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ASSOCIATED PROBABILITIES OF A FUZZY MEASURE IN THE AGGREGATIONS OF FUZZY PROBABILISTIC MEAN OPERATORS

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Abstract
The Ordered Weighted Averaging (OWA) operator was introduced by R.R. Yager [58] to provide a method for aggregating inputs that lie between the max and min operators. In this article several variants of the generalizations of fuzzy probabilistic OWA operators POWA and FPOWA (introduced by J.M. Merigo [27,28]) are presented in the environment of fuzzy uncertainty, where different monotone measures (fuzzy measure) are used as uncertainty measures. Monotone measures considered are: possibility measure, Sugeno $\lambda-$additive measure, monotone measure associated with Belief Structure and Choquet capacity of order two. New aggregation operators are introduced: AsPOWA, AsFPOWA, SA-AsPOWA and SA-AsFPOWA. Some properties of new aggregation operators and their information measures are proved. Concrete faces of new operators are presented with respect to different monotone measures and mean operators. Concrete operators are induced by the Monotone Expectation (Choquet integral) or Fuzzy Expected Value (Sugeno integral) and the Associated Probability Class (APC) of a monotone measure. For the classification of "classic" and new operators of aggregation presented here, the Information Structure is introduced where the incomplete available information in the general decision making system is presented as a concatenation of uncertainty measure + imprecision variable + objective function of weights. For the new operators the information measures – Orness, Entropy, Divergence and Balance are introduced as some extensions of the definitions presented in [28]. For the illustration of new constructions of AsFPOWA and SA-AsFPOWA operators the example of fuzzy decision making problem regarding the political management with possibility uncertainty is considered. Several aggregation operators ("classic" and new operators) are used for the comparing of the results of decision making.

Keywords *- mean aggregation operators, fuzzy aggregations, fuzzy measure, fuzzy numbers, fuzzy decision making, capacity of order, associated probabilities, most typical value, finite Sugeno averaging, finite Choquet averaging, body of evidence, possibility measure.*

INTRODUCTION

It is well recognized that intelligent decision support systems and technologies have been playing an important role in improving almost every aspect of human society. Intensive study over the past several years has resulted in significant progress in both the theory and applications of optimization and decision sciences.

Optimization and decision-making problems are traditionally handled by either the deterministic or the probabilistic approach. When working with complex systems in parallel with classical approaches of their modeling, the most important matter is to assume fuzziness ([3, 6, 14, 16-33, 36-44, 50-63] and others). All this is connected to the complexity of study of complex and vague processes and events in nature and society, which are caused by lack or shortage of objective information and when expert data are essential for construction of credible

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decisions. With the growth of complexity of information our ability to make credible decisions from possible alternatives with complex states of nature reduces to some level, below which some dual characteristics such as precision and certainty become mutually conflicting ([3, 11, 21-23, 37-39, 42, 50, 52, 55, 56] and others). When working on real, complex decision systems using an exact or some stochastic quantitative analysis is often less convenient, concluding that the use of fuzzy methods is necessary, because systems approach for development of information structure of investigated decision system [21, 37, 38] with combined fuzzy-stochastic uncertainty enables us to construct convenient intelligent decision support instruments. Obviously the source for obtaining combined objective + fuzzy + stochastic samplings is the populations of fuzzy-characteristics of experts knowledge ([23, 37, 39, 43, 52] and others). Our research is concerned with quantitative-information analysis of the complex uncertainty and its use for modeling of more précised decisions with minimal decision risks from the point of view of systems research. *The main objects of our attention* are 1) the analysis of Information Structures of experts knowledge and its uncertainty measure and imprecision variable; and 2) the construction of instruments of aggregation operators, which condense both characteristics of incomplete information - an uncertainty measure and an imprecision variable in the scalar ranking values of possible alternatives in the decision making system.

Making decisions under uncertainty is a pervasive task faced by many Decision Making Persons (DMP) experts, investigators or others. The main difficulty is that a selection must be made between alternatives in which the choice of alternative doesn't necessarily lead to well determined payoffs (experts valuations, utilities and so on) to be received as a result of selecting an alternative. In this case DMP is faced with the problem of comparing multifaceted objects whose complexity often exceeds his/her ability to compare of uncertain alternatives. One approach to addressing this problem is to use valuation functions (or aggregation operators). These valuation functions convert the multifaceted uncertain outcome associated with an alternative into a single (scalar) value. This value provides a characterization of the DMP or expert perception of the worth the possible uncertain alternative being evaluated. The problems of Decision Making Under Uncertainty (DMUU) [52] were discussed and investigated by many well-known authors ([1-4, 6, 9, 10,1 4, 16-19, 24-34, 36-61, 63] and others). In this work our focus is directed on the construction of new generalizations of the aggregation OWA operator in the fuzzy-probabilistic uncertainty environment.

In Section 2 some preliminary concepts are presented on the OWA operators; on the arithmetic of the triangular fuzzy numbers; on the some extensions of OWA operator – POWA and FPOWA operators and their information measures in the fuzzy probabilistic uncertainty (developed by J.M. Merigo [27, 28]). Subsection 2.3 considers probability representations – Associated Probability Class (APC) of a monotone measure [5, 37, 39, 42, 44]. Concepts of the Most Typical Value (MTV) [18, 19, 41, 42] of a compatibility function (membership function) of some imprecision variable with respect to some monotone measure is presented. The Fuzzy Expected value (FEV) [9] and Monotone Expectation (ME) [5] are interpreted as important MTVs of a compatibility function. The probability representations of ME and FEV are presented by the APC of a monotone measure. Also in this Subsection the associated probabilities representations are considered for the Choquet capacity of order two [7], possibility measure [11], Sugeno λ – additive measure [45] and a monotone measure associated with Dempster-Shafer Belief Structure [45].

In Section 3 a new conceptual Information Structure (IS) of a General Decision Making System (GDMS) with fuzzy probabilistic uncertainty is defined. This IS classifies some extensions of aggregation operators, e.g. new generalizations of the OWA operator defined in the paper.

In Sections 4 and 5 new generalizations of POWA and FPOWA operators are presented with respect to different monotone measures (insert of the probability measure) and different mean operators. New versions of POWA and FPOWA operators are defined. AsPOWA and AsFPOWA operators are induced by the ME; SA-AsPOWA and SA-AsFPOWA operators are induced by the FEV. In Subsection 4.3 the generalized variants of information measures – *Orness, Entropy,*

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Divergence and Balance are introduced for the new aggregation operators. Some properties of new operators and their information measures are proved in Subsections 4.1 – 4.3 and 5.1 – 5.2.

For the illustration of the applicability of the new generalizations of POWA and FPOWA operators an example of the fuzzy decision making problem regarding political management is considered (Section 6), where we study a country that is planning its fiscal policy for the next year analogously the example considered by J.M. Merigo in [28]. But we use the possibility distribution (possibility uncertainty) on the states of nature of decision making system instead of probability distribution (probability uncertainty) as considered in [28]. We think our approach is more natural and applicable then the case presented in [28]. In this example several aggregation operators were used for the comparing of the results in decision making: 1. SEV (Shapely Expected Value) operator, introduced be R.R. Yager [52]; 2. A new operator SEV-FOWA as a weighted combination of SEV and FOWA operators; 3. New operators – AsFPOWAmin, AsFPOWAmean, AsFPOWAmax, SA-AsFPOWAmin, SA-AsFPOWAmean and Sa-AsFPOWAmax operators introduced in Section 5. The resulting table (see table 7) is presented for ordering of the policies. The values of *Orness* parameter are calculated for all presented aggregation operators.

1. PRELIMINARY CONCEPTS

1.1. On the OWA operator and its some generalizations

In this type of problem the DMP has a collection $D=\{d_1,d_2,...,d_n\}$ of possible uncertain alternatives from which he must select one or some ranking of decisions by some expert's preference relation values. Associated with this problem is a variable of characteristics, activities, symptoms and so on, which acts on the decision procedure. This variable is normally called the state of nature, which affects the payoff, utilities, valuations and others to the DMP's preferences or subjective activities. This variable is assumed to take its values (states of nature) from some set $S=\{s_1,s_2,...,s_m\}$. As a result the DMP knows that if he selects d_i and the state of nature assumes the value s_j then his payoff (valuation, utility and so on) is a_{ij}. The objective of the decision is to select the "best" alternative, get the biggest payoff (valuation, utility and so on). But in DMUU [52] the selection procedure becomes more difficult. In this case each alternative can be seen as corresponding to a row vector of possible payoffs. To make a choice the DMP must compare these vectors, a problem which generally doesn't lead to a compelling solution. Assume d_i and d_k are two alternatives such that for all $j, j=1,2,....,m$ $a_{ij}\geq a_{kj}$ (Table 1). In this case there is no reason to select d_i. In this situation we shall say d_i dominates d_k $(d_i \succeq d_k)$. Furthermore if there exists one alternative that dominates all the alternatives then it will be optimal solution and as a result, we call this the *Pareto optimal*.

Table 1. Decision Matrix.

D \ S	s_1	s_2	…	s_k	…	s_m
d_1	a_{11}	a_{12}	…	a_{1k}	…	a_{1m}
d_2	a_{21}	a_{22}	…	a_{2k}	…	a_{2m}
…	…	…	…	…	…	…
d_i	a_{i1}	a_{i2}	…	a_{ik}	…	a_{im}
…	…	…	…	…	…	
d_n	a_{n1}	a_{n2}	…	a_{nk}	…	a_{nm}

Faced with the general difficulty of comparing vector payoffs we must provide some means of comparing these vectors. Our focus in this work is on the construction of valuation function (aggregation operator) F that can take a collection of m values and convert it into a single value,

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$$F : R^m \Rightarrow R^1.$$

Once we apply this function to each of the alternatives we select the alternative with the *largest scalar value*. The construction of F involves considerations of two aspects. The first being the satisfaction of some rational, objective properties naturally required of any function used to convert (aggregate) a vector of payoffs (valuations, utilities and so on) into an equivalent scalar value. The second aspect being the inclusion of characteristics particular to the DMP's subjective properties or preferences, dependences with respect to risks and other main external factors.

First we shall consider the objective properties required of the valuation function (aggregation operator) F [52].

1) The first property is the satisfaction of Pareto optimality. To insure this we require that if $a_{ij} \geq a_{kj}$ for $j = 1, 2, ..., m$, then

$$F(a_{i1}, a_{i2}, , ..., a_{im}) \geq F(a_{k1}, a_{k2}, ..., a_{km}) . \quad (1)$$

An aggregation operator satisfying this condition is said to be *monotonic*.

2) A second condition is that the value of an alternative should be bounded by its best payoffs (valuations, utilities) and worst possible one. $\forall i = 1, 2, ..., n$

$$\min_{j=1,m}\{a_{ij}\} \leq F(a_{i1}, a_{i2}, ..., a_{im}) \leq \max_{j=1,m}\{a_{ij}\} . \quad (2)$$

This condition is said to be *bounded*.

3) Remark: if $a_{ij} \equiv a_i$ for all j, then from (2)

$$\min_{j=1,m}\{a_{ij}\} = \max_{j=1,m}\{a_{ij}\} \text{ and } F(a_{i1}, a_{i2}, ..., a_{im}) = a_i .$$

This condition is said to be *idempotent*.

4) The final objective condition is that the indexing of the states of nature shouldn't affect the answer:

$$F(a_{i1}, a_{i2}, ..., a_{im}) = F(Permutation(a_{i1}, a_{i2}, ..., a_{im})), \quad (3)$$

where $Permutation(\cdot)$ is some permutation of the set $\{a_{i1}, a_{i2}, ..., a_{im}\}$. An aggregation function satisfying this is said to be *symmetric* (or commutative).

Finally, we have required that our aggregation function satisfy four conditions: *monotonicity, boundedness, idempotency and symmetricity*. Such functions are called *mean or averaging operators* [52].

In determining which of the many possible aggregation operators to select as our valuation function we need some *guidance* from the DMP. The choice of a valuation function, from among the aggregation operators is essentially a "subjective" act reflecting the preferences of the DMP for one vector of payoffs over another. What is needed are tools and procedure to enable a DMP to reflect their subjective preferences into valuations. There are important problems in expert knowledge engineering for which we often use such intelligent technologies as neural networks, machine learning, fuzzy logic control systems, knowledge representations and others.

These problems may be solved by introducing information measures of aggregation operators ([1, 2, 4, 13, 14, 16, 17, 27-34, 36, 39, 41-43, 46-61, 63] and others). In this paper we will present new extensions of information measures of operators constructed bellow.

As an example we present some mean aggregation operators. Assume we have an m-tuple of values $\{a_1, a_2, ..., a_m\}$. Then $F(a_1, a_2, ..., a_m) = \underset{i=1,m}{Min}\{a_i\}$ is one mean aggregation operator. The use of the operator Min corresponds to a pessimistic attitude, one in which the DMP assumes the worst thing will happen. Another example of a mean aggregation operator is $F(a_1, a_2, ..., a_m) = \underset{i=1,m}{Max}\{a_i\}$. Here we have very optimistic valuations. Another example is the simple average: $Mean(a_1, a_2, ..., a_m) = \frac{1}{m}\sum_{i=1}^{m} a_i$. In [58] R.R. Yager introduced a class of mean operators called Ordered Weighed Averaging (OWA) operator.

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DEFINITION 1 [58] : *An OWA operator of dimension* m *is mapping* $OWA: R^m \Rightarrow R^1$ *that has an associated weighting vector* W *of dimension* m *with* $w_j \in [0;1]$ *and* $\sum_{j=1}^{m} w_j = 1$*, such that*

$$OWA(a_1, ..., a_m) = \sum_{j=1}^{m} w_j b_j \ , \qquad (4)$$

where b_j *is the* j*th largest of the* $\{a_i\}$, $i = 1,2,...,m$.

Note that different properties could be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of the OWA operator ([1, 4, 27-34, 46, 48-53, 57, 58, 60, 61, 63] and others).

The OWA operator and its modifications are among the most known mean aggregation operators to the construction of DMUU valuation functions. These aggregations are generalizations of known instrument as Choquet Integral ([5, 7, 24, 39, 42, 52, 54, 55, 58] and others), Sugeno integral ([15, 18, 25, 26, 37, 43, 45] and others) or induced mean functions ([2, 13, 61, 63] and others).

The fuzzy numbers (FN) has been studied by many authors ([11,20] and others). It can represented in a more complete way as an imprecision variable of the incomplete information because it can consider the maximum and minimum and the possibility that the interval values may occur.

DEFINITION 2 [20]: $\tilde{a}(t): R^1 \to [0;1]$ *is called the FN which can be considered as a generalization of the interval number:*

$$\tilde{a}(t) = \begin{cases} 1 & if \ \ t \in [a_2'; a_2''] \\ \dfrac{t - a_1}{a_2' - a_1} & if \ \ t \in [a_1, a_2'] \\ \dfrac{a_3 - t}{a_3 - a_2''} & if \ \ t \in [a_2'', a_3] \\ 0 & otherwise \end{cases} , \qquad (5)$$

where $a_1 \le a_2' \le a_2'' \le a_3 \in R^1$.

In the following, we are going to review the triangular FN (TFN), [20] arithmetic operation as follows (in (5) $a_2' = a_2''$). Let $\tilde{a}$ and $\tilde{b}$ be two TFNs, where $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$. Then

1: $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

2: $\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

3: $\tilde{a} \times k = (ka_1, \ ka_2, ka_3), \ k > 0$

4: $\tilde{a}^k = (a_1^k, a_2^k, \ a_3^k), \ k > 0, a_i > 0$

5: $\tilde{a} \cdot \tilde{b} = (a_1 b_1, a_2 b_2, a_3 b_3), a_i > 0, b_i > 0$

6: $\tilde{b}^{-1} = \left\{ \dfrac{1}{b_3}, \dfrac{1}{b_2}, \dfrac{1}{b_1} \right\}, \ b_i > 0$ (6)

7: $\tilde{a} > \tilde{b}$ *if* $a_2 > b_2$ and *if* $a_2 = b_2$ *then* $\tilde{a} > \tilde{b}$ *if* $\dfrac{a_1 + a_3}{2} > \dfrac{b_1 + b_3}{2}$ *otherwise* $\tilde{a} = \tilde{b}$.

The set of all TFNs is denoted by Ψ and positive TFNs ($a_i > 0$) by Ψ^+.

Note that other operations and ranking methods could be studied ([20] and others).

Now we consider some extensions of the OWA operator, mainly developed by Merigo and others [27, 28, 30], because our future investigations concern with extensions of Merigo's aggregation operators constructed on the basis of the OWA operator.

DEFINITION 3 [30]: Let Ψ be the set of TFNs. A fuzzy OWA operator - FOWA of dimension m is a mapping $FOWA: \Psi^m \Rightarrow \Psi$ that has an associated weighting vector W of dimension m with $w_j \in [0,1]$, $\sum_{j=1}^{m} w_j = 1$ and $FOWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \sum_{j=1}^{m} w_j \tilde{b}_j$, (7) *where* $\tilde{b}_j$ *is the jth largest of the* $\{\tilde{a}_i\}_{i=1}^m$, *and* $a_i \in \Psi$, $i = 1,2,...,m.$

The FOWA operator is an extension of the OWA operator that uses imprecision information in the arguments represented in the form of TFNs. The reason for using this aggregation operator is that sometimes the available information presented by the DMP and formalized in payoffs (valuations, utilities and others) can't be assessed with exact numbers and it is necessary to use other techniques such as TFNs. So, in this aggregation incomplete information is presented by imprecision variable of experts reflections and formalized in TFNs. Sometimes the available information presented by the DMP (or expert) also has an uncertain character, which is presented by the probability distribution on the states of nature consequents on the payoffs of the DMP.

The fuzzy probability aggregations based on the OWA operator was constructed by J. M. Merigo and others. One of the variants we present here:

DEFINITION 4 [28]: *A probabilistic OWA operator - POWA of dimension* m *is a mapping* $POWA: R^m \Rightarrow R^1$ *that has an associated weighting vector* W *of dimension* m *such that* $w_j \in [0,1]$ *and* $\sum_{j=1}^m w_j = 1$ *according to the following formula:*

$$POWA(a_1, a_2, ..., a_m) = \sum_{j=1}^m \hat{p}_j b_j \ , \tag{8}$$

where b_j *is the jth largest of the* $\{a_i\}, i = 1,2,...,m$; *each argument* a_i *has an associated probability* p_i *with* $\sum_{i=1}^m p_i = 1$, $0 \le p_i \le 1$, $\hat{p}_j = \beta w_j + (1-\beta) p_j$ *with* $\beta \in [0,1]$ *and* p_j *is the probability* p_i *ordered according to* b_j, *that is according to the jth largest of the* a_i.

Note that if $\beta = 0$, we get the usual probabilistic mean aggregation (mathematical expectation - E_p with respect to probability distribution $\{p_i\}_{i=1}^m$), and if $\beta = 1$, we get the OWA operator. Equivalent representation of (8) may be defined as:

$$\begin{aligned} POWA(a_1, a_2, ..., a_m) &= \beta \sum_{j=1}^n w_j b_j + (1-\beta) \sum_{i=1}^m p_i a_i = \\ &= \beta \cdot OWA(a_1, a_2, ..., a_m) + (1-\beta) \cdot E_p(a_1, a_2, ..., a_m) \end{aligned} . \tag{9}$$

We often use probabilistic information in the decision making systems and consequently in their aggregation operators. Many fuzzy-probabilistic aggregations have been researched in OWA and other operators ([5, 18, 19, 27-33, 36-43, 50-54, 60, 61, 63] and others). In the following we present one of them defined in [28]:

DEFINITION 5 [28]: *Let* Ψ *be the set of TFNs. A fuzzy-probabilistic OWA operator - FPOWA of dimension* m *is a mapping* $FPOWA: \Psi^m \Rightarrow \Psi$ *that associated a weighting vector* W *of dimension* m *such that* $w_j \in [0,1]$, $\sum_{j=1}^m w_j = 1$, *according to the following formula:*

$$FPOWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \sum_{j=1}^n \hat{p}_j \tilde{b}_j \ , \tag{10}$$

where $\tilde{b}_j$ *is the jth largest of the* $\{a_i\}_{i=1}^m$ *are TFNs and each one has an associated probability* $p_i \equiv P(\tilde{a} = \tilde{a}_i)$, *with* $\sum_{j=1}^m p_j = 1$, $0 \le p_j \le 1$, $\hat{p}_j = \beta w_j + (1-\beta) p'_j$, $\beta \in [0,1]$ *and* p'_j *is the probability ordered according to* $\tilde{b}_j$ $(p'_j = P(\tilde{a} = \tilde{b}_j))$ *that is according to the jth largest of the* $\{\tilde{a}_i\}_{i=1}^m$.

Analogously to (9) we present the equivalent form of the FPOWA operator as a weighted sum of the OWA operator and the mathematical expectation - E_p :

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$$FPOWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \beta \sum_{j=1}^{n} w_j \tilde{b}_j + (1-\beta) \sum_{i=1}^{m} p_i \tilde{a}_i = \\ = \beta \cdot OWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) + (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) \quad (11)$$

In [28] the Semi-boundary condition of the aggregation operator (11) was proved. Semi-boundary condition of some operator F if defined as:

$$\beta \times \min_i \{\tilde{a}_i\} + (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) \le F(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) \le \\ \le \beta \times \max_i \{\tilde{a}_i\} + (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) \quad (12)$$

So the FPOWA operator is monotonic, bounded, idempotent, symmetric and semi-bounded.

1.2. On the information measures of the POWA and FPOWA operators

Now we present four probabilistic information measures of the POWA and FPOWA operators defined in [28] following similar methodology developed for the OWA operator ([1, 2, 3, 6, 48, 49, 51, 53] and others).

a) The *Orness* parameter classifies the POWA and FPOWA operators in regard to their location between *and* and *or*:

$$\alpha(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = \beta \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1} \right) + (1-\beta) \sum_{j=1}^{m} p'_j \left(\frac{m-j}{m-1} \right) \quad (13)$$

b) The *Entropy* (*dispersion*) measures the amount of information being used in the aggregation:

$$H(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = -\left\{ \beta \sum_{j=1}^{m} w_j \ln w_j + (1-\beta) \sum_{i=1}^{m} p_i \ln p_i \right\}$$

(14)

c) The *divergence* of weighted vector W measures the divergence of the weights against the degree of *Orness*:

$$Div(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = \beta \left\{ \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1} - \alpha(W) \right)^2 \right\} + \\ + (1-\beta) \left\{ \sum_{j=1}^{m} p'_j \left(\frac{m-j}{m-1} - \alpha(P) \right)^2 \right\}, \quad (15)$$

where $\alpha(W)$ is an *Orness* measure of the OWA or FOWA operators ($\beta = 1$):

$$\alpha(W) = \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1} \right), \quad (16)$$

and $\alpha(P)$ is an *Orness* measure of the fuzzy-probabilistic aggregation ($\beta = 0$):

$$\alpha(P) = \sum_{j=1}^{m} p'_j \cdot \left(\frac{m-j}{m-1} \right), \quad (17)$$

d) The *balance* parameter measures the balance of the weights against the *Orness* or the *Andness*:

$$Bal(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = \beta \left\{ \sum_{j=1}^{m} w_j \left(\frac{m+1-2j}{m-1} \right) \right\} + \\ + (1-\beta) \left\{ \sum_{j=1}^{m} p'_j \left(\frac{m+1-2j}{m-1} \right) \right\}. \quad (18)$$

1.3. On the Associated Probabilities of a Monotone Measure (Fuzzy Measure)

1.3.1. Associated Probabilities of a Monotone Measure

When trying to functionally describe insufficient expert data, in many real situations the property of additivity remains unrevealed for a measurable representation of a set and this creates an additional restriction. Hence, to study such data, it is frequently better to use monotone measures (estimators) instead of additive ones.

We introduce the definition of a monotone measure (fuzzy measure) [45] adapted to the case of a finite referential.

DEFINITION 6: *Let* $S=\{s_1,s_2,...,s_m\}$ *be a finite set and* g *be a set function* $g:2^S\to[0,1]$. *We say* g *is a monotone measure on* S *if it satisfies*

$$(i)g(\varnothing)=0;\ g(S)=1;\quad (ii)\forall A,B\subseteq S,\quad if\ \ A\subseteq B,\ then\ g(A)\le g(B).$$

A monotone measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity. Non-additive but monotone measures were first used in the fuzzy analysis in the 1980s [45] and is well investigated ([8, 15, 22, 23, 37-39, 44, 45, 54-56, 62] and others).

A fuzzy integral is a functional which assigns some number or a compatibility value to each fuzzy subset when the monotone measure is taken as an uncertainty measure. As known ([10, 15, 18, 19, 25, 26, 37, 38, 45, 63] and others), the concept of a fuzzy integral condenses the information provided by a compatibility (or membership) function of a fuzzy set and a monotone measure. Having the monotone measure determined, we can estimate a fuzzy subset by the most typical compatibility value - most typical value (MTV) ([18, 19, 41-45] and others) or a fuzzy average. As already known, fuzzy averages (MTVs) differ both in form and content from probabilistic–statistical averages and other numerical characteristics such as mode and median and others. Nevertheless, in some cases 'non-fuzzy' (objective) and 'fuzzy' (subjective) averages coincide ([18, 19, 41-45] and others). For a given set of fuzzy subsets with compatibility function values from the interval [0; 1], the fuzzy average determines the most typical representative compatibility value. From the point of our future presentations in the role of MTV we consider only two fuzzy statistics (integrals): 1. Monotone Expectation – ME (or Choquet Integral) and 2. Fuzzy Expected Value – FEV (or Sugeno Integral). So, we consider some aspects of a monotone measure in fuzzy statistics.

DEFINITION 7: *Assume* $S=\{s_1,s_2,...,s_m\}$ *is a set on which we have a monotone measure* g *and a function* $a:S\Rightarrow R_0^+$ *such that* $a(s_i)\equiv a_i\ge 0, i=1,2,...,m$. *Then*

a) The aggregation

$$ME_g(a_1,a_2,...,a_m)\equiv FCA(a_1,a_2,...,a_m)=\sum_{j=1}^{m}w_j a_{i(j)}\ , \tag{19}$$

where $w_j=g(\{s_{i(1)},.....,s_{i(j)}\})-g(\{s_{i(1)},.....,s_{i(j-1)}\}),\ g(\{s_{i(0)}\})\equiv 0,$ *is called a Finite Choquet Averaging (FCA) or Monotone Expectation (ME) operator. In the proceeding* $i(\cdot)$ *is index function such that* $a_{i(j)}$ *is the jth largest of the* $\{a_i\}_{i=1}^m$.

b) The aggregation

$$FEV_g(a_1,...,a_m)\equiv FSA(a_1,...,a_m)=a_{\max}\max_{j=\overline{1,m}}\min\{a_{i(j)}/a_{\max}\ ;\ \hat{w}_j\}, \tag{20}$$

where $\hat{w}_j=g(\{s_{i(1)},s_{i(2)},...,s_{i(j)}\})$, $a_{\max}=\max_{i=\overline{1,m}}\{a_i\}$ *is called a Finite Sugeno Averaging (FSA) or a Fuzzy Expected Value (FEV) operator.*

The ME always exists and is finite for each monotone measure g and some compatibility variable a. It is obvious that $ME_g(a)$ is a generalization of the mathematical expectation $E_P(a)$ and the ME of a non-negative function a with respect to a monotone measure g coincides with the mathematical expectation of a with respect to a probability measure that depends only on g and the ordering of the values of a.

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Following the definition 7 the maximum number of probability distributions in ME (formula 19) coincides with the number of possible orderings or permutations in a set with m elements, that is, $m!$. Thus, it makes sense to associate the $m!$ probabilities to each monotone measure, provided that they are deduced from this monotone measure through the different possible orderings.

In general, the possible orderings of the elements of S are given by the permutations of a set with m elements, which form the group S_m.

DEFINITION 8 [5]: *The probability functions* P_σ *defined by*

$$P_\sigma\left(s_{\sigma(1)}\right)=g\left(\left\{s_{\sigma(1)}\right\}\right),\ldots,$$
$$P_\sigma\left(s_{\sigma(i)}\right)=g\left(\left\{s_{\sigma(1)},\ldots,s_{\sigma(i)}\right\}\right)-g\left(\left\{s_{\sigma(1)},\ldots,s_{\sigma(i-1)}\right\}\right),\ldots, \quad (21)$$
$$P_\sigma\left(s_{\sigma(m)}\right)=1-g\left(\left\{s_{\sigma(1)},\ldots,s_{\sigma(m-1)}\right\}\right),$$

for each $\sigma=(\sigma(1),\sigma(2),\ldots,\sigma(m))\in S_m$, *are called the associated probabilities and the Associated Probability Class (APC) -*$\{P_\sigma\}_{\sigma\in S}$ *of the monotone measure* g.

An interesting case is when the monotone measure is a probability. It is easy to prove that in this case, all associated probabilities are equal.

PROPOSITION 1 [5]: *A monotone measure* g *is a probability measure* $(g=p)$ *if and only if its m! associated probabilities coincide.*

The concept of duality of monotone measures is very important, since it permits one to obtain alternative representations of a piece of information. So, we will consider a monotone measure and its dual measure to contain the same information, but codified in a different way.

The most remarkable case where different monotone measures provide the same $m!$ probabilities, but ordered in a different way, is the case of dual monotone measures. Before exposing it in the following proposition, we need a definition:

DEFINITION 9: *We will say that two permutations* $\sigma,\sigma^*\in S_m$ *are dual if*

$$\sigma^*(i)=\sigma(m-i+1),\ i=1,\ldots,m.$$

PROPOSITION 2 [5]: *A necessary and sufficient condition for two monotone measures* g_* *and* g^* *to be dual is to have the same m! associated probabilities corresponding to dual permutations, that is,* $P_{*\sigma}=P^*_{\sigma^*}$, *if* σ *and* σ^* *are dual, where* P_* *and* P^* *are associated probabilities for the measures* g_* *and* g^* *respectively.*

An especially interesting class of monotone measures is the capacities of order two [7], because they cover a great number of monotone measures.

DEFINITION 10: *Let* (g_*,g^*) *be a pair of dual monotone measures:*

g_* *is a lower capacity of order two if and only if*

$$\forall A,B\subseteq S,\quad g_*(A\cup B)+g_*(A\cap B)\geq g_*(A)+.g_*(B);$$

g^* *is an upper capacity of order two if and only if*

$$\forall A,B\subseteq S,\quad g^*(A\cup B)+g^*(A\cap B)\leq g^*(A)+.g^*(B).$$

The most used classes of monotone measures such as belief and plausibility measures [35], necessity and possibility ones [11], λ-measures [45] and probabilities are capacities of order two.

PROPOSITION 3 [5]: *Let* (g_*,g^*) *be a pair of dual monotone measures. Then* g_* *is a lower capacity of order two (*g_* *is an upper capacity of order two, respectively) if and only if*

$$g_*(A)=\min_{\sigma\in S_m}P_\sigma(A)\ \forall A\subseteq X,\quad (g^*(A)=\max_{\sigma\in S_m}P_\sigma(A)\ \forall A\subseteq X,\ ). \quad (22)$$

So the main characteristic of a capacity of order two is that it only depends on the probabilities associated to such a measure, but does not depend on the permutations that generate them: we can regenerate the initial monotone measure by only knowing its associated probabilities, without the necessity to know the corresponding permutations. This characteristic makes the use of capacities of order two by means of associated probabilities especially easy.

Starting from this property, the following result is evident and valid for every monotone measure:

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PROPOSITION 4 [5]: *If* $P_\sigma, \sigma \in S_m$*, are the associated probabilities to a monotone measure* g*, then for every* $a: X \to R_0^+$ *, it holds*

$$\min_{\sigma\in S_m} E_{P_\sigma}(a) \le ME_g(a) \le \max_{\sigma\in S_m} E_{P_\sigma}(a). \tag{23}$$

PROPOSITION 5 [39]: *A necessary and sufficient condition for a pair of dual fuzzy measures* (g_*, g^*) *to be lower and upper capacities of order two, respectively, is that* $\forall\, a: X \to R_0^+$,

$$ME_{g_*}(a) = \min_{\sigma\in S_m} E_{P_\sigma}(a),\ \ ME_{g^*}(a) = \max_{\sigma\in S_m} E_{P_\sigma}(a). \tag{24}$$

Let $S_m^{(a)}$ $(S_m^{(a)} \subset S_m)$ be the subgroup of all permutations such that $\forall\, \sigma \in S_m^{(a)}$

$$a(s_{\sigma(1)}) \ge a(s_{\sigma(2)}) \ge ... \ge a(s_{\sigma(m)}). \tag{25}$$

Following Proposition 2 and Definitions 7 and 8 there exist some connections of mathematical expectations with respect to dual associated probability $P_{*\sigma}; P^*{}_\sigma (\sigma \in S_m^{(a)})$:

$$\begin{aligned} ME_{g_*}(a) &= E_{P_{*\sigma}}(a) = \sum_{i=1}^{m} P_{*\sigma}(s_{\sigma(i)}) a(s_{\sigma(i)}), \\ ME_{g^*}(a) &= E_{P_\sigma^*}(a) = \sum_{i=1}^{m} P_\sigma^*(s_{\sigma(i)}) a(s_{\sigma(i)}) = \sum_{j=1}^{m} P_{*\sigma_*}(s_{\sigma_*(m-i+1)}) a(s_{\sigma_*(m-i+1)}) = E_{P_{*\sigma_*}}(a) \end{aligned}, \tag{26}$$

where $P_{*\sigma}$ and P_σ^* are associated probabilities for g_* and g^* monotone measures, respectively; σ and σ_* are dual permutations and a is symmetric.

1.3.2. FEV's probability representation

It clearly follows that (definition 7b) the FEV somehow 'averages' the values of the compatibility function a not in the sense of a statistical average but by cutting subsets of the α level, whose values of monotone measure g are either sufficiently 'high' or sufficiently 'low'. The FEV gives a concrete value of the compatibility function a, this value being the most typical characteristic of all possible values with respect to the monotone measure g, obtained by cutting off the 'upper' and 'lower' strips on the graph of $g(H_\alpha) = g(\{s / a(s) \ge \alpha\})$. Thus, the incomplete information carried by an imprecision variable a and an uncertain measure g is condensed in the FEV, which is the MTV of all compatibility levels of a. Following definition 7b for all permutation such that $\sigma \in S_m^{(a)}$ the FEV can be written by the associated probabilities of a monotone measure g as

$$FEV_{g_*}(a) = a_{\max} \min_{j=\overline{1,m}} \min_{\sigma'\in S_m} \max\{a(s_{\sigma(i)} / a_{\max}); P_{*\sigma'}(A_i^{(\sigma)})\}, \tag{27}$$

where $A_i^{(\sigma)} = \{s_{\sigma(1)}, s_{\sigma(2)}, ..., s_{\sigma(i)}\}$, $i = 1,..,m$.

Let (g_*, g^*) be a pair of a dual lower and upper capacities of order two. Using propositions 2, 3 and formulas (27) the FEV can be written, $\forall\, \sigma \in S_m^{(a)}$:

$$\begin{aligned} FEV_{g_*}(a) &= a_{\max} \min_{j=\overline{1,m}} \min_{\sigma'\in S_m} \max\{a(s_{\sigma(i)} / a_{\max}); P_{*\sigma'}(A_i^{(\sigma)})\}, \\ FEV_{g^*}(a) &= a_{\max} \max_{i=\overline{1,m}} \max_{\sigma'\in S_m} \min\{a(s_{\sigma(i)}) / a_{\max}; P_{\sigma'}^*(A_i^{(\sigma)})\} = \\ &= a_{\max} \max_{i=\overline{1,m}} \max_{\sigma'\in S_m} \min\{a(s_{\sigma(i)}) / a_{\max}; P_{*\sigma'^*}(A_i^{(\sigma)})\}. \end{aligned} \tag{28}$$

1.3.3. Dempster–Shafer belief structure and its associated probabilities

The Theory of Evidence (Dempster–Shafer Belief Structure) ([11, 15, 22, 23, 25, 32, 37, 43, 56, 59, 62] and others) is a powerful tool which enables one to build: 1. models of decisions and their risks' measures; 2. Aggregation operators in an uncertain environment and so on.

The Theory of Evidence is based on two dual monotone measures: Belief measures and Plausibility measures. These classes of monotone measures are subclasses of classes of dual lower and upper capacities of order two. This is easily provable after introduction of Belief and Plausibility measures ([22,23] and others). Belief and Plausibility measures can be characterized by the set function:

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$$\mathrm{m}: 2^S \to [0;1],$$

which is required to satisfy two conditions:

$$\text{(a)} \quad \mathrm{m}(\varnothing) = 0,$$
$$\text{(b)} \quad \sum_{\mathrm{B}\in 2^S} \mathrm{m(B)} = 1. \tag{29}$$

This function is called a *Basic Probability Assignment* (BPA). For each set $B \in 2^S$, the value $m(B)$ expresses the proportion that all available and relevant evidence supporting the claim that a particular element of S, whose characterization in terms of relevant attributes is deficient, belongs to the set B. This value $m(B)$, pertains solely to one set $-B$; it does not imply any additional claims regarding subsets of B. If there is some additional evidence supporting the claim that the element belongs to a subset of B, say $B_1 \subseteq B$, it must be expressed by another value $m(B_1)$ [23].
Let m be a PBA on S. The *plausibility measure* Pl associated to m is given by

$$\mathrm{Pl(A)} = \sum_{\mathrm{B}\subset \mathrm{S}:\, \mathrm{A}\cap \mathrm{B}\neq \varnothing} \mathrm{m(B)}, \ \ \forall A \in 2^S$$

and the *Belief measure* Bel associated to m is given by

$$\mathrm{Bel(A)} = \sum_{\mathrm{B}:\, \mathrm{B}\subset \mathrm{A}} \mathrm{m(B)}, \ \ \forall A \in 2^S.$$

Inverse procedures are also possible. Given, for example, a Belief measure Bel, the corresponding BPA is determined for all $A \in 2^S$ by formula

$$\mathrm{m(A)} = \sum_{\mathrm{B}:\, \mathrm{B}\subseteq \mathrm{A}} (-1)^{|\mathrm{A}\setminus \mathrm{B}|}\, \mathrm{Bel(B)}, \tag{30}$$

where $|A \setminus B|$ is the cardinality of the set difference of A and B. If the Belief measure is also additive that is

$$Bel(A \cup B) = Bel(A) + Bel(B), \ \ if \ \ A \cap B = \varnothing, \ \ A, B \in 2^S, \tag{31}$$

then we obtain the classical probability measure [23].

Given a BPA, every set $A \in 2^S$ for which $m(B) > 0$ is called a focal element. The pair $\langle F_S, m\rangle$ where F_S denotes the set of all focal elements induced by m is called a *Body of Evidence*. Because Bel is a lower capacity of order two, then using proposition 3 and formulas (29) and (30) we receive probability representation of the BPA, $\forall A \in 2^S$, $\sigma \in S_m$:

$$\mathrm{P}_\sigma^{(Bel)}(s_{\sigma(i)}) = \sum_{\substack{\mathrm{B}\in \mathrm{F_S}:\, \mathrm{B}\subseteq \{s_{\sigma(1)},..,s_{\sigma(i)}\} \\ B\cap \{s_{\sigma(i)}\}\neq \varnothing}} \mathrm{m(B)}, \qquad \mathrm{m(A)} = \sum_{\mathrm{B}\in \mathrm{F_S}:\, \mathrm{B}\subseteq \mathrm{A}} (-1)^{|\mathrm{A}\setminus \mathrm{B}|} \min_{\sigma\in S_m} \mathrm{P}_\sigma^{(Bel)}(B), \tag{32}$$

where $\{P_\sigma^{(Bel)}\}_{\sigma\in S_m}$ are the associated probabilities of the monotone measure Bel.

1.3.4. Possibility measure and its associated probabilities

When the focal elements of a body of evidence $\langle F_S, m\rangle$ are required to be nested, $F = \{A_{j_1} \subset A_{j_2} \subset ... \subset A_{j_l}\}$, the associated belief and plausibility measures are called consonant [23]. The special branch of the evidence theory that deals only with bodies of evidence whose focal elements are nested is referred to as the possibility theory [11].

Special counterparts of Bel measures and Pl measures in the possibility theory are called *necessity* (Nec) measures and *possibility* (Pos) measures, respectively:

PROPOSITION 6 [23]: *Given a consonant body of evidence $\langle F_S, m\rangle$, the associated consonant belief (necessity) and plausibility (possibility) measures possess the following properties:*

$$Nec(A \cap B) = \min\{Nec(A); Nec(B)\} \ \ for \ \ all \ \ A, B \in 2^S,$$
$$Pos(A \cup B) = \max\{Pos(A); Pos(B)\} \ \ for \ \ all \ \ A, B \in 2^S. \tag{33}$$

PROPOSITION 7 **[**23]: *Every possibility measure Pos on 2^S can be uniquely determined by its possibility distribution function $\pi: S \to [0,1]$; $\max_{s\in S} \pi(s) = 1$ via the formula: $\forall A \in 2^S$*

$$Pos(A)=\max_{s\in A}\pi(s). \quad (34)$$

Assume the finite universe $S=\{s_1,s_2,...,s_m\}$ is given and let $F_S=\{A_{j_1}\subset A_{j_2}\subset...\subset A_{j_l}\}$ be some consonant body of evidence.

Let $m_{j_i}\equiv m(A_{j_i}),\ i=1,...,l;\ \pi_i\equiv\pi(s_i),\ \pi_i\geq\pi_{i+1};i=1,...,m;\pi_1=1.$

Then, we have the l-tuple

$$m=\langle m_{j_1},\ m_{j_2},...,m_{j_l}\rangle \quad (35)$$

and m-tuple

$$\pi=\langle\pi_1,\ \pi_2,...,\pi_m\rangle. \quad (36)$$

It is easy to show that

$$\begin{cases}\pi_i=\sum\limits_{\nu:s_i\in A_{j_\nu}\in F_S}m_{j_\nu},\ i=1,2,..,m\\ m_{j_i}=\pi_{j_i}-\pi_{j_{i+1}},\ \pi_{j_{l+1}}\equiv 0,\ i=1,2,...,l.\end{cases} \quad (37)$$

Let $\{P_\sigma^{(Pos)}\}_{\sigma\in S_m}$ be the associated probabilities class of a possibility measure Pos. Then, we have the following connection between $\{\pi_i\}$, $\{m_{j_i}\}$ and $\{P_\sigma\}_{\sigma\in S_m}:\forall\sigma\in S_m$

$$\begin{aligned}P_\sigma^{(Pos)}(s_{\sigma(i)})&=Pos(\{s_{\sigma(1)},...,s_{\sigma(i)}\})-Pos(\{s_{\sigma(1)},...,s_{\sigma(i-1)}\})=\\&=\max_{\nu=1,i}\pi(s_{\sigma(\nu)})-\max_{\nu=1,i-1}\pi(s_{\sigma(\nu)})=\\&=\max_{\nu=1,i}\sum_{q:s_{\sigma(\nu)}\in A_{j_q}\in F_S}m_{j_q}-\max_{\nu=1,i-1}\sum_{q:s_{\sigma(\nu)}\in A_{j_q}\in F_S}m_{j_q}=\\&=\begin{cases}0,\quad otherwise\\ \sum\limits_{q:s_{\sigma(i)}\in A_{j_q}\in F_S}m_{j_q}-\sum\limits_{q:s_{\sigma(i')}\in A_{j_q}\in F_S}m_{j_q},\quad if\ \sigma(i')<\sigma(i)\end{cases}\end{aligned} \quad (38)$$

Since Pos is a capacity of order two, using proposition 5 we receive:

$$\pi_i=Pos(\{s_i\})=\max_{\sigma\in S_m}P_\sigma^{(Pos)}(\{s\}_i),\quad i=1,2,..,m, \quad (39)$$

$$m_{j_i}=\pi_{j_i}-\pi_{j_{i+1}}=\max_{\sigma\in S_m}P_\sigma^{(Pos)}(\{s_{j_i}\})-\max_{\sigma\in S_m}P_\sigma^{(Pos)}(\{s_{j_{i+1}}\}),\quad i=1,2,..,l. \quad (40)$$

1.3.5. Monotone measures associated with a belief structure and its associated probabilities

Let m be a BPA with a body of evidence $F_S=\{A_1,A_2,...,A_q\}$. For each focal element $A_j,\ j=1,..,q,$ let W_j^0 be a weighting vector of dimension $|A_j|$ whose components $w_j^0(i)$ $\left(W_j^0\equiv\langle w_j^0(1),...,w_j^0(|A_j|)\rangle\right)$ satisfy the conditions $w_j^0(i)\in[0,1]$, $\sum_{i=1}^{|A_j|}w_j^0(i)=1$. We shall call these the allocation vectors. In [56], it was shown that a set function $g:2^S\to[0,1]$ defined by

$$g(A)=\sum_{j=1}^{q}\left[m(A_j)\cdot\sum_{i=1}^{|A_j\cap A|}w_j^0(i)\right],....\forall A\in 2^S \quad (41)$$

is a monotone measure associated with the belief structure. Thus, by selecting a collection $W^0=\{W_1^0,W_2^0,...,W_q^0\}$ of allocation vectors, we can define a unique monotone measure associated with a belief structure. For example: if all the W_j^0 are such that $w_j^0(1)=1$, then the resulting monotone measure is the plausibility measure Pl. If all W_j^0 are selected such that $w_j^0(|A_j|)=1$, then this results in the belief measure Bel.

We have the following important proposition concerning all associated monotone measures with a belief structure.

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PROPOSITION 8 [56]: *If* g *is any monotone measure generated from a collection of allocation vectors:*

(a) $Bel(A) \le g(A) \le Pl(A) \qquad \forall A \in 2^S$;

(b)The Shapley Entropy of generated monotone measures coincide

$$E_{Shapley}(Bel) = E_{Shapley}(g) = E_{Shapley}(Pl).$$

i.e. generated monotone measures have the same information but codified in a different way.

Now, we shall compute the associated probabilities of a monotone measure g associated with the belief structure: $\forall \sigma \in S_m, \forall i = 1,2,...,m$.

$$P_\sigma(s_{\sigma(i)}) = g(\{s_{\sigma(1)},...,s_{\sigma(i)}\}) - g(\{s_{\sigma(1)},...,s_{\sigma(i-1)}\}) =$$

$$= \sum_{j=1}^{q} m(A_j)\left[\sum_{\nu=1}^{|A_j \cap \{s_{\sigma(1)},...,s_{\sigma(i)}\}|} w_j^0(\nu) - \sum_{\nu=1}^{|A_j \cap \{s_{\sigma(1)},...,s_{\sigma(i-1)}\}|} w_j^0(\nu)\right] = \tag{42}$$

$$= \sum_{A_j \in F_S : A_j \cap \{s_{\sigma(i)}\} \ne 0} m(A_j) w_j^0\left(\left|A_j \cap \{s_{\sigma(1)},...,s_{\sigma(i)}\}\right|\right).$$

1.3.6. Sugeno λ-additive monotone measure and its associated probabilities

DEFINITION 11 [45]: *A monotone measure* $g_\lambda : 2^S \to [0,1]\ (\lambda > -1)$ *is called a* λ*-additive monotone measure if for any* $A, B \in 2^S$, $A \cap B = \varnothing$,

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) \cdot g_\lambda(B). \tag{43}$$

It is easy to verify that for any $A \in 2^S$

$$g_\lambda(A) = \frac{1}{\lambda}\left\{\prod_{s_i \in A}(1 + \lambda g_i) - 1\right\}, \tag{44}$$

where $0 < g_i \equiv g(\{s_i\}),\ i = 1,...,m;\ \lambda > -1$ is the parameter with following normalization condition:

$$\frac{1}{\lambda}\left\{\prod_{s_i \in S}(1 + \lambda g_i) - 1\right\} = 1. \tag{45}$$

Note, that $g_0\ (\lambda = 0)$ is a probability measure if $\sum_{s_i \in S} g_i = 1.$

It is easy to prove that the λ-additive monotone measure g_λ is a capacity of order two and $g_\lambda^* = g_{-\lambda/(1+\lambda)}$.

Due to (44), (45) and (21), we can write the class of associated probabilities for the λ-additive monotone measure g_λ for any $\sigma \in S_m$ as

$$\mathrm{P}_\sigma(s_{\sigma(i)}) = g_\lambda(\{s_{\sigma(i)}\}) \prod_{j=1}^{i-1}\left(1 + \lambda g_\lambda(\{s_{\sigma(j)}\})\right), \tag{46}$$

or, more exactly, as

$$\mathrm{P}_\sigma(s_i) = g_\lambda(\{s_i\}) \prod_{j=1}^{i(\sigma)-1}\left(1 + \lambda g_\lambda(\{s_{\sigma(j)}\})\right), \tag{47}$$

where $i = 1,2,...,m,\ \sigma \in S_m$; $i(\sigma)$ is the location of s_i in the permutation σ (if $i(\sigma) = 1$, then $\prod_{j=1}^{0} \equiv 1$).

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2. GENERAL DECISION MAKING SYSTEM (GDMS) AND ITS INFORMATION STRUCTURE (IS)

Our focus is directed for the construction of a new POWA and FPOWA fuzzy-probabilistic aggregation operators induced by the ME or the FEV with respect to different monotone measures. The different cases of incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) will be considered in new aggregation operators. Therefore from the point of view of systems research it is necessary to describe and formally present the scheme of general decision making system (GDMS) in uncertain – objective environment. GDMS gives us the possibility to identify the different cases of levels of incompleteness and objectivity of available information which in whole defines the aggregation procedure.

DEFINITION 12: *The general decision making system (GDMS) that will combine decision-making technologies and methods of construction of decision functions (aggregation operators) may be presented by the following 8-tuple*

$$\langle D,\ S,\ a,\ g,\ W,\ I,\ F,\ I_m \rangle, \tag{48}$$

where

1) $D = \{d_1, d_2, ..., d_n\}$ is a set of all possible alternatives (decisions, diagnosis and so on) that are made by a Decision-Making Person (DMP).
2) $S = \{s_1, s_2, ..., s_m\}$ is a set of systems states of nature (actions, activities, factors, symptoms and so on) that are act on the possible alternatives in the decision procedure.
3) a - is an imprecision on precision variable of payoffs (utilities, valuations, some degrees of satisfaction to a fuzzy set, prices and so on), which will by defined by DMP's subjective properties of preferences, dependences with respect to risks and other external factors. As a result a constructs some decision matrix (binary relation) on $D \times S$.
4) g is an uncertainty measure on 2^S $(g : 2^S \to [0,1])$. In our case it may be some monotone measure.
5) W is an objective weighted function (or vector) on the states of nature - S.
6) I is the Information Structure on the data of states of nature. Cases of different levels of information incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) on the states of nature will be considered as:

   I = Information Structure (on S): =imprecision (on S) + uncertainty (on S) + objectivity (on S), where:

   a) Imprecision on S may be presented by some inexact (stochastic, fuzzy, fuzzy-stochastic or other) variable.

   b) Uncertainty on S may be presented by the levels of belief, credibility, probability, possibility and other monotone measures on 2^S. These levels identify the possibility of occurrence of some groups (events, focal elements and others) on the states of nature.

   c) Objectivity on S is defined by the objective importance of states of nature in the procedure of decision making. As usual the objective function is presented by a weighted function (vector) $W : S \Rightarrow R_0^+$.

   Now we may classify cases of the Information Structure – I:

   I1: The case:

   a) – imprecision is presented by the some exact variable $a : S \Rightarrow R^1$;

   b) – the measure of uncertainty does not exists;

   c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

   Examples: OWA and MEAN operators belong to I1.

   I2: The case:

   a) – imprecision is presented by the some fuzzy variable: $\tilde{a} \in \psi$; $\tilde{a} : S \Rightarrow [0,1]$;

   b) – The measure of uncertainty does not exist;

   c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

Examples: OWA and FOWA operators belong to I2.

I3: The case:

a) – imprecision is presented by the some stochastic variable: $a: S \Rightarrow R^1$;

b) – the measure of uncertainty is presented by concerning probability distribution on S ($P: 2^S \Rightarrow [0,1]$) $p_i = P\{s_i\},\ i = 1,2,...,m.$

c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

Example: POWA operator belongs to I3.

I4: The case:

a) – imprecision is presented by the some fuzzy-stochastic variable: $\tilde{a} \in \psi;\ \tilde{a}: S \Rightarrow [0,1]$;

b) – uncertainty measure is presented by the concerning probability distribution on S ($P: 2^S \Rightarrow [0,1]$) $p_i = P\{s_i\},\ i = 1,2,...,m.$

c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

Example: FPOWA operator belongs to I4.

I5: The case:

a) – imprecision is presented by the some exact variable: $a: S \Rightarrow R^1$;

b) – the measure of uncertainty defined by some monotone measure (possibility measure, λ-additive measure and so on) $g: 2^S \Rightarrow [0,1]$.

c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

Examples: SEV (R.R. Yager [52]) operator belongs to I5; SEV-POWA, AsPOWA, SA-POWA, SA-AsPOWA (will be defined in the following sections) operators belong to I5.

I6: The case:

a) – imprecision is presented by the some fuzzy variable: $\tilde{a} \in \psi;\ \tilde{a}: S \Rightarrow [0,1]$;

b) – the measure of uncertainty is presented by some monotone measure $g: 2^S \Rightarrow [0,1]$;

c) – Objectivity is presented by the weights $W = \{w_1, w_2, ..., w_m\}$;

Examples: SEV-FOWA, AsFPOWA, and SA-AsFPOWA operators belong to I6.

Note that some other cases may be considered in the Information Structure – I (for an example, the cases when the weights in structure are not present and others).

7) F – is an aggregation (in our case OWA-type) operator for ranking of possible alternatives by its outcome values calculated by the F. Following the Information Structure I on the states of nature for all possible alternatives $d \in D,\ F(d)$ is a ranking value. In general $F(d)$ is defined as a converted (or condensed) information of imprecision values plus uncertainty measure and objective weights.

$$F(d) = aggregation(a(d),\ g,\ W). \tag{49}$$

We say – that alternative d_j is more prefered (dominated) than d_k, $d_j \succ d_k$, $if\ F(d_j) > F(d_k)$, and d_j is equivalent to d_k, $d_j \equiv d_k$, if $F(d_j) = F(d_k)$. So the aggregation operator F induces some preference binary relation $\succeq$ on the all possible alternatives - D .

8) Im is a set of information measures of an aggregation operator F :

$$\text{Im} = \{Orness,\ Dispersion,\ Divergence,\ Balance\}. \tag{50}$$

In order to classify OWA-type aggregation operators $\{F\}$ it is necessary to investigate information measures (50). This analysis also gives us some information on the inherent subjectivity of the choice of the decision aggregation operator by DMP [6].

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3. ASSOCIATED PROBABILITIES' AGGREGATIONS IN THE POWA OPERATOR

Different approaches were developed by the authors, which constructed aggregation operators with respect to a monotone measure, where I1-I6 and other levels of IS were considered. But for the POWA or FPOWA-type operators Information Structures on the levels I5 and I6 (or weighted OWA operators constructed on the basis of a monotone measure) were not investigated. So, we leave the Information Structures I1-I4 and go to the levels of I5 and I6. In this Section we consider the level I5.

It is important that in the aggregation operators POWA and FPOWA defined in Section 2 the both nature of incomplete information: 1. an uncertain measure (probability distribution $\{p_i\}$) and 2. An imprecision variable (random variable (a) or fuzzy variable ($\tilde{a}$)) are condensed in the outcome values, which get us more credibility for use of these aggregation operators in applications. In this Section we define new generalization of POWA operator where more general measure of uncertainty – monotone measure (fuzzy measure) is used instead of probability measure in the role of uncertainty measure.

3.1. AsPOWA operators induced by the ME

Let on the states of nature $S=\{s_1,s_2,...,s_m\}$ be given some monotone measure $g:2^S \Rightarrow [0,1]$ instead of probability measure $P=\{p_1,p_2,...,p_m\}$, $p_i=\mathrm{P}(s_i)$. There exist many aggregations in the decision making systems when we use monotone measure g as a measure of fuzzy uncertainty ([10, 15, 18, 19, 24-26, 36, 37, 39, 40-43] and others) the definition of which was given in Section 2. In Section 2 the FEV and ME were defined along with their probability representations by associated probability class (APC) $\{\mathrm{P}_\sigma\}_{\sigma\in S_m}$, where the number of probability distributions on S is equal to $k=m!$. We have k values of mathematical expectations for random or fuzzy-random variable a - $\{\mathrm{E}_{\mathrm{P}_\sigma}(a)\}_{\sigma\in S_m}$, where

$$\mathrm{E}_{\mathrm{P}_\sigma}(a)=\sum_{i=1}^{m} a_i P_\sigma(s_i),\ \ \sigma\in S_m. \tag{51}$$

So, we will focus on the use of $m!$ mathematical expectations in the POWA operator, instead of one expectation $\mathrm{E}_{\mathrm{P}}(a)=\sum a_i p_i$, as a more usual extension of this operator.

Let $M:R^k \Rightarrow R^1$, $k=m!$ be some deterministic mean aggregation function with symmetricity, boundedness, monotonicity and idempodency properties (see the definition in the Section 1). Let $a:S\to R_0^+$ be some random variable.

DEFINITION 13: *An associated POWA operator - AsPOWA of dimension m is a mapping* $AsPOWA:R^m \Rightarrow R^1$, *that has an associated objective weighted vector* W *of dimension* m *such that* $w_j\in[0,1]$ *and* $\sum_{i=1}^{m} w_j=1,$ *some uncertainty measure - monotone measure* $g:2^S \Rightarrow [0,1]$ *with associated probability class* $\{\mathrm{P}_\sigma\}_{\sigma\in S_m}$ *, and is defined according the following formula:*

$$AsPOWA(a_1,a_2,...,a_m)=\beta\sum_{j=1}^{m} w_j b_j+(1-\beta)\cdot M\left(\sum_{i=1}^{m} a_i \mathrm{P}_\sigma(s_i)\Big/\sigma\in S_m\right)=$$

$$=\beta\sum_{j=1}^{m} w_j b_j+(1-\beta)\cdot M\left(E_{P_{\sigma_1}}(a),\ E_{P_{\sigma_2}}(a),...,\ E_{P_{\sigma_k}}(a)\right), \tag{52}$$

where b_j *is the jth largest of the* $\{a_i\}, i=1,...,m.$

It is easy proved that in general cases of operator M the AsPOWA operator is induced by the ME:

PROPOSITION 9: *Let* M *be the Min operator, then AsPOWA operator may be written as:*

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$$AsPOWA\min(a_1, a_2, ..., a_m) = \beta\sum_{j=1}^{m} w_j b_j + (1-\beta)\cdot\min_{\sigma\in S_m}\left(\sum_{i=1}^{m} a_i P_\sigma(s_i) \Big/ \sigma \in S_m\right), \qquad (53)$$

and if monotone measure g is a lower capacity of order two, then in the AsPOWAmin operator the second addend coincides with ME_g :

$$AsPOWA\min(a_1, a_2, ..., a_m) = \beta\cdot OWA(a_1, a_2, ..., a_m) + (1-\beta)\cdot ME_g(a_1, a_2, ..., a_m). \qquad (54)$$

PROPOSITION 10: *Let M be the Max operator, then AsPOWA operator may be written as:*

$$AsPOWA\max(a_1, a_2, ..., a_m) = \beta\sum_{j=1}^{m} w_j b_j + (1-\beta)\cdot\max_{\sigma\in S_m}\left(\sum_{i=1}^{m} a_i P_\sigma(s_i)\right), \qquad (55)$$

and if monotone measure g is an upper Choquet capacity of order two, then in the AsPOWAmax operator the second addend coincides with ME_g :

$$AsPOWA\max(a_1, a_2, ..., a_m) = \beta\cdot OWA(a_1, a_2, ..., a_m) + (1-\beta)\cdot ME_g(a_1, a_2, ..., a_m). \qquad (56)$$

These proofs are easy if we use the results of proposition 5 (formula (24)).

PROPOSITION 11: *Let M be any mean aggregation operator and in AsPOWA operator monotone measure g is a probability measure. Then AsPOWA and POWA operators coincide.*

$$AsPOWA(a_1, a_2, ..., a_m) = POWA(a_1, a_2, ..., a_m). \qquad (57)$$

Proof: As known the associated probabilities of probability measure coincide (see proposition 1). Using the property of idempotency of operator M $\left(M(E_{P_1}, E_{P_2}, ..., E_{P_m}) \equiv E_P\right)$, because $p_i \equiv p,\ \ i = 1,...,k;\ \ E_{p_i} = E_p$ and $M(E_p, E_p, ..., E_p) = E_p$, then AsPOWA removes to the POWA (formula 11).

PROPOSITION 12: *If g_* and g^* are dual monotone measures on 2^S, then AsPOWA operators constructed on basis g_* and g^* coincide:*

Proof: Using simmetricity of operator M and results of proposition 2 it is easy to prove this proposition. Consider AsPOWA operator for the lower monotone measure g_*:

$$AsPOWA_*(a_1, a_2, ..., a_m) = \beta\sum_{j=1}^{m} w_j b_j + (1-\beta)M\left(E_{P_{*\sigma_1}}(a), E_{P_{*\sigma_2}}(a), ..., E_{P_{*\sigma_k}}(a)\right) =$$

$$= \beta\sum_{j=1}^{m} w_j b_j + (1-\beta)M\left(E_{P^*_{\sigma_1}}(a), E_{P^*_{\sigma_2}}(a), ..., E_{P^*_{\sigma_k}}(a)\right) =$$

$$= AsPOWA^*(a_1, a_2, ..., a_m),$$

where $\{P_{*\sigma_i}\}_{i=1}^{k}$ is the associated probability class for the measure g_* and $\{P^*_{\sigma_i}\}_{i=1}^{k}$ is the associated probability class for the measure g^*.

Now we consider different variants of the AsPOWA operator induced by the ME with respect to different classes of monotone measures. Following the Subsection 2.3 associated probabilities' formulas were presented for different classes of monotone measures. For example: a) possibility measure (Subsection 2.3.4); b) monotone measure associated with a belief structure (Subsection 2.3.5); c) Sugeno λ-additive monotone measure (Subsection 2.3.6). Therefore there exist many combinatorial possibilities for the analytical construction of concrete faces of the AsPOWA operator for concrete classes of a monotone measure and concrete operator M induced by the ME. But this procedure is omitted here. We will consider some of them:

1) Consider AsPOWAmax for the Sugeno λ-additive monotone measure g_λ. Using formulas (55) and (46), we receive:

$$AsPOWA\max(a_1, a_2, ..., a_m) = \beta\cdot\sum_{j=1}^{m} b_j w_j +$$
$$+ (1-\beta)\cdot\max_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda(\{s_{\sigma(i)}\})\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot a_{\sigma(i)}\right\}. \qquad (58)$$

2) Analogously we may construct the face of AsPOWAmin:

$$AsPOWA\min(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}b_j w_j + \\ +(1-\beta)\cdot\min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda(\{s_{\sigma(i)}\}))\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot a_{\sigma(i)}\right\}. \qquad (59)$$

3) Following Subsection 2.3.1 we consider the AsPOWAmin and AsPOWAmax operators for the monotone measure associated with the belief structure. Using formulas (53) and (42) we construct new variants of the AsPOWA operator:

$$AsPOWA\max(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}b_j w_j + \\ +(1-\beta)\cdot\max_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[\sum_{F_j\in F_S:F_j\cap\{s_{\sigma(i)}\}\neq\varnothing}m(F_j)w_j^0(|F_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|)\right]\cdot a_{\sigma(i)}\right\}, \qquad (60)$$

$$AsPOWA\min(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}b_j w_j + \\ +(1-\beta)\cdot\min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[\sum_{F_j\in\Im:F_j\cap\{s_{\sigma(i)}\}\neq\varnothing}m(F_j)w_j^0(|F_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|)\right]\cdot a_{\sigma(i)}\right\}. \qquad (61)$$

3.2. AsPOWA operators induced by the FEV

In this Subsection we define new generalizations of the POWA operator induced by the Sugeno Averaging Operator - Fuzzy Expected Value (FEV) with respect to probability measure - P. Analogously definition 13 (formula 52) but difference is that Mathematical Expectation operator $E_p(.)$ is changed by the $FEV_P(.)$.

DEFINITION 14: *A Sugeno Averaging POWA operator SA-POWA of dimension* m *is a mapping* $SA-POWA:R^m\Rightarrow R_0^+$ *that has an associated weighting vector* W *of dimension* m *such that* $w_j\in[0,1]$ *and* $\sum_{j=1}^{m}w_j=1$ *according to the following formula:*

$$SA-POWA(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}b_j w_j+(1-\beta)\cdot FEV_P(a_1,a_2,...,a_m)= \\ \overset{\Delta}{=}\beta\cdot\sum_{j=1}^{m}w_j a_{i(j)}+(1-\beta)\cdot\max_{l=\overline{1,m}}\{a_l\}\max_{j=\overline{1,m}}\{\min[a'_{i(j)},w_j^P]\}, \qquad (62)$$

where $b_j=a_{i(j)}$ *is the j-th largest of the* $\{a_i=a(s_i)\geq 0\}$, $i=1,2,...,m$*; on* S *there exists probability distribution* $\{p_i=P(s_i)\}$ *with* $\sum_{j=1}^{m}p_i=1,\ 0\leq p_i\leq 1$*;*

$$w_j^P=P(\{s_{i(1)},s_{i(2)},...,s_{i(j)}\})=\sum_{l=1}^{j}p_{i(l)} \quad \textit{and} \quad a'_{i(j)}=\frac{a_{i(j)}}{\max_{l=\overline{1,m}}\{a_l\}}.$$

On the basis of the definition 14 analogously to the definition 13 we may generalize the POWA operator induced by the FEV with respect to some monotone measure g.

DEFINITION 15: *A Sugeno Averaging AsPOWA operator SA-AsPOWA of dimension* m *is mapping* $SA-AsPOWA:R^m\Rightarrow R_0^+$*, that has an associated objective weighted vector* W *of dimension m such that* $w_j\in[0,1]$ *and* $\sum_{j=1}^{m}w_j=1$ *; some monotone measure* $g:2^S\to[0,1]$ *with associated probability class* $\{P_\sigma\}_{\sigma\in S_m}$*, according the following formula:*

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$$SA-AsPOWA(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}w_j a_{i(j)} + \\ +(1-\beta)\cdot M\left(FEV_{P_{\sigma_1}}(a_1,a_2,...,a_m),FEV_{P_{\sigma_2}}(a_1,a_2,...,a_m),...,FEV_{P_{\sigma_k}}(a_1,a_2,...,a_m)\right), \quad (63)$$

where

$$FEV_{P_\sigma}(a_1,a_2,...,a_m)=\max_{l=1,m}\{a_l\}\max_{j=1,m}\min\{a'_{i(j)};w_j^{P_\sigma}\}, \quad (64)$$

and $w_j^{P_\sigma}=P_\sigma(\{s_{i(1)},...,s_{i(j)}\})=\sum_{i=1}^{m}P_\sigma(\{s_{i(j)}\})$, $a'_{i(j)}=\dfrac{a_{i(j)}}{\max_l\{a_l\}}$, $\forall\sigma\in S_m$.

Now we consider SA-AsPOWA operators induced by the FEV with respect to $M=Max$ and $M=Min$ averaging operators:

$$SA-AsPOWA\max(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}w_j a_{i(j)} + \\ +(1-\beta)\cdot\max_{l=1,m}\{a_l\}\max_{\sigma\in S_m}\left[\max_{j=1,m}\left\{\min\left[a'_{i(j)},w_j^{P_\sigma}\right]\right\}\right], \quad (65)$$

$$SA-AsPOWA\min(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m}w_j a_{i(j)} + \\ +(1-\beta)\cdot\max_{l=1,m}\{a_l\}\min_{\sigma\in S_m}\left[\min_{j=1,m}\left\{\max\left[a'_{i(j)},w_j^{P_\sigma}\right]\right\}\right]. \quad (66)$$

It is easy to prove the propositions analogously to propositions 9-12. But these propositions are omitted here.

3.3. Information measures of the AsPOWA and SA-AsPOWA operators

Analogously to [28] (see Subsection 2.2) now we extend the definitions of the information measures for the AsPOWA and SA-AsPOWA operators:

DEFINITION 16: *The Orness measure of the AsPOWA operator is the extension of the formula (15):*

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot M\left[\sum_{j=1}^{m}P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\Big/\sigma\in S_m\right]. \quad (67)$$

For AsPOWAmax we receive:

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot\max_{\sigma\in S_m}\left[\sum_{j=1}^{m}P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\right], \quad (68)$$

but for AsPOWAmin we have:

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot\min_{\sigma\in S_m}\left[\sum_{j=1}^{m}P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\right]. \quad (69)$$

Constructing the *Orness* measure of the SA-AsPOWA operator induced by the FEV we receive the analogous extension.

DEFINITION 17: *The Orness measure of the SA-AsPOWA operator is the extension of the formula (15):*

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot M\left[\max_{j=1,m}\min\left\{\frac{m-\sigma(j)}{m-1};w_j^{P_\sigma}\right\}\Big/\sigma\in S_m\right]. \quad (70)$$

For example, for the AsPOWAmax operator we have:

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot\max_{\sigma\in S_m}\left[\max_{j=1,m}\min\left\{\frac{m-\sigma(j)}{m-1},w_j^{P_\sigma}\right\}\right], \quad (71)$$

and for AsPOWAmin :

$$\alpha\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=\beta\cdot\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot\min_{\sigma\in S_m}\left[\min_{j=1,m}\max\left\{\frac{m-\sigma(j)}{m-1},w_j^{P_\sigma}\right\}\right]. \quad (72)$$

DEFINITION 18: *The entropy (the dispersion) H of the AsPOWA operator of the amount of information is defined as:*

$$H\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=-\left\{\beta\cdot\sum_{j=1}^{m}w_j\ln\left(w_j\right)+(1-\beta)\cdot M\left[\sum_{j=1}^{m}P_{\sigma(j)}\ln\left(P_{\sigma(j)}\right)\Big/\sigma\in S_m\right]\right\}. \tag{73}$$

For example, if we have AsPOWAmax operator, then

$$H\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=-\left\{\beta\cdot\sum_{j=1}^{m}w_j\ln\left(w_j\right)+(1-\beta)\cdot\max_{\sigma\in S_m}\left[\sum_{j=1}^{m}P_{\sigma(j)}\ln\left(P_{\sigma(j)}\right)\Big/\sigma\in S_m\right]\right\} \tag{74}$$

and for AsPOWAmin:

$$H\left(\hat{p}_1,\hat{p}_2,...,\hat{p}_m\right)=-\left\{\beta\cdot\sum_{j=1}^{m}w_j\ln\left(w_j\right)+(1-\beta)\cdot\min_{\sigma\in S_m}\left[\sum_{j=1}^{m}P_{\sigma(j)}\ln\left(P_{\sigma(j)}\right)\Big/\sigma\in S_m\right]\right\}. \tag{75}$$

DEFINITION 19: *The divergence measure Div has the following face:*

$$\begin{aligned}Div\left(\hat{P}_1,\hat{P}_2,...,\hat{P}_m\right)&=\beta\left\{\sum_{j=1}^{m}\left(\frac{m-j}{m-1}-\alpha(W)\right)^2\right\}+\\&+(1-\beta)\left\{M\left[\sum_{j=1}^{m}P_{\sigma(j)}\cdot\left(\frac{m-\sigma(j)}{m-1}-\alpha(P_\sigma)\right)^2/\sigma\in S_m\right]\right\},\end{aligned} \tag{76}$$

where $\alpha(W)$ *is an Orness measure of the OWA operator*

$$\alpha(W)=\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)$$

and $\alpha(P)$ *is an Orness measure of associated probabilities' aggregations:*

$$\alpha(P_\sigma)=\sum_{j=1}^{m}P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right). \tag{77}$$

Analogously to definition 19 we may construct the concrete analytical forms of the measure Div for AsPOWAmax and AsOWAmin and other operators with respect to different monotone measures (Here these formulas are omitted).

DEFINITION 20: *The Balance parameter of the AsPOWA operator has the following extension*

$$\begin{aligned}Bal\left(\hat{P}_1,\hat{P}_2,...,\hat{P}_m\right)&=\beta\sum_{j=1}^{m}w_j\left(\frac{m+1-2j}{m-1}\right)+\\&+(1-\beta)M\left[\sum_{j=1}^{m}P_{\sigma(j)}\left(\frac{m+1-2\sigma(j)}{m-1}\right)\Big/\sigma\in S_m\right].\end{aligned} \tag{78}$$

The Bal of the AsPOWAmax and AsPOWAmin operators and the H, Div, Bal parameters of the SA-AsPOWA operator may be written analogously definitions 9-20, but are omitted here.

4. ASSOCIATED PROBABILITIES AGGREGATIONS IN THE FPOWA OPERATOR

In this Section we construct new aggregations in the FPOWA operator (definition 5) by monotone measure's associated probabilities when the imprecision variable is presented by the fuzzy triangular number. So we consider the Information Structure I6.

Let on the states of nature $S=\{s_1,s_2,...,s_m\}$ be given some monotone measure $g:2^S\Rightarrow[0,1]$ as a uncertainty measure of incomplete information and on S defined some payoffs (utilities and so on) which are presented by triangular fuzzy numbers as expert reflections on possible alternatives. I.e. for every alternative and for every state of nature s_i there exists $\tilde{a}_i=\tilde{a}(s_i)$ - positive triangular fuzzy number as some payoff. So vector $\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\}$ is imprecision values of expert reflections on states of nature with respect to alternatives.

Using the arithmetic operations on the triangular fuzzy numbers, presented in section 1, we may define new aggregations in the FPOWA operator (definition of the FPOWA see in Section 2) as induced functions by monotone measures' associated probabilities.

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4.1 AsFPOWA operators induced by the ME

Let $M:\Psi^{+k} \Rightarrow \Psi^{+}$ $(k = m!)$ be some deterministic mean aggregation function with symmetricity, boundedness, monotonicity and idempotency properties.

DEFINITION 21: *An associated FPOWA operator AsFPOWA of dimension m is mapping* $AsFPOWA:\Psi^{+m} \Rightarrow \Psi^{+}$ *, that has an associated objective weighted vector W of dimension m such that* $w_j \in (0,1)$ *and* $\sum_{j=1}^{m} w_j = 1$*, and some uncertainty measure – monotone measure* $g:2^S \Rightarrow [0,1]$ *with associated probability class* $\{P_\sigma\}_{\sigma\in S_m}$ *, and is defined according to the following formula:*

$$AsFPOWA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)M\left\{\sum_{i=1}^{m}\tilde{a}_i P_\sigma(s_i) \Big/ \sigma\in S_m\right\} = \tag{79}$$

$$= \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)M\left(E_{P_{\sigma_1}}(\tilde{a}), E_{P_{\sigma_2}}(\tilde{a}),...,E_{P_{\sigma_k}}(\tilde{a})\right),$$

where $\tilde{b}_j$ *is the jth largest of the* $\{\tilde{a}_i\}, i=1,...,m$.

Now we consider concrete AsFPOWA operators for concrete mean functions M and induced by the ME.

DEFINITION 22:

1) *Let* M *be the* Min*-operator dimension of k=m! then*

$$AsFPOWA\min(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\underset{\sigma\in S_m}{Min}\left\{\sum_{i=1}^{m}\tilde{a}_i P_\sigma(s_i)\right\}; \tag{80}$$

2) *Let* M *be the* Max*-operator dimension of k=m! then*

$$AsFPOWA\max(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\underset{\sigma\in S_m}{Max}\left\{\sum_{i=1}^{m}\tilde{a}_i P_\sigma(s_i)\right\}; \tag{81}$$

3) *Let* M *be the averaging operator dimension of k=m!,* $M(c_1,c_2,...,c_m) = \frac{1}{k}\sum_{i=1}^{k} c_i$*, then*

$$AsFPOWAmean(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\left\{\frac{1}{m!}\sum_{\sigma\in S_m}\sum_{i=1}^{m}\tilde{a}_i P_\sigma(s_i)\right\}; \tag{82}$$

4) *Let* M *be the* α*-averaging operator dimension of k=m!,* $M(c_1,c_2,...,c_m) = \left\{\frac{1}{k}\sum_{i=1}^{k} c_i^{\alpha}\right\}^{\frac{1}{\alpha}}$*,*

then

$$AsFPOWAmean\alpha(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\left\{\frac{1}{m!}\sum_{\sigma\in S_m}\{\sum_{i=1}^{m}\tilde{a}_i P_\sigma(s_i)\}^{\alpha}\right\}^{\frac{1}{\alpha}}. \tag{83}$$

The propositions analogous to propositions 9-12 are true (we omitted this propositions here).

Now we define concrete AsFPOWA operators for concrete monotone measures analogously to Section 4. Consider AsFPOWAmax for Sugeno λ-additive monotone measure - g_λ. Analogously to (58) we have:

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$$AsFPOWA\max(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}\tilde{b}_j w_j + \\ +(1-\beta)\underset{\sigma\in S_m}{Max}\left\{\sum_{i=1}^{m}\left[g_\lambda\left(\{s_{\sigma(i)}\}\right)\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot\tilde{a}_{\sigma(i)}\right\}. \tag{84}$$

Analogously we may construct the face of the AsFPOWAmin:

$$AsFPOWA\min(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}\tilde{b}_j w_j + \\ +(1-\beta)\min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda\left(\{s_{\sigma(i)}\}\right)\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot\tilde{a}_{\sigma(i)}\right\}. \tag{85}$$

Analogously to Section 4 (formulas (60)-(61)) we may construct AsFPOWAmin and AsFPOWAmax operators induced by the belief structure's associated monotone measure (omitted here). We also may define some other combinations of different monotone measures and averaging operator M. So, there exist many cases of Information Structures on the level I6 for the constructions of the AsFPOWA operator. For example - $AsFPOWAmean\alpha$ induced by the belief structure:

$$AsFPOWAmean\alpha(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}w_j\tilde{b}_j + \\ +(1-\beta)\cdot\left[\sum_{\sigma\in S_m}\left\{\frac{1}{m!}\sum_{i=1}^{m}\{\left[\sum_{A_j\in F_S:A_j\cap\{s_{\sigma(i)}\}\neq\varnothing}m(A_j)w_j^0(|A_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|\right]\tilde{a}(s_{\sigma(i)})\}^\alpha\right\}^{\frac{1}{\alpha}}\right] \tag{86}$$

and others.

Note the information measures of the AsFPOWA operator - *Orness, Entropy, Div* and *Bal* are defined analogously to Subsection 4.3 (omitted here). We may add the proposition concerning the dual monotone measures g_* and g^*, which is general for AsPOWA and AsFPOWA operators.

PROPOSITION 13: *Let g_* and g^* be dual monotone measures on $2^S \Rightarrow [0,1]$; let AsPOWA$_*$ and AsPOWA* (or AsFPOWA$_*$ and AsFPOWA*) be AsFPOWA (or AsFPOWA) operators constructed on the basis of the measures g_* and g^* respectively. Then corresponding information measures coincide:*

$$\alpha_*=\alpha^*;\ H_*=H^*;\ Div_*=Div^*;\ \ and\ \ Bal_*=Bal^*.$$

Proof: We prove the equality $\alpha_*=\alpha^*$. Other proofs are analogous.

Consider

$$\alpha_*=\beta\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)M\left[\sum_{j=1}^{m}P_{*\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\Big/\sigma\in S_m\right]=$$

$$=\beta\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)M\left[\sum_{j=1}^{m}P^*_{\sigma_*(m-j+1)}\left(\frac{m-\sigma_*(m-j+1)}{m-1}\right)\Big/\sigma_*\in S_m\right]=$$

$$=\beta\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)M\left[\sum_{j=1}^{m}P^*_{\sigma_*(j)}\left(\frac{m-\sigma_*(j)}{m-1}\right)\Big/\sigma_*\in S_m\right]=\alpha^*.$$

In this proof we use the property of symmetry of the function M; the fact, that Associated Probability Classes of g_* and g^* coincide $\{P_{*\sigma}\}_{\sigma\in S_m}\equiv\{P^*_{\sigma_*}\}_{\sigma_*\in S_m}$ and $P_{*\sigma(j)}\equiv P^*_{\sigma_*(m-j+1)}$, where σ and σ_* are dual permutations.

4.2. AsFPOWA operators induced by the FEV

Now we define new generalizations of the FPOWA operator induced by the $FEV_P(\ )$. The values of imprecision of the incomplete information on S are presented by the fuzzy variable $\tilde{a}\in TFN,\ \tilde{a}:S\Rightarrow\Psi^+$, (*or* $\tilde{a}_i=\tilde{a}(s_i)\in\Psi^+$ *for every* $i=1,2,..,m$).

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DEFINITION 23: *A Sugeno Averaging FPOWA operator SA-FPOWA of dimension* m *is mapping* $SA-FPOWA:\Psi^{+m}\Rightarrow\Psi^{+}$*, that has an associated weighting vector* W *of dimension* m *such that* $w_j\in[0,1]$ *,* $\sum_{j=1}^{m}w_j=1$ *and is defined according to the following formula:*

$$SA-FPOWA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}w_j\tilde{b}_j+(1-\beta)FEV_P(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=$$
$$=\beta\sum_{j=1}^{m}w_j\tilde{b}_j+(1-\beta)\max_{l=1,m}\{a_l\}\max_{j=1,m}[\min\{\tilde{b}'_j,w_j^P\}], \qquad (87)$$

where $\tilde{b}_j$ *is the jth largest of the* $\{\tilde{a}_i\}$, $i=1,...,m$ *;* $\tilde{b}'_j=\dfrac{\tilde{b}_j}{\max_l\{\tilde{a}_l\}}$*; on the* S *there exist a probability distribution* $p_i=P\{s_i\}$, $i=1,...,m$ *with* $\sum_{i=1}^{m}p_i=1,\ 0\le p_i\le 1$ *and* $w_j^P\overset{\Delta}{\equiv}P\{s_{i(1)},...,s_{i(j)}\}=\sum_{i=1}^{j}p_{i(j)}$.

On the basis of definition 23 and analogously to definition 21 we present a definition of the operator AsFPOWA induced by the FEV with respect to some monotone measure $g:2^S\Rightarrow[0,1]$.

DEFINITION 24: *A Sugeno Averaging AsFPOWA operator SA-AsFPOWA of dimension* m *is mapping* $SA-AsFPOWA:\Psi^{+m}\Rightarrow\Psi^{+}$*, that has an associated objective weighted vector* W *of dimension* m *such that* $w_j\in[0,1]$ *and* $\sum_{j=1}^{m}w_j=1$*; some uncertain measure – monotone measure* $g:2^S\Rightarrow[0,1]$ *with associated probability class* $\{P_\sigma\}_{\sigma\in S_m}$ *defined according the following formula:*

$$SA-AsFPOWA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}w_j\tilde{a}_{i(j)}+(1-\beta)M\left\{FEV_{P_{\sigma_1}}(\tilde{a}'),...,FEV_{P_{\sigma_k}}(\tilde{a}')\right\}, \qquad (88)$$

where $FEV_{P_\sigma}(\tilde{a})\equiv FEV_{P_\sigma}(\tilde{a}_1,...,\tilde{a}_m)=\max_{l=1,m}\{\tilde{a}_l\}\max_{j=1,m}\left[\min\{\tilde{a}'_{i(j)};w_j^{P_\sigma}\}\right]$*;* $\tilde{a}'_{i(j)}=\dfrac{\tilde{a}_{i(j)}}{\max_{l=1,m}\{a_l\}}$ *and* $w_j^{P_\sigma}=P_\sigma(\{s_{i(1)},...,s_{i(j)}\})=\sum_{l=1}^{j}P_\sigma(s_{i(l)})$, $\forall\sigma\in S_m$, $j=1,2,..,m$ *;* M *is some averaging operator.*

Analogously to Subjection 4.2 (formulas 65-66) we may define new SA-AsFPOWA operators induced by the FEV with respect to concrete monotone measures: Sugeno λ-additive measure, possibility measure, believe structure's associated monotone measure and others (but these procedures are omitted here).

5. EXAMPLE

Analogously to [28] we analyze an illustrative example on the use of new AsFPOWA and SA-AsFPOWA operators in a fuzzy decision-making problem regarding political management. We study a country that is planning its fiscal policy for the next year.

Assume that government of a country has to decide on the type of optimal fiscal policy for the next year. They consider five alternatives:

d_1: "Development a strong expansive fiscal policy";

d_2: "Development an expansive fiscal policy";

d_3: "Do not make any changes in the fiscal policy";

d_4: "Development of a contractive fiscal policy";

d_5: "Development a strong contractive fiscal policy".

In order to analyze these fiscal policies, the government has brought together a group of experts. This group considers that the key factors are the economic situations of the world (external) and country (internal) economy for the next period. They consider 3 possible states of nature that in whole could occur in the future.

s_1: "Bad economic situation";

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s_2: "Regular economic situation";

s_3: "Good economic situation".

As a result the group of experts gives us their opinions and results. The results depending on the state of nature s_i and alternative d_k that the government selects are presented in the Table 2:

Table 2: Expert's valuations in TFNs

D \ S	s_1	s_2	s_3
d_1	(60,70,80)	(40,50,60)	(50,60,70)
d_2	(30,40,50)	(60,70,80)	(70,80,90)
d_3	(50,60,70)	(50,60,70)	(60,70,80)
d_4	(70,80,90)	(40,50,60)	(40,50,60)
d_5	(60,70,80)	(70,80,90)	(50,60,70)

Following the expert's knowledge on the world economy for the next period, experts decided that the objective weights (as an external factor) of states of nature must be $W=(0{,}5;0{,}3;0{,}2)$, while for the economy of the country for the next period the occurrence of presented states of nature is defined by some possibilities (as an internal factor). So, there exists some possibilities (internal levels), as an uncertainty measure, of the occurrence of states of nature in the country. This decision making model (Information Structure I6) is more detailed than the model (Information Structure I4) presented in [28]. In another words in decision model we cannot define the objective probabilities $p_i=P(s_i)$ for the future events, but we can define subjective possibilities $\pi_i=Pos(s_i)$ based on the experts' knowledge. Let on the basis of some fuzzy term of internal factor – country economy experts define the possibility levels of states of nature:

$$poss(s_1)\equiv\pi_1=0{,}7;\quad poss(s_2)\equiv\pi_2=1;\quad poss(s_3)\equiv\pi_3=0{,}5.$$

So, we have the Information Structure I6 of decision making system (48) (definition 12), where $g:=Pos(.):2^S\Rightarrow[0,1]$ $Pos(A)=\max_{s_i\in A}\pi_i,\ \forall A\subseteq S$; (a monotone measure is a possibility measure). In this model as in [28] $\beta\equiv 0{,}3$. Decision procedure is equivalent to the detalization of GDMS as the Information Structure I6 (but in [28] the author had the IS as I4). So, for every decision d payoffs' values are the column from Table 2; $g:=Pos$; $W=(0{,}5;0{,}3;0{,}2)$; $I=I6$; $F=AsFPOWA$ or $F=SA-AsFPOWA$ and others. Im is the quadruple structure (definition 12). For ranking of alternatives $\{d_1,..,d_5\}$ we must calculate its AsFOWA or other operators. For $\tilde{a}=(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)$ we have: $AsFPOWA(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)=\beta\sum_{j=1}^{3}\tilde{b}_j w_j+(1-\beta)M\left(E_{P_{\sigma_1}}(\tilde{a}),E_{P_{\sigma_2}}(\tilde{a}),...,E_{P_{\sigma_6}}(\tilde{a})\right)$. It is clear that k=m!=3!=6 and for calculation of the AsFPOWA operator we firstly define the associated probability class $\{P_\sigma\}_{\sigma\in S_3}$ for the $Pos:2^S\Rightarrow[0,1]$.

For every

$$\sigma=\{\sigma(1),\sigma(2),\sigma(3)\}\in S_3\quad E_{P_\sigma}(d)=E_{P_\sigma}(\tilde{a})=\sum_{i=1}^{3}P_{\sigma(i)}\cdot\tilde{a}_{\sigma(i)},$$

where

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$$P_\sigma\left(s_{\sigma(i)}\right) = Poss(\{s_{\sigma(1)},...,s_{\sigma(i)}\}) - Poss(\{s_{\sigma(1)},...,s_{\sigma(i-1)}\}) =$$
$$= \max_{j=1,i} \pi_{\sigma(j)} - \max_{j=1,i-1} \pi_{\sigma(j)}, \qquad \pi_{\sigma(0)} \equiv 0.$$

The results are presented in the Table 3.

Table 3: Associated Probability Class - $\{P_\sigma\}_{\sigma\in S_3}$

$\sigma = (\sigma(1),\sigma(2),\sigma(3))$	$P_{\sigma(1)}$	$P_{\sigma(2)}$	$P_{\sigma(3)}$
$(1,2,3)=\sigma_1$	$P_1 = 0{,}7$	$P_2 = 0{,}3$	$P_3 = 0$
$(1,3,2)=\sigma_2$	$P_1 = 0{,}7$	$P_3 = 0$	$P_2 = 0{,}3$
$(2,1,3)=\sigma_3$	$P_2 = 1$	$P_1 = 0$	$P_3 = 0$
$(2,3,1)=\sigma_4$	$P_2 = 1$	$P_3 = 0$	$P_1 = 0$
$(3,1,2)=\sigma_5$	$P_3 = 0{,}5$	$P_1 = 0{,}2$	$P_2 = 0{,}3$
$(3,2,1)=\sigma_6$	$P_3 = 0{,}5$	$P_2 = 0{,}5$	$P_1 = 0$

Following the Table 3 we calculate Mathematical Expectations -$\{E_{P_\sigma}(\cdot)\}_{\sigma\in S_3}$ (Table 4) and Fuzzy Expected Values -$\{FEV_{P_\sigma}(.)\}_{\sigma\in S_m}$ (Table 5).

Table 4: Mathematical Expectations - $\{E_{P_\sigma}(\cdot)\}_{\sigma\in S_3}$

$E_{P_\sigma}(\cdot)$ σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$E_{P_\sigma}(d_1)$	(54,64,74)	(54,64,74)	(40,50,60)	(40,50,60)	(49,59,69)	(45,55,65)
$E_{P_\sigma}(d_2)$	(39,49,59)	(39,49,59)	(60,70,80)	(60,70,80)	(59,69,79)	(65,75,85)
$E_{P_\sigma}(d_3)$	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(55,65,75)	(55,65,75)
$E_{P_\sigma}(d_4)$	(61,71,81)	(61,71,81)	(40,50,60)	(40,50,60)	(46,56,66)	(40,50,60)
$E_{P_\sigma}(d_5)$	(63,73,83)	(63,73,83)	(70,80,90)	(70,80,90)	(58,68,78)	(60,70,80)

Table 5: Fuzzy Expected Values - $\{FEV_{P_\sigma}(.)\}_{\sigma\in S_m}$

$E_{P_\sigma}(\cdot)$ σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$E_{P_\sigma}(d_1)$	(70,70,70)	(70,70,70)	(50,60,70)	(50,60,70)	(40,50,60)	(40,50,60)
$E_{P_\sigma}(d_2)$	(30,40,50)	(30,40,50)	(60,70,80)	(60,70,80)	(60,70,80)	(60,70,80)
$E_{P_\sigma}(d_3)$	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)
$E_{P_\sigma}(d_4)$	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)
$E_{P_\sigma}(d_5)$	(60,70,80)	(60,70,80)	(70,80,90)	(70,80,90)	(40,50,60)	(40,50,60)

Now we may calculate the values of different variants of the AsFPOWA and SA-AsFPOWA operators with respect to different averaging operators M (Tables 6 and 7):

Table 6: Aggregation results

D/Ag. Op.	FOWA	SEV	SEV-FOWA	$\text{AsFPOWA}_{\text{min}}$	$\text{AsFPOWA}_{\text{max}}$	$\text{AsFPOWA}_{\text{mean}}$
d_1	(53,63,73)	(46,57,68)	(48,59,70)	(44,54,64)	(54,64,74)	(49,59,69)
d_2	(59,69,73)	(53,64,75)	(55,66,77)	(45,55,65)	(64,74,84)	(57,66,75)
d_3	(55,65,75)	(51,62,73)	(52,63,74)	(52,62,72)	(56,66,76)	(53,63,73)
d_4	(63,73,83)	(47,58,69)	(52,63,74)	(45,55,65)	(60,70,80)	(51,61,71)
d_5	(63,73,83)	(63,74,85)	(63,74,85)	(60,70,80)	(68,78,88)	(64,74,84)

Table 7: Aggregation results

D/Ag. Op.	$\text{SA-AsFPOWA}_{\text{min}}$	$\text{SA-AsFPOWA}_{\text{max}}$	$\text{SA-AsFPOWA}_{\text{mean}}$
d_1	(44,54,64)	(65,68,71)	(55,61,69)
d_2	(37,47,57)	(58,68,77)	(51,61,70)
d_3	(51,61,71)	(51,61,71)	(51,61,71)
d_4	(44,54,64)	(44,54,64)	(44,54,64)
d_5	(44,54,64)	(65,75,85)	(56,66,76)

For possibility distribution $\{\pi_i\}_{i=1}^m$ and payoff vector $\tilde{a} = (\tilde{a}_1,...,\tilde{a}_m)$ R.R. Yager in [52] defined the aggregation mean operator - Shapely Expected Value (SEV) for possibility uncertainty:

$$SEV(\tilde{a}_1,...,\tilde{a}_m) = \sum_{i=1}^{m} a_{\sigma(i)} P^{\pi}_{\sigma(i)} , \qquad (89)$$

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where $\{P^{\pi}_{\sigma(i)}\}_{i=1}^{m}$ is the probability distribution on $S=(s_1,...,s_m)$ induced by possibility distribution $\{\pi_i\}_{i=1}^{m}$

$$P^{\pi}_{\sigma(i)}=\sum_{j=1}^{\sigma(i)}\frac{\pi_{\sigma(j)}-\pi_{\sigma(j-1)}}{m+1-j}$$

where $\sigma=\{\sigma(1),...,\sigma(m)\}$ is some permutation from S_m form which $0=\pi_{\sigma(0)}\le\pi_{\sigma(1)}\le...\le\pi_{\sigma(m)}=1$. On the other hand this values are Shapley Indexes of a possibility measure with a possibility distribution $\{\pi_i\}_{i=1}^{m}$. It was proved [52] that the $SEV(\cdot)$ coincides with the ME for possibility measure:

$$SEV(\tilde{a}_1,...,\tilde{a}_m)=ME_{Poss}(\tilde{a}_1,...,\tilde{a}_m)=\int_0^1 Poss(\tilde{a}_i\ge\alpha/i=1,...,m)d\alpha .$$

On the basis of definition SEV we connect the SEV operator to the OWA operator as weighted sum. So we consider new generalization of the FOWA operator in Information Structure I6:

$$SEV-FOWA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}w_j\tilde{b}_j+(1-\beta)\sum_{i=1}^{m}\tilde{a}_{\sigma(i)}\left[\sum_{j=1}^{\sigma(i)}\frac{\pi_{\sigma(j)}-\pi_{\sigma(j-1)}}{m+1-j}\right].$$

Calculating numerical values of FOWA, SEV, SEV-FOWA, AsFPOWAmin, AsFPOWAmax, AsFPOWAmean, SA-AsFPOWAmin, SA-AsFPOWAmax, SA-AsFPOWAmean operators we constructed the Decision Comparing Matrix (Table 9). Firstly we calculated Shapely Indexes - $\{P_i^{\pi}\}, j=\overline{1,3}$ for the possibility measure (Table 8).

***Table 8*: Shapley Indexes of the possibility distribution**

P_i^{π}	$4/15$	$17/30$	$1/6$
s_i	s_1	s_2	s_3

According to the information received in this Section, we can rank the alternatives from the most prefered to the less prefered. The results are shown in table 9.

Table 9: Ordering of the policies

N	Aggreg. Operator	Ordering	Information Structure
1	FOWA	$d_5=d_4\succ d_2\succ d_3\succ d_1$	I2
2	SEV	$d_5\succ d_2\succ d_3\succ d_4=d_1$	I6 (without weights)
3	SEV-FOWA	$d_5\succ d_2\succ d_4=d_3\succ d_1$	I6
4	$AsFPOWA_{min}$	$d_5\succ d_3\succ d_2=d_4\succ d_1$	I6
5	$AsFPOWA_{max}$	$d_5\succ d_2\succ d_4\succ d_3\succ d_1$	I6
6	$AsFPOWA_{mean}$	$d_5\succ d_2\succ d_3\succ d_4\succ d_1$	I6
7	$SA\text{-}AsFPOWA_{min}$	$d_3\succ d_5=d_4=d_1\succ d_2$	I6
8	$SA\text{-}AsFPOWA_{max}$	$d_5\succ d_2\succ d_1\succ d_3\succ d_4$	I6
9	$SA\text{-}AsFPOWA_{mean}$	$d_5\succ d_3\succ d_2\succ d_1\succ d_4$	I6

We also calculated values of the *Orness* parameter of the aggregation operators presented in Table 10.

Table 10: Orness values

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α\Ag.Op.	FOWA	SEV	SEV-FOWA	$AsFPOWA_{min}$	$AsFPOWA_{max}$	$AsFPOWA_{mean}$	$SA\text{-}AsFPOWA_{min}$	$SA\text{-}AsFPOWA_{max}$	$SA\text{-}AsFPOWA_{mean}$
α	0,65	0,55	0,58	0,37	0,79	0,58	0,68	0,89	0,79

Following Table 10 we see that for the nearer of SA-AsFPOWA operators is to on *or*, the closer its measure is to one, while AsFPOWAmin operator is to on *and*, the closer is to zero. Calculations of other information measures are omitted here. More on these measures of new aggregation operators we will present in our future investigations.

CONCLUSIONS

New generalizations of the POWA and FPOWA operators were presented with respect to monotone measure's associated probability class (APC) and induced by the Choquet and Sugeno integrals (finite cases). There exist many combinatorial variants to construct faces or expressions of generalized operators: AsPOWA, AsFPOWA, SA-AsPOWA and SA-AsFPOWA for concrete mean operators (Mean, Max, Min and so on) and concrete monotone measures (Choquet capacity of order two, monotone measure associated with belief structure, possibility measure and Sugeno $\lambda-$additive measure). Some properties of new operators and their information measures (*Orness, Enropy, Divergence and Balance*) are proved. But only some variants (AsPOWAmax, AsPOWAmin and others) are presented, the list of which may be longer that it is presented in the paper. So, other presentations of new operators and properties of information measures will be considered in our future research. The example was constructed for the illustration of the properties of generalized operators in the problems of political management.

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REFERENCES

1. Beliakov G. Learning Weights in the Generalized OWA Operators. Fuzzy Optimization and Decision Making, 2005; 4 : 119-130.
2. Beliakov G, Pradera A, Calvo I. Aggregation Functions: A Guide for Practitioners. Berlin- Heidelberg: Springer-Verlag; 2007. 361 p.
3. Bellman RE, Zadeh LA. Decision making in a fuzzy environment. Management Science 1970; 17 (4): B-141- B-164.
4. Calvo T, Beliakov G. Identification of weights in Aggregation Operators. In: Bustince H, Herrera I, Montero J. (Eds.). Fuzzy sets and their extensions: representation, aggregation and models: intelligent systems from decision making to data mining, web intelligence and computer vision. Berlin: Springer; 2008. 145-162.
5. De Campos Ibanez LM, Bolanos Carmona MN. Representation of fuzzy measures through probabilities. Fuzzy sets and systems 1989; 31 (1) : 23–36.
6. Carlson C, Fuller R. Fuzzy Reasoning in Decision Making and Optimization, Studies in fuzziness and Soft Computing 82, Heidelberg-New York: Physica-Verlag; 2002. 341 p.
7. Choquet G. Theory of capacities. Annals d'Institute Fourier 1954; 5: 131–295.

ISSN 1512-1232

8. Denneberg D. Non-Additive Measure and Integral. Norwell, MA: Kluwer Academic; 1994. 185 p.
9. Dong YC, Xu YF, Li HY, Feng B. The OWA-Bases consensus operator under linguistic representation models using position indexes. Eurp. Journal of Operational Research 2010; 203(2): 453-463.
10. Dubois D, Marichal JL, Prade H, Roubens M, Sabbadin R. The use of the discrete Sugeno integral in decision-making: a survey. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 2001; 9 (5): 539–561.
11. Dubois D, Prade H. Possibility Theory. New York: Plenum Press; 1988.
12. Ehrgott M. Multicriteria optimization. Berlin-Heidelberg: Springer-Verlag; 2005. 323 p.
13. Galvo T, Magor G, Mesier R. Aggregation Operators: New Trends and Application. New York: Physica-Verlag; 2002. 368 p.
14. Gil-Lafuente AM, Merigo JM. Computational Intelligence in Bussiness and Economics, Singapore: World Scientific; 2010. 835 p.
15. Grabisch M, Murofushi T, Sugeno M. Fuzzy measures and integrals: Theory and applications, Studies in Fuzziness and Soft Computing 40. Heidelberg: Physica-Verlag; 2000. 477 p.
16. Greco S, Marques Pereiza RA, Sguillante M, Yager RR, Kacprzyk J. (Eds). Preferences and Decisions: Models and Applications, Studies in Fuzziness and Soft Computing 257, 1st Edition. Berlin, Heidelberg: Springer-Verlag; 2010. 430 p.
17. Kacprzyk J, Zadrozny S. Towards a generalized and unified characterization of individual and collective choice functions under fuzzy and nonfuzzy preferences and majority via ordered weighted average operators. Int. Journal of Intelligent Systems 2009; 24 (1): 4-26.
18. Kandel A. On the control and evaluation of uncertain processes. IEEE Transactions on Automatic Control 1980; 25 (6) : 1182–1187.
19. Kandel A. Fuzzy statistics and forecast evaluation. IEEE Transactions on Systems, Man and Cybernetics 1978; SMC-8 (5): 396–401.
20. Kaufman A, Gupta MM. Introduction to fuzzy arithmetic. New York: Van Nostrad Reinhold Co; 1985. 384 p.
21. Klir GJ, Elias D. Architecture of Systems Problem Solving, IFSR International Series on Systems Science and Engineering 21. New York: Kluwer Academic/Plenum; 2003. 354 p.
22. Klir GJ, Folger TA. Fuzzy Sets, Uncertainty and Information. New York: Prentice-Hall, Englewood Cliffs; 1988. 356 p.
23. Klir GJ, Wierman MJ. Uncertainty-Based Information: Elements of Generalized Information Theory, Studies in Fuzziness and Soft Computing 15, 2nd ed. Heidelberg: Physica-Verlag; 1999. 170 p.
24. Marichal JL. An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. IEEE Transactions on Fuzzy Systems 2000; 8(6): 800–807.
25. Marichal JL. On Choquet and Sugeno Integrals as Aggregation Functions. In: Grabisch M, Murofushi T, Sugeno M (Eds). Fuzzy Measures and Integrals. Heidelberg: Physica-Verlag; 2000. pp. 247–272.
26. Marichal JL. On Sugeno integral as an aggregation function. Fuzzy sets and systems 2000; 114 (3): 347–365.
27. Merigo JM. The uncertain probabilistic weighted average and its application in the theory of expertons. African Journal of Business Management 2011; 5(15): 6092-6102.
28. Merigo JM. Fuzzy Multi-Person Decision Making with Fuzzy Probabilistic Aggregation Operators. Int. Journal of Fuzzy Systems 2011; 13 (3): 163-174.
29. Merigo JM, Casanovas M. The uncertain induced quasi-arithmetic OWA operators, Int. Journal of Intelligent Systems 2011; 26 (1): 1-24.

ISSN 1512-1232

30. Merigo JM, Casanovas M. Fuzzy Generalized Hybrid Aggregation Operators and its Application in Decision Making. Int. Journal of Fuzzy Systems 2010; 12 (1) : 15-24.
31. Merigo JM, Casanovas M. The Fuzzy Generalized OWA Operators and its Application in strategic Decision Making. Cybernetics and Systems 2010; 41 (5) : 359-370.
32. Merigo JM, Casanovas M. Induced Aggregation Operators in Decision Making with Dempster-Shafer Belief Structure. Int. Journal of Intelligent Systems 2009; 24 (8): 934-954.
33. Merigo JM, Casanovas M, Martiner L. Linguistic Aggregation Operators for Linguistic Decision Making based on the Dempster-Shafer Theory of Evidence. Int. Journal of uncertainty, Fuzziness and Knowledge-Bases Systems 2010; 18 (3): 287-304.
34. Mesiar R, Spirkov J. Weighted means and weighting functions. Kybernetika 2006; 42 (2): 151-160.
35. Shafer G. A mathematical theory of evidence. Princeton, NJ: Princeton University Press; 1976. 297 p.
36. Sikharulidze A, Sirbiladze G. Average misbelief criterion in the minimal fuzzy covering problem. Proceedings of the 9th WSEAS International Conference on Fuzzy Systems (FS'08); 2008, 42–48.
37. Sirbiladze G. Extremal Fuzzy Dynamic Systems: Theory and Applications, IFSR International Series on Systems Science and Engineering 28, 1st Edition. Springer; 2013. 396 p.
38. Sirbiladze G. Modeling of extremal fuzzy dynamic systems. Parts I, II, III. International Journal of General Systems 2005, 34 (2): 107–198.
39. Sirbiladze G, Gachechiladze T. Restored fuzzy measures in expert decision-making. Information sciences 2005; 169 (1/2) : 71–95.
40. Sirbiladze G, Ghvaberidze B, Latsabidze T, Matsaberidze B. Using a minimal fuzzy covering in decision-making problems. Information sciences 2009; 179: 2022–2027.
41. Sirbiladze G, Sikharulidze A. Generalized Weighted Fuzzy Expected Values in Uncertainty Environment, Proceeding of the 9-th WSEAS International Conference on Artificial Intelligence, Knowledge Engineering and Data Bases; 2010. pp. 54-64.
42. Sirbiladze G, Sikharulidze A. Weighted fuzzy averages in fuzzy environment. Parts I, II. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 2003; 11(2) : 139–157, 159–172.
43. Sirbiladze G, Sikharulidze A, Ghvaberidze B, Matsaberidze B. Fuzzy-probabilistic Aggregations in the Discrete Covering Problem. International Journal of General Systems 2011; 40 (2): 169 -196.
44. Sirbiladze G, Zaporozhets N. About two probability representations of fuzzy measures on a finite set. Journal of fuzzy mathematics 2003; 11 (2): 1147–1163.
45. Sugeno M. Theory of fuzzy integrals and its applications, Thesis (PhD). Tokio Institute of Technology 1974.
46. Torra V. The weighted OWA operator. Int. Journal of Intelligent Systems 1997; 12 (2) : 153-166.
47. Torra V, Narukawa Y. Modeling Decisions: Information Fusion and Aggregation Operators. Berlin: Springer-Verlag; 2007. 284 p.
48. Yager RR. Weighted Maximum Entropy OWA Aggregation with Applications to Decision Making under Risk. IEEE Trans. On Systems, Man and Cybernetics 2009, Part A; 39 (3): 555-564.
49. Yager RR. On the dispersion measures of OWA operators. Information Sciences 2009; 179 (22): 3908-3919.
50. Yager RR. Aggregation of ordinal information. Fuzzy Optimization and Decision Making 2007; 6 (3): 199-219.
51. Yager RR. Generalized OWA Aggregation Operators. Fuzzy Optimization and Decision Making 2004; 3 (1): 93-107.
52. Yager RR. On the Evaluation of Uncertain Courses of Action. Fuzzy Optimization and Decision Making 2002; 1 (1): 13-41.

ISSN 1512-1232

53. Yager RR. Heavy OWA Operators. Fuzzy Optimization and Decision Making 2002; 1 (4): 379-397.
54. Yager RR. On the cardinality index and attitudinal character of fuzzy measures. International Journal of General Systems 2002; 31 (3): 303-329.
55. Yager RR. On the entropy of fuzzy measure. IEEE Transaction on Fuzzy Sets and Systems 2000; 8 (4): 453-461.
56. Yager RR. A class of fuzzy measures generated from a Dempster-Shafer Belief Structure. International Journal of Intelligent Systems 1999; 14 (12): 1239-1247.
57. Yager RR. Families of OWA operators. Fuzzy Sets and Systems 1993; 59 (2): 125-148.
58. Yager RR. On Ordered Weighted Averaging aggregation operators in multicriteria decision making. IEEE Trans. On Systems, Man and Cybernetics 1988; 18 (1): 183-190.
59. Yager RR, Kacprzyk J, Fedrizzi M (Eds). Advances in the Dempster-Shafer Theory of Evidence, New York: John Wiley & Sons; 1994. 597 p.
60. Yager RR, Kacprzyk J (Eds). The Ordered Weighted Averaging Operators: Theory and Applications. Norwell: Kluwer Academic Publishers; 1997. 357 p.
61. Yager RR, Kacprzyk J, Beliakov G (Eds). Recent Development in the ordered Weighted Averaging Operations: Theory and Practice, Studies in Fuzziness and Soft Computing 265, 1st Edition. Springer; 2011. 298 p.
62. Wang Z, Klir GJ. Generalized Measure Theory. IFSR International Series of Systems Science and Engineering 25, 1st Edition. Springer; 2009. 384 p.
63. Xu ZS, Da QL. An Overview of operators for aggregating information. Int. Journal of Intelligent Systems 2003; 18 (9) : 953-969.

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