\pm)} = \Phi_{11}^{(\pm)} \cdot \Phi_{12}^{(\pm)} \cdot \Phi_{13}^{(\pm)} , \\ \Phi_8^{(\pm)} = \Phi_{81}^{(\pm)} \cdot \Phi_{82}^{(\pm)} \cdot \Phi_{83}^{(\pm)} . \end{cases} \tag{135}$$

From comparisons of the right sides of (134) and (135) the simple relations follow:

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$$\begin{cases} \Phi_{1i}^{(\pm)} = \pm \dfrac{\vec{\sigma}_i \cdot \vec{k}_i}{\omega_i \mp m_i} \Phi_{8i}^{(\pm)}, \\ \Phi_{8i}^{(\pm)} = \pm \dfrac{\vec{\sigma}_i \cdot \vec{k}_i}{\omega_i \pm m_i} \Phi_{1i}^{(\pm)}, \quad i = 1,2,3. \end{cases} \tag{136}$$

So, we have a 16×1 dimensional spinor:

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_8 \end{pmatrix}, \tag{137}$$

with Φ_1 and Φ_8 two spinor every 8×1 dimensions and spinors of dimension 4×1:

$$\Phi_{[i]} = \begin{pmatrix} \Phi_{1i} \\ \Phi_{2i} \end{pmatrix} = \begin{pmatrix} \Phi_{1i} \\ \dfrac{\vec{\sigma}_i \cdot \vec{k}_i}{\omega_i \pm m_i} \Phi_{1i} \end{pmatrix}, \tag{138}$$

where Φ_{1i} – usual spin function, $i = 1,2,3$ and we consider a case with $M = +\Omega$. As well as has to be, the relativity is presented both by spin and energy degrees of freedom.

Let's consider now the nonrelativistic limit of the Salpeter equation with an oscillatory potential. From expression (121) it is possible to make the following estimates of order of magnitudes:

$$V_0 \sim E, \quad \frac{\vec{\kappa}^2}{M_0} \sim E, \quad \frac{M_0 \Omega_0^2}{2} \frac{1}{\vec{\kappa}^2} \sim E. \tag{139}$$

From here two mutually agreed ratios follow

$$\vec{\kappa}^2 / M_0^2 \ll 1, \quad E \ll M_0. \tag{140}$$

Let's make assumptions:

$$M = E + M_0, \quad \left| E - \int V \right| \ll M_0. \tag{141}$$

Equations (129) we will write down in another (symbolical) form:

$$\begin{cases} \left(M - \Omega \dfrac{B}{A} - \Pi \int V \right) \Phi_1 = \Omega \dfrac{\hat{C}}{A} \Phi_8, \\ \left(M + \Omega \dfrac{B}{A} + \Pi \int V \right) \Phi_8 = \Omega \dfrac{\hat{C}}{A} \Phi_1. \end{cases} \tag{142}$$

Relations (141) we will substitute into equations (142) and pass to the limit $\vec{\kappa}^2 / M_0^2 \to 0$:

$$E \Phi_1 = \Omega \frac{\hat{C}}{A} \Phi_8 - \left(M_0 - \Omega \frac{B}{A} - \Pi \int V \right) \Phi_1, \tag{143}$$

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$$\Phi_8 \approx \frac{1}{2M_0}\Omega\frac{\hat{C}}{A}\Phi_1. \,, \tag{144}$$

The spinor Φ_8 from (144) we insert in equation (143) and use relations

$$\begin{cases} \dfrac{\hat{C}^2}{A} = \dfrac{4\varepsilon_1\varepsilon_2\varepsilon_3}{(1+\varepsilon_1\varepsilon_2\varepsilon_3)^2} \approx \dfrac{1}{2}\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2 \sim \left(\dfrac{\kappa}{M_0}\right)^6 \approx \mathbf{0}, \\ \Pi \approx 1, \quad \Omega \approx M_0 + \dfrac{\vec{\kappa}^2}{2M_0}. \end{cases} \tag{145}$$

and expression (121) for oscillatory potential. As a result we obtain the nonrelativistic equation for a 6-oscillator (see (A29)):

$$\left\{-\frac{M_0\Omega_0^2}{2}\Delta_{\vec{\kappa}} + \frac{\vec{\kappa}^2}{2M_0} - (E - \mathbf{V}_0)\right\}\dot{\Phi}_1(\vec{\kappa}) = 0 \,. \tag{146}$$

It is obvious from (144) that the order of magnitude of the "small" component Φ_8 is $(\kappa/M_0)^3$ in comparison with the "big" component Φ_1. It is also visible from relations (93) and (141) that in the nonrelativistic limit $\vec{\kappa}^2/M_0^2 \to 0$

$$\mathbf{N}\ (\vec{\kappa}) \to 1/I^{1/4} \,, \quad M \to M_0 \tag{147}$$

and normalizing condition (92) assumes the following form:

$$\frac{1}{I^{1/4}2M_0}\int\frac{d\vec{\kappa}}{(2\pi)^6}\left|\Phi_1(\vec{\kappa})\right|^2 = 1 \,, \tag{148}$$

or

$$\int\frac{d\vec{\kappa}}{(2\pi)^6}\left|\bar{\Phi}_1'(\vec{\kappa})\right|^2 = 1 \,, \quad \bar{\Phi}_1'(\vec{\kappa}) = \Phi_1(\vec{\kappa})\Big/\sqrt{I^{1/4}2M_0} \,. \tag{149}$$

As it was stated above, at construction of functions $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathsf{R}_L}$ we use expansion on functions $\Psi_{NK}^{l_q l_p L M_L}(\vec{\kappa})$ (see.(A30)), which are the same solutions $\Phi_1(\vec{\kappa})$ of the nonrelativistic equation (146).

In the conclusion it should be noted that reduction of the solution of Salpeter equation for three-particle bound systems to determination of two 8-component spinors without additional "relativization" of wave function, effective use of three-particle interaction (in a condition of quarks confinement) and a powerful method of expansions on hyperspherical harmonics (the most suitable for the description of the bound systems), allows us hope to simplify calculations of the main features of baryons considerably.

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Appendix A

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The Jacobi coordinates

The 4-dimensional coordinates of particles with masses m_1, m_2 and m_3 is symbolized by x_1, x_2 and x_3, and 4-momenta of the same particles is symbolized by k_1, k_2 and k_3. The 4-dimensional Jacobi coordinates are defined as follows:
In x-representation:

$$\begin{cases} X=\mu_1 x_1+\mu_2 x_2+\mu_3 x_3, \\ \eta=\sqrt{\dfrac{\mu_1\mu_2}{\mu_1+\mu_2}}(x_1-x_2), \\ \xi=-\sqrt{\mu_3(\mu_1+\mu_2)}\left(x_3-\dfrac{\mu_1 x_1+\mu_2 x_2}{\mu_1+\mu_2}\right). \end{cases} \qquad \begin{cases} x_1=X+\sqrt{\dfrac{\mu_2}{\mu_1(\mu_1+\mu_2)}}\eta+\sqrt{\dfrac{\mu_3}{\mu_1+\mu_2}}\xi, \\ x_2=X-\sqrt{\dfrac{\mu_1}{\mu_2(\mu_1+\mu_2)}}\eta+\sqrt{\dfrac{\mu_3}{\mu_1+\mu_2}}\xi, \\ x_3=X-\sqrt{\dfrac{\mu_1+\mu_2}{\mu_3}}\xi\ . \end{cases} \tag{A1}$$

In κ-representation:

$$\begin{cases} K=k_1+k_2+k_3, \\ q=\sqrt{\dfrac{(\mu_1+\mu_2)/}{\mu_1\mu_2}}\ \dfrac{\mu_2 k_1-\mu_1 k_2}{\mu_1+\mu_2}, \\ p=\dfrac{[\mu_3(k_1+k_2)-(\mu_1+\mu_2)k_3]}{\sqrt{\mu_3(\mu_1+\mu_2)}}. \end{cases} \qquad \begin{cases} k_1=\mu_1 K+\sqrt{\dfrac{\mu_1\mu_2}{\mu_1+\mu_2}}\,q+\mu_1\sqrt{\dfrac{\mu_3}{\mu_1+\mu_2}}\,p, \\ k_2=\mu_2 K-\sqrt{\dfrac{\mu_1\mu_2}{\mu_1+\mu_2}}\,q+\mu_2\sqrt{\dfrac{\mu_3}{\mu_1+\mu_2}}\,p, \\ k_3=\mu_3 K-\sqrt{\mu_3(\mu_1+\mu_2)}\,p. \end{cases} \tag{A2}$$

The Jacobian of transformation from Cartesian 4-momenta k_1, k_2, k_3 to Jacobi 4-momenta K, q, p is equal:

$$I=\left|\frac{\partial(x_1,x_2,x_3)}{\partial(X,\eta,\xi)}\right|=\frac{1}{\left|\dfrac{\partial(k_1,k_2,k_3)}{\partial(K,q,p)}\right|}=1/(\mu_1\mu_2\mu_3)^2\ . \tag{A3}$$

Everywhere we have:

$$\mu_i=m_i/M_0,\qquad M_0=m_1+m_2+m_3\ . \tag{A4}$$

For convenience we will unite Jacobi's corresponding 3-dimensional vectors in one 6-dimensional vector

$$\vec\rho=(\vec\eta,\vec\xi),\qquad \vec\kappa=(\vec q,\vec p) \tag{A5}$$

with radiuses

$$\rho=\sqrt{\eta^2+\xi^2},\qquad \kappa=\sqrt{q^2+p^2}. \tag{A6}$$

As a result we will obtain compact entries in $\vec\rho$- and $\vec\kappa$-representations. The completeness conditions:

$$\int|\vec\rho\rangle d\vec\rho\langle\vec\rho|=1,\qquad \int|\vec\kappa\rangle\frac{d\vec\kappa}{(2\pi)^6}\langle\vec\kappa|=1\ , \tag{A7}$$

where

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$$d\vec{\rho} = d\vec{\eta} \cdot d\vec{\xi} \ , \quad d\vec{\kappa} = d\vec{q} \cdot d\vec{p} \tag{A8}$$

are the elements of 6-volumes in coordinate and momentum spaces. The plane wave:

$$\langle \vec{\rho} | \vec{\kappa} \rangle = \langle \vec{\kappa} | \vec{\rho} \rangle^* = e^{-i\vec{\kappa}\cdot\vec{\rho}} \ . \tag{A9}$$

The orthogonality (6-dimensional δ-function):

$$\left. \begin{aligned} \langle \vec{\rho}' | \vec{\rho} \rangle &= \int e^{-i\vec{\kappa}\cdot(\vec{\rho}'-\vec{\rho})} \frac{d\vec{\kappa}}{(2\pi)^6} = \delta(\vec{\rho}'-\vec{\rho}) \ , \\ \langle \vec{\kappa}' | \vec{\kappa} \rangle &= \int e^{-i\vec{\rho}\cdot(\vec{\kappa}'-\vec{\kappa})} d\vec{\rho} = (2\pi)^6 \delta(\vec{\rho}'-\vec{\rho}) \ . \end{aligned} \right\} \tag{A10}$$

The Hyperspherical Harmonics

Here all formulas are presented in $\vec{\kappa}$-representation. Formulas in $\vec{\rho}$-representation are the same. Let's choose angular variables $\Omega_{\vec{\kappa}} \equiv (\alpha, \Omega_{\vec{q}}, \Omega_{\vec{p}})$ according to "tree" in a Fig. 1 [19, 20]. Hyperspherical harmonics (HH) are defined as follows:

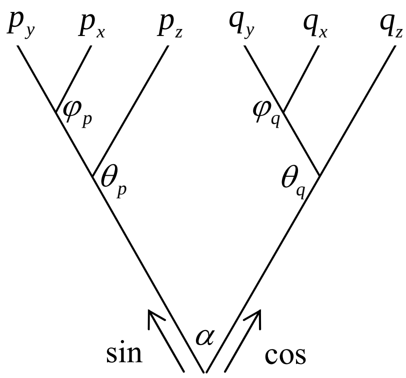

Fig. 1

$$\begin{aligned} \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) = N_K^{l_q l_p} (\cos\alpha)^{l_q} (\sin\alpha)^{l_p} P_n^{(l_p+1/2, l_q+1/2)}(\cos 2\alpha) \cdot \\ \cdot Y_{l_q m_q}(\Omega_{\vec{q}}) Y_{l_p m_p}(\Omega_{\vec{p}}) \ , \end{aligned} \tag{A11}$$

where $P_n^{(l_p+1/2, l_q+1/2)}(\cos 2\alpha)$ is Jacobi polynomial, $Y_{l_q m_q}(\Omega_{\vec{q}})$ and $Y_{l_p m_p}(\Omega_{\vec{p}})$ are usual 3-dimensional spherical harmonics,

$$N_K^{l_q l_p} = \sqrt{\frac{2 \cdot n!(K+2)(n+l_q+l_p+1)!}{\Gamma(n+l_q+3/2)\Gamma(n+l_p+3/2)}} \tag{A12}$$

–normalizing coefficient,

$$n = (K - l_q - l_p)/2 \tag{A13}$$

–an integer and therefore the 6-dimensional orbital moment K and the sum of the usual 3-dimensional orbital moments $l_\eta + l_\xi$ have identical parity. Parity of HH (A11) is equal to $(-)^K = (-)^{l_q + l_p}$ and satisfies to the equation:

$$\hat{K}^2 \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) = K(K+4) \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) \ , \tag{A14}$$

where

$$\hat{K}^2 = -\frac{\partial^2}{\partial\alpha^2} - 4ctg 2\alpha \frac{\partial}{\partial\alpha} + \frac{\hat{l}_\eta^2}{\cos^2\alpha} + \frac{\hat{l}_\xi^2}{\sin^2\alpha} \tag{A15}$$

is square of 6-dimensional orbital momentum K or an angular part of 6-dimensional Laplacian:

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$$\Delta \equiv \frac{\partial^2}{\partial \vec{\kappa}^2} = \frac{\partial^2}{\partial \kappa^2} + \frac{5}{\kappa}\frac{\partial}{\partial \kappa} - \frac{\hat{K}^2}{\kappa^2} . \tag{A16}$$

HH are orthonormal functions:

$$\int \left(\phi_{K'}^{l'_q l'_p m'_q m'_p}(\Omega_{\vec{\kappa}}) \right)^* \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) d\Omega_{\vec{\kappa}} = \delta_{K'K} \delta_{l'_q l_q} \delta_{l'_p l_p} \delta_{m'_q m_q} \delta_{m'_p m_p} . \tag{A17}$$

Here

$$d\Omega_{\vec{\kappa}} = \cos^2 \alpha \sin^2 \alpha d\alpha d\Omega_{\vec{q}} d\Omega_{\vec{p}} \tag{A18}$$

– angular element of 6-volume in momentum space , $\int d\Omega_{\vec{\kappa}} = \pi^3$. As usual, construction of HH $\phi_K^{l_q l_p L M_L}(\Omega_{\vec{\kappa}})$ with the fixed orbital moment L we fulfill by means of Klebsh-Gordon coefficients:

$$\phi_K^{l_q l_p L M_L}(\Omega_{\vec{\kappa}}) = \sum_{m_q m_p} \left\langle l_q l_p m_q m_p \middle| L M_L \right\rangle \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) . \tag{A19}$$

For a 6-dimensional plane wave (A9) the following expansion on the HH takes place:

$$e^{\pm i \vec{\kappa} \cdot \vec{\rho}} = (2\pi)^3 \sum_{K l_1 m_1 l_2 m_2} (\pm i)^K \frac{J_{K+2}(\kappa\rho)}{(\kappa\rho)^2} \left(\phi_K^{l_1 l_2 m_1 m_2}(\Omega_{\vec{\kappa}}) \right)^* \phi_K^{l_1 l_2 m_1 m_2}(\Omega_{\vec{\rho}}) . \tag{A20}$$

A 6-dimensional delta function can be presented in a form:

$$\delta(\vec{\kappa} - \vec{\kappa}') = \frac{\delta(\kappa - \kappa')}{\kappa^5} \sum_{K l_1 m_1 l_2 m_2} \left(\phi_K^{l_1 l_2 m_1 m_2}(\Omega_{\vec{\kappa}}) \right)^* \phi_K^{l_1 l_2 m_1 m_2}(\Omega_{\vec{\kappa}'}). \tag{A21}$$

Nonrelativistic 6-oscillator in $\vec{\kappa}$-representation

The Schrodinger equation

$$(H_0 - E) \left| \Psi \right\rangle = -V \left| \Psi \right\rangle \tag{A22}$$

in $\vec{\kappa}$-representation assumes the following form:

$$\int \left\langle \vec{\kappa} \middle| H_0 \middle| \vec{\kappa}' \right\rangle \frac{d\vec{\kappa}'}{(2\pi)^6} \left\langle \vec{\kappa}' \middle| \Psi \right\rangle - E \left\langle \vec{\kappa} \middle| \Psi \right\rangle = -\int \left\langle \vec{\kappa} \middle| V \middle| \vec{\kappa}' \right\rangle \frac{d\vec{\kappa}'}{(2\pi)^6} \left\langle \vec{\kappa}' \middle| \Psi \right\rangle . \tag{A23}$$

The Matrix elements entering into this equation in $\vec{\rho}$-representation have a form:

$$\left\langle \vec{\rho} \middle| H_0 \middle| \vec{\rho}' \right\rangle = \delta(\vec{\rho} - \vec{\rho}') \left[-\Delta / (2M_0) \right] , \tag{A24}$$

$$\left\langle \vec{\rho} \middle| V \middle| \vec{\rho}' \right\rangle = \delta(\vec{\rho} - \vec{\rho}') \left[\mathbf{V}_0 + M_0 \Omega_0^2 \rho^2 / 2 \right] . \tag{A25}$$

Using a universal relation

$$\left\langle \vec{\kappa} \middle| O \middle| \vec{\kappa}' \right\rangle = \int \left\langle \vec{\kappa} \middle| \vec{\rho} \right\rangle d\vec{\rho} \left\langle \vec{\rho} \middle| O \middle| \vec{\rho}' \right\rangle d\vec{\rho}' \left\langle \vec{\rho}' \middle| \vec{\kappa}' \right\rangle \tag{A26}$$

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and equalities (A9) and (A10), in $\vec{\kappa}$ -representation we will obtain matrix elements:

$$\langle\vec{\kappa}|H_0|\vec{\kappa}'\rangle = (2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\frac{\vec{\kappa}^2}{2M_0}, \tag{A27}$$

$$\langle\vec{\kappa}|V|\vec{\kappa}'\rangle = \mathbf{V}(\vec{\kappa},\vec{\kappa}') = (2\pi)^6\left\{\delta(\vec{\kappa}-\vec{\kappa}')\mathbf{V}_0 - M_0\Omega_0^2\left[\Delta_{\vec{\kappa}'}\delta(\vec{\kappa}-\vec{\kappa}')\right]/2\right\}. \tag{A28}$$

Now substituting matrix elements (A27) and (A28) in the equation (A23) we come to known oscillatory equation:

$$\left\{-\frac{M_0\Omega_0^2}{2}\Delta_{\vec{\kappa}} + \frac{\vec{\kappa}^2}{2M_0} - (E-\mathbf{V}_0)\right\}\langle\vec{\kappa}|\Psi\rangle = 0. \tag{A29}$$

The solution of this equation is the following expression:

$$\begin{cases} \Psi_{NK}^{l_q l_p L M_L}(\vec{\kappa}) = (2\pi)^3 a\dfrac{\chi_{NK}(a\kappa)}{\kappa^{5/2}}\phi_K^{l_q l_p L M_L}(\Omega_{\vec{\kappa}}), \\ \chi_{NK}(x) = \sqrt{\dfrac{2\Gamma(N+1)}{a\Gamma(N+K+3)}}e^{-x^2/2}x^{K+5/2}L_N^{K+2}(x^2), \\ a = 1/\sqrt{M_0\Omega_0}\ . \end{cases} \tag{A30}$$

$L_N^{K+2}(x^2)$ is Laguerre polynomial, N is a positive integer, the energy eigenvalue is equal:

$$E_{NK} = \mathbf{V}_0 + \Omega_0(2N+K+3). \tag{A31}$$

Oscillatory basis functions $\Psi_{NK}^{l_q l_p L M_L}(\vec{\kappa})$ represent the total set of orthonormal functions. Introducing designation $\nu \equiv_{NK}^{l_q l_p L M_L}$ for a set of oscillatory quantum numbers one can write a completeness condition:

$$\sum_\nu \langle\vec{\kappa}|\nu\rangle\langle\nu|\vec{\kappa}'\rangle = (2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}'). \tag{A32}$$

This condition together with equation (A29) and with energy expression (A31) allows to calculate expression $\Delta_{\vec{\kappa}'}\delta(\vec{\kappa}-\vec{\kappa}')$ entering into matrix element (A28). As a result this matrix element takes the following simple form:

$$\langle\vec{\kappa}|V|\vec{\kappa}'\rangle = (2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\left[\mathbf{V}_0 - \vec{\kappa}^2/(2M_0)\right] + \sum_\nu \Omega_0(2N+K+3)\langle\nu|\vec{\kappa}'\rangle\langle\vec{\kappa}|\nu\rangle. \tag{A33}$$

The set ν defines values of quantum numbers N and K.

Appendix B
Quark structure of baryon states

Multiplet	Particle	Hypercharge, Isospin and its projection	Quark structure 1	Quark structure 2	Symmetrization operator

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		Y	T	T_3			
10_S	Δ^{++}	1	3/2	3/2	*uuu*		$\omega^{[3]}$
	Δ^{+}	1	3/2	1/2	*uud*		
	Δ^{0}	1	3/2	-1/2	*udd*		
	Δ^{-}	1	3/2	-3/2	*ddd*		
	Σ^{*+}	0	1	1	*uus*		
	Σ^{*0}	0	1	0	*uds*		
	Σ^{*-}	0	1	-1	*dds*		
	Ξ^{*0}	-1	1/2	1/2	*uss*		
	Ξ^{*-}	-1	-1/2	-1/2	*dss*		
	Ω^{-}	-2	0	0	*sss*		
8_S	p	1	1/2	1/2	*uud*	*udu*	$\omega^{[21]}_{11}$ ($\omega^{[21]}_{12}$)
	n	1	1/2	-1/2	*udd*	*udd*	
	Σ^{+}	0	1	1	*uus*	*usu*	
	Σ^{0}	0	1	0	*uds*	$usd+dsu$	
	Σ^{-}	0	1	-1	*dds*	*dsd*	
	Λ^{0}	0	0	0	$usd-dsu$	*uds*	
	Ξ^{0}	-1	1/2	1/2	*uss*	*uss*	
	Ξ^{-}	-1	1/2	-1/2	*dss*	*dss*	
8_A	p	1	1/2	1/2	*uud*	*udu*	$\omega^{[21]}_{21}$ ($\omega^{[21]}_{22}$)
	n	1	1/2	-1/2	*udd*	*udd*	
	Σ^{+}	0	1	1	*uus*	*usu*	
	Σ^{0}	0	1	0	*uds*	$usd+dsu$	
	Σ^{-}	0	1	-1	*dds*	*dsd*	
	Λ^{0}	0	0	0	$usd-dsu$	*uds*	
	Ξ^{0}	-1	1/2	1/2	*uss*	*uss*	
	Ξ^{-}	-1	1/2	-1/2	*dss*	*dss*	
1_A	Λ	0	0	0	*uds*		$\omega^{[1^3]}$

Young symmetrization operators:

$$\omega^{[3]} = (1+P_{23}+P_{13})(1+P_{12})/6 \qquad \omega^{[1^3]} = (2+P_{23}+P_{13})(1-P_{12})/6$$

$$\left.\begin{aligned}\omega^{[21]}_{11} &= (2-P_{23}-P_{13})(1+P_{12})/6\\ \omega^{[21]}_{21} &= (P_{23}-P_{13})(1+P_{12})/\sqrt{12}\end{aligned}\right\} \qquad \left.\begin{aligned}\omega^{[21]}_{12} &= (P_{23}-P_{13})(1-P_{12})/\sqrt{12}\\ \omega^{[21]}_{22} &= (2+P_{23}+P_{13})(1-P_{12})/6\end{aligned}\right\}.$$

Action of Young operators with the first lower indices 1 and 2 give the functions of multiplets 8_S and 8_S , respectively. The second lower indices specify the number of structure on which Young operator acts. It is easy to check that basis functions of irreducible representations of this $SU(3)$ group don't depend on a choice of the second lower index. The obtained function can be characterized with a quantum numbers set (Y,T,T_z) and also (T_{12},T,T_z). Examples of construction of normalized state vector for a proton p from a multiplets 8_S :

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$$\left.\begin{array}{l}\omega_{11}^{[21]}|uud\rangle \\ \omega_{12}^{[21]}|udu\rangle\end{array}\right\} \rightarrow |Y=1,1/2,1/2\rangle = |T_{12}=1,1/2,1/2\rangle = (2|uud\rangle - |udu\rangle - |duu\rangle)/\sqrt{6}$$

and 8_A :

$$\left.\begin{array}{l}\omega_{21}^{[21]}|uud\rangle \\ \omega_{22}^{[21]}|udu\rangle\end{array}\right\} \rightarrow |Y=1,1/2,1/2\rangle = |T_{12}=0,1/2,1/2\rangle = (|udu\rangle - |duu\rangle)/\sqrt{2}$$

Construction of any other functions is quite simple.

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