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ISING MODEL ON A 3-7 LATTICE: ORDER AND DISORDER

Viktor Urumov

Partenij Zografski 46, Skopje, Macedonia
v.urumov@gmail.com

ABSTRACT

The analytic solution of the model is obtained using the method of mapping. One observes ferromagnetic and ferromagnetic phases, and the reentrance phenomenon. For a certain range of interaction parameters between nearest neighbors, as a result of geometrical frustration, the ground state is degenerate, but nevertheless the system exhibits phase transition at a finite critical temperature accompanied by coexistence of order and disorder.

Keywords: *Ising model, lattice estimation*

INTRODUCTION

The model proposed by Lenz [1] to his student Ising [2] has an important place in the theory of phase transitions, as shown by the huge number of published papers accumulated over the years. The model has been applied to magnetic systems, lattice gas, binary alloys, systems with random interactions and random fields, and different types of lattices
with nearest and further neighbor interactions. An introductory guide to the literature in the field has been prepared by Tobochnik [3].

Geometrical frustration first appeared when the triangular lattice was considered [4]. The spin system is said to be frustrated if its minimum energy does not incorporate the minimum of all local interactions of each pair of spins. In an elementary triangle with Ising spins with two possible orientations at each vertex, when the interaction is antiferromagnetic, all three bonds cannot be simultaneously in the state of lower energy.

ISING MODEL ON THE 3-7 LATTICE AND ITS TRANSFORMATION

The elementary plaquette of the lattice in the shape of a square is depicted in Fig. 1a. There are five internal sites, connected in the shape of bow tie and four at the corners of the square. The spin at each site can be in one of two possible states, denoted as upwards and downwards orientations, or plus and minus. The lattice has a structure similar to a chessboard pattern. Each plaquette is surounded by four plaquettes, all of them rotated by 90° with respect to the one in the middle. This gives rise to heptagons from the elements of each pair of eighboring plaquettes. Hence the name 3-7 or bow tie lattice. The Cairo lattice has a similar structure, containing only pentagons [5, 6]. There are sites with two different coordination numbers, three and four. Only pair interactions between nearest neighbors are assumed, with an interaction strength, J, J_1, or J_2, depending on the type of bond (Fig. 1a). The contribution to the Hamiltonian of the system from each elementary plaquette is given by

$$H = -J(s_1 s_2 + s_3 s_4) - J(s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 + s_4\sigma_4)\,. \tag{1}$$

There are several ways to proceed. It can be verified that the partial summation in the partition function leads to a system that satisfies the free fermion condition [7] and subsequently to an equation for the critical temperature. Another method is to use star-triangle and dedecoration

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transformations [8] to map the model to the centered square lattice with nearest and next-nearest neighbor diagonal noncrossing interactions solved by Vaks et al. [9]. It is simpler to achieve the mapping to the above mentioned lattice by using the general transformation [10]. Thus, summation over all possible orientations of the internal spins of an elementary plaquette and the subsequent identity in Eq. 2, provide the effective interactions $P(K, K_1, K_2)$ and $Q(K, K_1, K_2)$ of the centered square lattice (Fig. 1b)

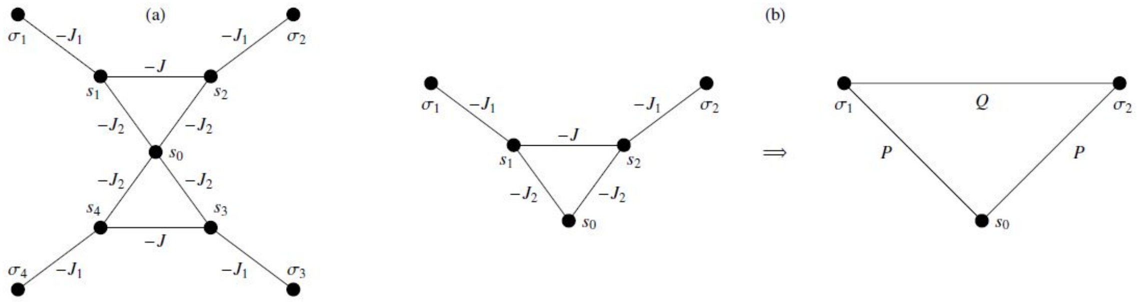

FIGURE 1. (a) Elementary plaquette of 3-7 lattice, (b) Transformation by elimination of the internal spin variables s_1 and s_2.

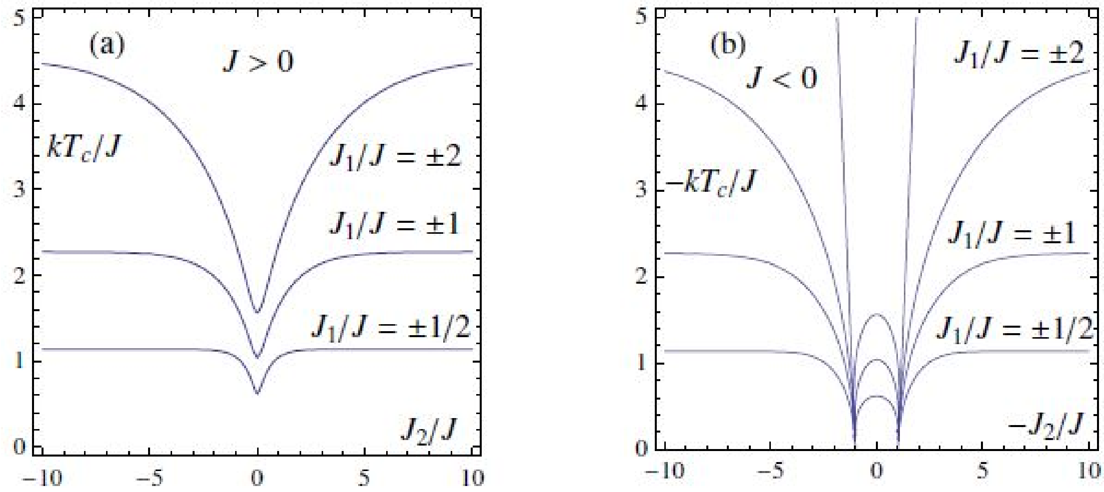

FIGURE 2. Critical temperature as a function of J_2/J for several values of J_1/J : (a) $J > 0$, (b) $J < 0$. The slightly inclined lines are the asymptotes in the limit $T_c \to 0$ indicating reentrance in a tiny interval.

$$\sum_{s_1, s_2} \exp(-H/kT) = \sum_{s_1, s_2} \exp[K s_1 s_2 + K_1 s_0 (s_1 + s_2) + K_2 (s_1 \sigma_1 + s_2 \sigma_2)] =$$

$$= A \exp[P s_0 (\sigma_1 + \sigma_2) + Q \sigma_1 \sigma_2], \qquad (2)$$

where

$$\exp(4P) = [\exp(-K) + \exp\cosh 2(K_1 + K_2)] / [\exp(-K) + \exp\cosh 2(K_1 - K_2)], \qquad (3)$$

$$\exp(4Q) = [\exp(K) + \exp\cosh 2(K_1 + K_2)] \times$$

$$[\exp(-K) + \exp\cosh 2(K_1 - K_2)] / [\exp K \cosh 2K_1 + \exp(-K) \cosh 2K_2]^2 \qquad (4)$$

$$A^4 =$$

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$$=16\left[\exp(-K)+\exp\cosh 2(K_1+K_2)\right]\left[\exp(-K)+\exp\cosh 2(K_1-K_2)\right]\left[\exp K\cosh 2K_1+\exp(-K)\cosh 2K_2\right]^2 \quad (5)$$

$K=J/kT$, $K_1=J_1/kT$, $K_2=J_2/kT$, k is the Boltzmann constant and T is the absolute temperature. Here A represents a factor contributed to the partition function from each elementary plaquette. The effective interaction $P(K,K_1,K_2)$ is even function of K_1 and K_2, while $Q(K,K_1,K_2)$ is odd function of the same arguments. Therefore $P(K,-K_1,-K_2)=P(K,K_1,K_2)$ and $Q(K,-K_1,-K_2)=Q(K,K_1,K_2)$, and the space of parameters to be examined can be reduced to $J_1>0$ or $J_2>0$.

CRITICAL TEMPERATURE

There are two equivalent ways for determination of the critical temperature. In the first case it can be obtained from one or both of the following two equations [9]

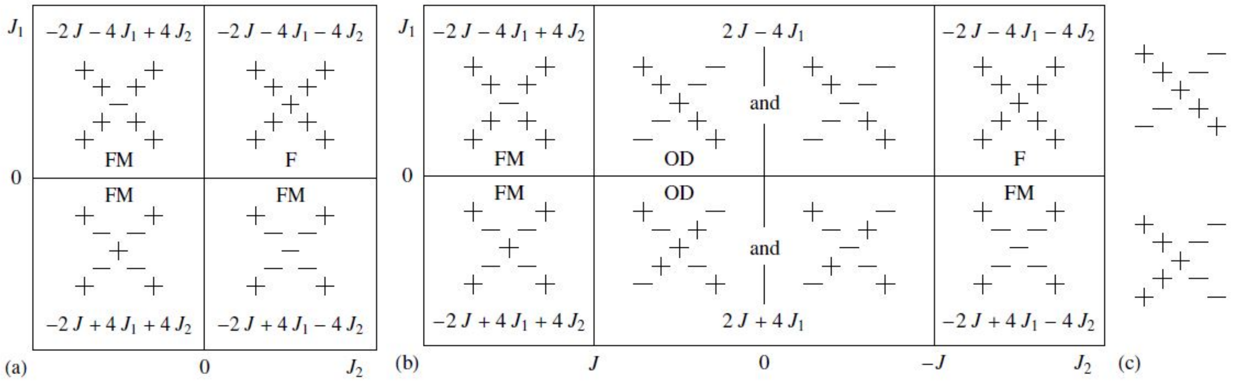

FIGURE 3. Ground state energy and spin orientations of the elementary plaquette: ((a) $J>0$, (b) $J<0$, (c) two basic ground states when $J<0$ and $|J_2|<|J|$. *FM* - ferrimagnetic state, *F* - ferromagnetic state, *OD* - coexistence of order and disorder.

$$(y+1)^2(x^2+1)^2=2(1-x^2)^2, \qquad (y+1)^2(x^2+1)^2=2y(1-x^2)^2, \quad (6)$$

where $x=\tanh P$ and $y=\tanh Q$. Alternatively the equation for T_c is given by [9, 11]

$$(1-x^4)^2+4x^4(1-y^2)^2=4\left[xy(1+x^2)\right]^2 \quad (7)$$

where $x=\exp(-2P)$ and $y=\exp(-2Q)$. The latter equation can be factorized, which leads to the following simplified expressions

$$y_{1,2}=\pm 1+\frac{1+x^2}{\sqrt{2}x}, \qquad y_{3,4}=\pm 1-\frac{1+x^2}{\sqrt{2}x} \quad (8)$$

Only y_1 and y_2 are positive and provide the expressions for determination of the critical temperature.

In Fig. 2a the dependence of T_c on the exchange interaction parameters is shown for the case $J>0$. The critical temperature increases with the increase of the strength of J_1 independently of its sign. The minima of the curves correspond to the critical temperature of the doubly decorated square lattice which is obtained when the interaction J_2 vanishes. The analogous curves for $J<0$ are shown in Fig. 2b. The critical temperature, similarly to the case of the triangular lattice, vanishes due to frustration when $J_2=\pm|J|$. The approach to zero, for $J_2>0$, follows the asymptotic law

$$kT_c/J=(4/\ln 2)(1+J_2/J), \quad (9)$$

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or its symmetric expression when $J_2 < 0$. The same asymptotic law is found for the dependence of T_c on J, for a given J_2, when $J \to -|J_2|$.

In a narrow interval for J_2 when $|J_2| > |J|$, increasing the temperature from zero, a disordered phase appears, which is followed by an ordered phase that disappears with further increase of the temperature. Such a behavior is known as reentrance phenomenon.

GROUND STATE

The ground states of the system are shown in Fig. 3a and 3b. When $J < 0$ and $|J_2| < |J|$, the ground state is degenerate. There are altogether 8 different arrangements of the spins on an elementary plaquette with a minimal energy. They arise from the two basic configurations (Fig. 3c) by mirror symmetry with respect to vertical line and from interchange between up and down spin orientations. Only the configurations obtained from the upper arrangement in Fig. 3c can cover the whole plane with plaquettes at their lowest energy. All possible coverages can be obtained by the quadruplets containing four elementary plaquettes shown in Fig. 4a and 4b for $J < 0$ and $|J_2| < |J|$, and $J_1 < 0$ or $J_1 < 0$, respectively.

Depending on the interaction parameters, the ground state is ferromagnetic (F) or ferrimagnetic (FM), except for the case when $J < 0$ and $|J_2| < |J|$. In the latter case, at *T = 0*, one observes coexistence of order and disorder (OD).

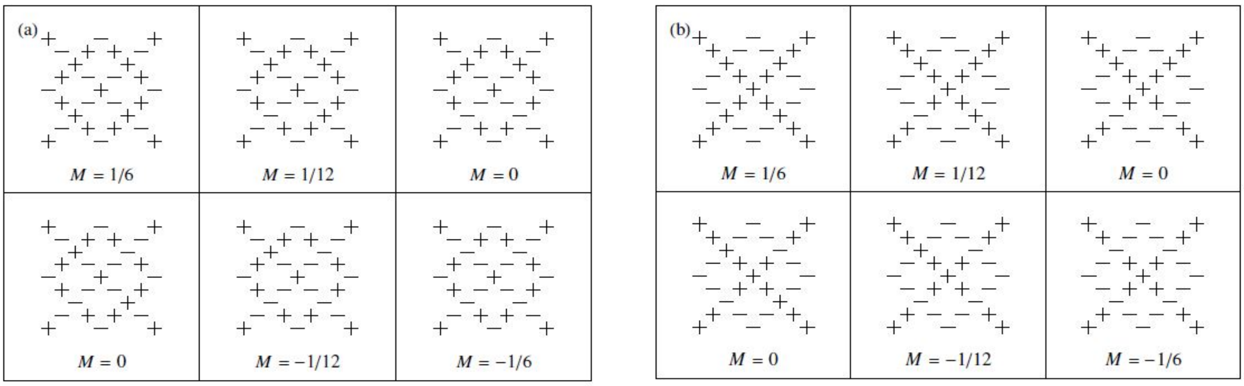

FIGURE 4. Quadruplets of elementary plaquettes in the ground state: (a) $J < 0$, $J_1 < 0$, $-|J| < J_2 < |J|$, (b) $J < 0$, $J_1 > ($ $-|J| < J_2 < |J|$. Plaquettes on each diagonal have their own specific orientation which is rotated by 90° in comparison to th orientation of plaquettes on the other diagonal. Central spins are disordered, the remaining spins have antiferromagnetic (a) an superantiferromagnetic (b) order.

The central spins of each plaquette have arbitrary orientation, while the remaining spins are ordered antiferromagnetically
(Fig. 4a) or superantiferromagnetically (Fig. 4b). The entropy at *T = 0* for the OD state is $S_0 = \ln(2)/6$. The magnetization of the quadruplets with lowest energy takes one of the following values: $0, \pm 1/12$ and $\pm 1/6$. Despite the degeneracy of the ground state for the case under consideration, the system has a finite critical temperature (Fig. 2b). Similar behavior was observed in other two-dimensional Ising models [12].

DISCUSSION

The analysis of the model was performed without any approximation. It shows phase transition to some ordered phase or transition to a state leading to coexistence of order and partial disorder. For $J \to 0$ the model is reduced to decorated square lattice, for $J \to \infty$ it becomes equivalent to partly decorated pentagonal Cairo lattice. Other limiting cases considered previously are: $J_1 \to \infty$, $J_2 \to 0$ and $J_2 \to \infty$.

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The model is exactly solvable in the more general case with higher spins included, not necessarily equal, at the intermediate locations between the central and corner spins. Also, the symmetry of the interactions could be avoided and one can introduce further neighbor interactions between spins in the first part of Fig. 1b.

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REFERENCES

1. W. Lenz, Z. Phys. (1920), 21, 613-615.
2. E. Ising, Z. Phys. (1925), 31, 253-258.
3. J. Tobochnik, Am. J. Phys. (2001), 69, 255-263.
4. R. M. F. Houtappel, Physica (1950), 16, 425-455.
5. V. Urumov, J. of Phys. A: Math. Gen. (2002), 35, 7317-7321.
6. M. Rojas, O. Rojas, and S. M. de Souza, Phys. Rev. E(2012), 86, 051116-1-11.
7. C. Fan, and F. Y. Wu, Phys. Rev. B(1970), 2, 723-733.
8. I. Syozi. ”Transformation of Ising Models” in Phase Transitions and Critical Phenomena Vol. 1, edited by C. Domb and M. S. Green, Academic Press, New York, (1972).
9. V. G. Vaks, A. I. Larkin, and Yu. N. Ovchinnikov, Zh. Eksp. Theor. Phys. (1965) 45, 1180-1189 [Sov. Phys. JETP(1966), 22, 820-826].
10. O. Rojas, and S. M. de Souza, J. of Phys. A: Math. Theor.(2011),44, 245001-1-17.
11. T. C. Choy, and R. J. Baxter, Phys. Lett. A, (1987), 125, 365-368.
12. P. Azaria, H. T. Diep, and H. Giacomini, Phys. Rev. Lett. (1987), 59, 1629-1632.
13. H. T. Diep, M. Debauche, and H. Giacomini, Phys. Rev. B , (1991), 43, 8759-8762.

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