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GROUP THEORETIC APPROACH TO STUDY TRANSFER MATRIX METHOD IN ONE-DIMENSIONAL PHOTONIC CRYSTALS

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Abstract:
The transfer-matrix method is one of the well known methods used in photonics to analyze the propagation of electromagnetic waves through the layered periodic medium of a one-dimensional photonic crystal. The transfer matrix method was developed in accordance to Maxwell's equations which stated there exists, simple continuity conditions for the electric field across boundaries from one medium to the next. The TMM models a single period cell which can then be infinitely extended to include the whole structure. In this paper, we propose a formula which obtains the transfer matrix of the N cell structure using group theory. The novelty of the proposed formula lies in the fact that it calculates the system transfer matrix of the N cell periodic dielectric array directly easing the burden of not repeatedly performing the matrix multiplication operation.

Keywords: *Photonic Crystal, Transfer Matrix Method, Maxwell's Equations, Group Theory*

1. Introduction

One-dimensional photonic crystal is an arrangement of materials with special periodicity in dielectric constant along the direction of propagation of electromagnetic wave [1].In one dimensional photonic crystals, according to Maxwell's equations; there are simple continuity conditions for the electric field across boundaries separating the medium. If the field is assumed to be known at the beginning of a layer, the field at the end of the layer can be derived from a simple matrix operation. The modeling of the transfer matrix is based on such principle. Obtaining the total system matrix is to multiply each individual matrix that we obtain.
In the field of mathematics, group theory basically studies the algebraic structures known as groups and its symmetry. Except for some band assignments and related discussions, group theory hasn't been used in the field of photonic crystals [2]. The group theory for photonic crystals, which is important to describe the property of symmetry, was formulated by Ohtaka and Tanabe [3]. They investigated the symmetry properties of photonic crystals by an array of dielectric spheres. The calculated band structure agreed perfectly with the predication made using group theory. K. Sakoda [4] used group theory to analyze the symmetry of the eigen-modes in photonic crystals as well as showed the existence of uncoupled modes that cannot be excited by external plane waves. M. Lokke et al. [5] made particular prediction of spatial symmetries of the crystal modes using group theory in triangular air-silica photonic crystal. More recently, Zang et al. [6] developed an algorithm using group theory based on the plane wave expansion method to calculate the photonic band structure.
In the present paper we seek to extend the usage of the theory of groups to model our system transfer matrix without actually doing a sing matrix operation yet making full use of the advantages offered by both group theory and transfer matrix method. The paper is organized as follows, in section 2; we develop the formula which is the basis of our proposed claim. In section 3, we follow up with discussion of the formula in section 4, we conclude our paper.

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2. Mathematical Model

Considering the smallest unit of 1D photonic crystal structure the forward and backward propagating waves are given by-

$$a_2 = t_{21}a_1 + r_{12}b_2 \tag{1}$$

$$b_1 = t_{12}b_2 + r_{21}a_1 \tag{2}$$

where r_{ij} and t_{ij} are reflectivity and transmissivity in passing from layer *i* to layer *j*.(fig. 1)

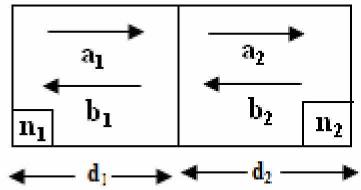

Figure 1: Schematic picture of forward and backward waves in smallest unit of 1D photonic crystal

From the wave equations, transfer matrix corresponding to the interface can be obtained as

$$M^T{}_{1,2} = \frac{1}{t}\begin{pmatrix} 1 & r_{21,12} \\ r_{21,12} & 1 \end{pmatrix} \tag{3}$$

Now, taking into account the phase factor of the field propagating through uniform medium, propagation matrix is given as

$$P_{1,2} = \begin{pmatrix} \exp[jk_{1,2}d_{1,2}] & 0 \\ 0 & -\exp[jk_{1,2}d_{1,2}] \end{pmatrix} \tag{4}$$

where d_i is the propagation length in i[th] layer, and k_i is the wave vector in that layer. Thus, transfer matrix for the elementary cell is

$$M = M^T{}_1 P_1 M^T{}_2 P_2 \tag{5}$$

For a perfectly periodic medium composed of N such elementary cells, the system transfer matrix for such a structure is

$$M_{tot} = M_N \tag{6}$$

Once the system matrix for the entire structure is determined, ω−k dispersion relation for the infinite periodic structure can easily be computed which in turn leads to the effective computation of the photonic band structure. However to compute the transfer matrix of the N cell structure i.e. the system matrix we follow a different approach than the conventional matrix multiplication.

We now consider an $n \times n$ matrix $A = (a_{i,j})$, where the entry $a_{i,j}$ is also denoted as $A_{i,j}$ and A^k denotes the k-th power of A; thus, the $n \times n$ matrix A^k has entries:

$$(A^k)_{i,j} = \sum_{i_1 i_2 \ldots\ldots i_{k-1}} a_{i,i_1} a_{i_1,i_2} \ldots\ldots a_{i_{k-1},j} = \sum_{i_1 i_2 \ldots\ldots i_{k-1}} A_{i,i_1} A_{i_1,i_2} \ldots\ldots A_{i_{k-1},j} \tag{7}$$

where A (i, j) being already defined, we denote the (n−1) × (n−1) sub-matrix obtained from A by deleting the ith row and jth column. Now, we denote the unit matrix, $I_{i,j} = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's symbol (= 1 if i = j and = 0 else).

Following from the above definitions what follow is a closed form equation of the single periodic cell which instead of representing in the matrix notation can be written in the following form:

$$\sum_{k \ge 0} (A^k)_{i,j} x^k = \frac{(-1)^{i+j} \det((I - xA)(j,i))}{\det(I - xA)} \tag{8}$$

Now eqn. (8) can legitimately be written as:

$$\left(\sum_{k \ge 0} (A^k)_{i,j} x^k\right)_{i,j=1}^{n} = \sum_{k \ge 0} (A^k)_{i,j} x^k \tag{9}$$

Similarly,

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$$\left(\sum_{k\geq 0}(A^k)_{i,j}\,x^k\right)_{i,j=1}^{n} = (Ix^0 - Ax)^{-1} \tag{10}$$

Now, once again,

$$\left(\sum_{k\geq 0}(A^k)_{i,j}\,x^k\right)_{i,j=1}^{n} = (I - Ax)^{-1} \tag{11}$$

Finally,

$$\left(\sum_{k\geq 0}(A^k)_{i,j}\,x^k\right)_{i,j=1}^{n} = \left(\frac{(-1)^{i+j}\det((I - xA)(j,i))}{\det(I - xA)}\right)_{i,j=1}^{n} \tag{12}$$

which we denote as the Transfer Matrix Method Formula

3. Analysis and Discussion

The group theoretic approach to solve for the system transfer matrix has yielded a closed form solution in the form of eqn. (12). The name transfer matrix method formula may be a little misleading but since the formulation relies on relating the properties derived from both the conventional group theory principles and the transfer matrix method technique, the naming is quite appropriate.

The symmetry property associated with the one-dimensional stratified periodic structures was exploited using the theory of groups. Actually, what we have done is nothing but to partition or represent each of the terms in the transfer matrix with the corresponding representation groups to obtain a single formula. Based on the value of the i^{th} index and the j^{th} index we can find the corresponding value in the system transfer matrix.

Another significant thing to look for in the derivation is the similarity in both eqns. (10) and (11). We note that in eqn. (11) the "=" sign is not really an equality. In essence, it "equate" matrices with power series and requires to be understood in the sense of isomorphism. In contrary to that, the second "=" sign can be accepted as an equality since it has on both sides, structures of the same type i.e. power series. This provides an advantage in mathematical calculations.

4. Conclusion

In this paper, a new approach was proposed to obtain the system transfer matrix of an N cell periodic one dimensional photonic crystal instead of the conventional transfer matrix method. The method accurately models the transfer matrix, which is essential to effectively study the dispersion relations as well as the photonic band patterns in photonic crystals.

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