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CONTRIBUTIONS ON S- EDGE REGULAR BIPOLAR FUZZY GRAPHS

Ch. Ramprasad [1 *], N. Srinivasarao [2], S. Satyanarayana [3], G. Srinivasarao [4]
[1 *] Research scholar, Department of Mathematics, VFSTR University, Vadlamudi, Andhra Pradesh, India
Email id: ramprasadchegu1984@gmail.com, Phone number: +918019399557
[2, 4] Department of Mathematics, Tirumala Engineering College, Narasaraopet, Andhra Pradesh, India
[3] Department of Computer Science & Engineering, KL University, Vaddeswaram, Andhra Pradesh, India

Abstract
Bipolar fuzzy graphs are revolutionized the analysis of probabilistic data to arrive at a judicious decision making power. In this paper, we introduced the notion of s-edge regular bipolar fuzzy graph, strongly regular bipolar fuzzy graph and biregular bipolar fuzzy graph, describe various methods of their construction and discuss some of their important theorems related to these graphs and also investigate on equivalence condition theorem of these graphs.

Keywords: *s-Edge regular bipolar fuzzy graph, strongly regular bipolar fuzzy graph, biregular bipolar fuzzy graph, bipartite bipolar fuzzy graph.*

1. Introduction

The moment science is involved in finding solutions to theoretical problems mathematical dependency increases. It has been proved by researchers that frameworks of analysis developed using mathematical models, especially those based on fuzzy logic were capable of handling uncertain data sets. The new mathematical models have shown their superiority over the conventional fuzzy logic based sets. Issues of doubt, data inconsistency and wrong or mismatched data were effectively analyzed with the help of graph theory in the various domains like medical, life sciences, management sciences, mechanical engineering, electrical engineering, with special focus on machine learning systems. The self-learning feature of these new mathematical models has given the possibility of scaling the size of the operations to suit the industrial requirement. In short, mathematical models based on fuzzy graph theory have simplified the handling of probabilistic to arrive at rational conclusions.

The problem of Konigsberg bridge, in eighteenth century laid the foundation to graph theory, where Euler strongly suggested that there is a solution using graph theory. In 1994, Zhang [16-18] extended the fuzzy set theory to bipolar fuzzy sets in which the first point predicts the reality condition while the second point represents a decimal opposite to the first point.

In 2011, using the concepts of bipolar fuzzy sets, Akram et al. [1, 2] studied the bipolar fuzzy graph notion and vague hyper graphs are discussed various operations on it. Based on this concept much research work is done in this line till date. In 2011, Yang et al. [15] studied a note on bipolar fuzzy graphs. In 2015, Ghorai and Pal [3] presented certain types of product bipolar fuzzy graphs. Rashmanlou et al. [4-6] studied balanced interval-valued fuzzy graph, some properties of highly irregular interval-valued fuzzy graphs, bipolar fuzzy graphs with categorical properties. Samanta and Pal have presented a few results on fuzzy graphs like threshold [7], fuzzy tolerance graphs [8], irregular bipolar fuzzy graphs [9], k-competition, p-competition [10] and m-step fuzzy competition graphs [11], Fuzzy planar graphs [12]. Talebi and Rashmanlou [13, 14] studied on complement, homomorphism, weak, and co weak isomorphism on bipolar fuzzy graphs. Bipolar fuzzy graph representation is more suitable to reality than crisp graph version.

2. Preliminaries

Some definitions and conventions used in this paper are discussed in this section. Literature review is available in [1, 6, 13-14].

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Definition 2.1 A graph $G=(V,E)$ is an ordered pair consisting of a non-empty vertex set V , an edge set E and a connection that associates with every edge between two edges (not as a matter of course particular) called its end points.

Definition 2.2 Let $G=(V,E)$ be a graph. Then $S=(N,L)$ is said to be a subgraph of G if $N \subseteq V$ and $L \subseteq E$.

Definition 2.3 A fuzzy set A on a universal set X is characterized by function $m: X \to [0,1]$, which is called the membership function. A fuzzy set is denoted by $A=(X,m)$.

Definition 2.4 A fuzzy graph $\delta=(V,\sigma,\mu)$ is a non-empty set V together with a pair of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ such that for all $m,n \in V, \mu(mn) \leq$ min $\{\sigma(m),\sigma(n)\}$, where $\sigma(m)$ and $\mu(mn)$ represent the membership values of the vertex m and of the edge mn in δ respectively. The underlying crisp graph of the fuzzy graph $\delta=(V,\sigma,\mu)$ is denoted as $\delta^*=(V,\sigma^*,\mu^*)$ where $\sigma^*=\{x \in V / \sigma(x)>0\}$ and $\mu^*=\{xy \in V \times V / \mu(xy)>0\}$. Thus for underlying fuzzy graph, $\sigma^*=V$.

Definition 2.5 A fuzzy graph $\delta=(V,\sigma,\mu)$ is called complete if $\mu(xy)=\min\{\sigma(x),\sigma(y)\}$ $\forall x,y \in V$ where xy denotes the edge between the vertices x and y. The fuzzy graph $\delta_a=(V,\sigma_a,\mu_a)$ is called a fuzzy subgraph of $\delta=(V,\sigma,\mu)$ if $\sigma_a(m) \leq \sigma(m)$ for all m and $\mu_a(mn) \leq \mu(mn)$ for all edges $mn, m,n \in V$.

Definition 2.6 The bipolar fuzzy set W in V is defined by $W=(\mu_W^+,\mu_W^-): X \times X \to [0,1] \times [-1,0]$ a bipolar fuzzy relation on X such that $\mu_W^+(x,y) \in [0,1]$ and $\mu_W^-(x,y) \in [-1,0]$.

Definition 2.7 Let $W=(\mu_W^+,\mu_W^-)$ and $F=(\mu_F^+,\mu_F^-)$ be a bipolar fuzzy sets on a set V. If $W=(\mu_W^+,\mu_W^-)$ is a bipolar fuzzy set on V , then $F=(\mu_F^+,\mu_F^-)$ is called a bipolar fuzzy relation on $E \subseteq V \times V$ if $\mu_F^+(xy) \leq \min\{\mu_W{}^+(x),\mu_W{}^+(y)\}$, $\mu_F^-(xy) \geq \max\{\mu_W^-(x),\mu_W^-(y)\}$, for all $x,\ y \in V$.

Definition 2.8 The bipolar fuzzy graph is a pair $G=(W,F)$ of a graph $G^*=(V,E)$, where $W=(\mu_W^+,\mu_W^-)$ is a bipolar fuzzy set on V and $F=(\mu_F^+,\mu_F^-)$ is a bipolar fuzzy relation on E such that $\mu_F^+(xy) \leq \min\{\mu_W^+(x),\mu_W^+(y)\}$, $\mu_F^-(xy) \geq \max\{\mu_W^-(x),\mu_W^-(y)\}$ for all $xy \in E$.

The underlying crisp graph of $G=(W,F)$ is the crisp graph $G^*=(V,E)$ where $V=\{v / \mu_W^+(v)>0 \ or \ \mu_W^-(v)<0\}$ and $E=\{uv / \mu_F^+(uv)>0 \ or \ \mu_F^-(uv)<0\}$.

Definition 2.9 Let $G=(W,F)$ be a bipolar fuzzy graph of $G^*=(V,E)$.Then $G=(W,F)$ is said to be strong if $\mu_F^+(xy)=\min\{\mu_W^+(x),\mu_W^+(y)\}$, $\mu_F^-(xy)=\max\{\mu_W^-(x),\mu_W^-(y)\}$, for all $xy \in E$.

Definition 2.10 Let $G=(W,F)$ be a bipolar fuzzy graph of $G^*=(V,E)$.Then $G=(W,F)$ is said to be complete if $\mu_F^+(xy)=\min\{\mu_W^+(x),\mu_W^+(y)\}$, $\mu_F^-(xy)=\max\{\mu_W^-(x),\mu_W^-(y)\}$, for all $x,y \in V$.

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Definition 2.11 The complement of a bipolar fuzzy graph $G=(W,F)$ of a graph $G^*=(V,E)$ is a bipolar fuzzy graph $\overline{G}=\left(\overline{W},\overline{F}\right)$, where $\overline{W}=W=\left(\mu_W^+(x),\mu_W^-(x)\right)$ and $\overline{F}=\left(\overline{\mu_F^+}(x),\overline{\mu_F^-}(x)\right)$. Here

$\overline{\mu_F^+}(xy)=\min\left\{\mu_W^+(x),\mu_W^+(y)\right\}-\mu_F^+(xy)$, $\overline{\mu_F^-}(xy)=\max\left\{\mu_W^-(x),\mu_W^-(y)\right\}-\mu_F^-(xy)$ for all $x,\ y\in V$.

Definition 2.12 Let $G=(W,F)$ be a bipolar fuzzy graph, where $W=\left(\mu_W^+,\mu_W^-\right)$ and $F=\left(\mu_F^+,\mu_F^-\right)$. The degree of a vertex is defined as $d_G=\left(d_G^+(w),d_G^-(w)\right)$, where $d_G^+(w)=\sum\mu_F^+(wx)$ $w\neq x, wx\in E$ is the positive degree of a vertex w and $d_G^-(w)=\sum\mu_F^-(wx)$ $w\neq x, wx\in E$ is the negative degree of a vertex.

Definition 2.13 Let $G^*=(V,E)$ be a crisp graph and let $e=w_rw_s$ be an edge in G^*.Then, the degree of an edge $e=w_rw_s\in E$ is defined as $d_{G^*}(w_rw_s)=d_{G^*}(w_r)+d_{G^*}(w_s)-2$.

Definition 2.14 Let $G=(W,F)$ be a bipolar fuzzy graph. The neighborhood degree of a vertex y is defined as $d_N(y)=\left(d_N^+(y),d_N^-(y)\right)$, where $d_N^+(y)=\sum_{x\in N(y)}\mu_W^+(x)$ and $d_N^-(x)=\sum_{x\in N(y)}\mu_W^-(x)$.

Definition 2.15 Bipartite bipolar fuzzy graph is a bipolar fuzzy graph $G=(W,F)$ in which the vertex set V can be partitioned in to two sets V_1 and V_2 where $V_1\neq\phi$ and $V_2\neq\phi$ such that

(a) $\mu_F^+(w_rw_s)=0$ and $\mu_F^-(w_rw_s)=0$, if $w_r,\ w_s\in V_1$ or $w_r,\ w_s\in V_2$.

(b) $\mu_F^+(w_rw_s)=0$, $\mu_F^-(w_rw_s)<0$, if $w_r\in V_1$ or $w_s\in V_2$.

(c) $\mu_F^+(w_rw_s)>0$, $\mu_F^-(w_rw_s)=0$ if $w_r\in V_1$ or $w_s\in V_2$, for some r and s.

3. Contributions on s-edge regular bipolar fuzzy graph, strongly regular bipolar fuzzy graph, biregular bipolar fuzzy graph

In this section, we introduced s-edge regular bipolar fuzzy graph, strongly regular bipolar fuzzy graph and biregular bipolar fuzzy graph and studied some of its properties.

Definition 3.1 Let $G=(W,F)$ be a bipolar fuzzy graph on G^*. The open neighborhood degree of an edge $xy\in E$ in a bipolar fuzzy graph $G=(W,F)$ is defined as $d_G(xy)=\left(d_G^+(xy),d_G^-(xy)\right)$, where $d_G^+(xy)=d_G^+(x)+d_G^+(y)-2\mu_F^+(xy)$ is the positive open neighborhood degree of an edge and $d_G^-(xy)=d_G^-(x)+d_G^-(y)-2\mu_F^-(xy)$ is the negative open neighborhood degree of an edge.

The minimum open neighborhood degree of an edge is defined as $\delta_E(G)=\min\left\{d_G(xy)/xy\in E\right\}$.

The maximum open neighborhood degree of an edge is defined as $\Delta_E(G)=\max\left\{d_G(xy)/xy\in E\right\}$.

Definition 3.2 The total open neighborhood degree of an edge $xy\in E$ in a bipolar fuzzy graph $G=(W,F)$ is defined as $td_G(xy)=\left(td_G^+(xy),td_G^-(xy)\right)$, where $td_G^+(xy)=d_G^+(x)+d_G^+(y)-\mu_F^+(xy)$ is

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the total positive open neighborhood degree of an edge and $td_G^-(xy) = d_G^-(x) + d_G^-(y) - \mu_F^-(xy)$ is the total negative open neighborhood degree of an edge.

The minimum total open neighborhood degree of an edge is defined as $\delta_{tE}(G) = \min\{td_G(xy) / xy \in E\}$.

The maximum total open neighborhood degree of an edge is defined as $\Delta_{tE}(G) = \max\{td_G(xy) / xy \in E\}$.

Example 3.1 Consider the bipolar fuzzy graph $G = (V, W, F)$ of $G^* = (V, E)$ as shown in Figure 1., where

$W = \{(K, (0.5, -0.8)), (L, (0.2, -0.4)), (M, (0.3, -0.5)), (N, (0.4, -0.6)), (O, (0.5, -0.7))\}$ and

$F = \{(KL, (0.1, -0.3)), (LM, (0.2, -0.4)), (MN, (0.3, -0.5)), (NO, (0.4, -0.6)), (OK, (0.5, -0.7))\}$.

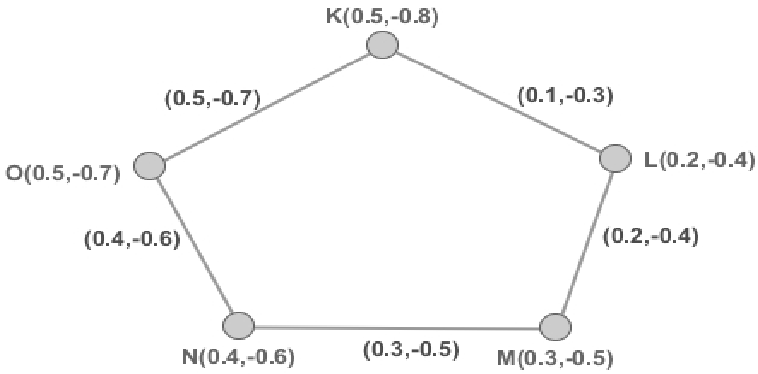

Figure 1: Bipolar fuzzy graph

In Figure 1. the degrees of the vertices are as follows:

$d_G(K) = (0.6, -1), d_G(L) = (0.3, -0.7), d_G(M) = (0.5, -0.9), d_G(N) = (0.7, -1.1), d_G(O) = (0.9, -1.3).$

Further, the degrees of the edges are as follows:

1. $d_G^+(KL) = d_G^+(K) + d_G^+(L) - 2\mu_F^+(KL) = 0.6 + 0.3 - 2(0.1) = 0.7,$

   $d_G^-(KL) = d_G^-(K) + d_G^-(L) - 2\mu_F^-(KL) = -1.0 - 0.7 - 2(-0.3) = -1.1,$

   $d_G(KL) = (d_G^+(KL), d_G^-(KL)) = (0.7, -1.1).$

2. $d_G^+(LM) = d_G^+(L) + d_G^+(M) - 2\mu_F^+(LM) = 0.3 + 0.5 - 2(0.2) = 0.4,$

   $d_G^-(LM) = d_G^-(L) + d_G^-(M) - 2\mu_F^-(LM) = -0.7 - 0.9 - 2(-0.4) = -0.8,$

   $d_G(LM) = (d_G^+(LM), d_G^-(LM)) = (0.4, -0.8).$

3. $d_G^+(MN) = d_G^+(M) + d_G^+(N) - 2\mu_G^+(MN) = 0.5 + 0.7 - 2(0.3) = 0.6,$

   $d_G^-(MN) = d_G^-(M) + d_G^-(N) - 2\mu_F^-(MN) = -0.9 - 1.1 - 2(-0.5) = -1,$

   $d_G(MN) = (d_G^+(MN), d_G^-(MN)) = (0.6, -1).$

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4. $d_G^+(NO) = d_G^+(N) + d_G^+(O) - 2\mu_F^+(NO) = 0.7 + 0.9 - 2(0.4) = 0.8,$

$d_G^-(NO) = d_G^-(N) + d_G^-(O) - 2\mu_F^-(NO) = -1.1 - 1.3 - 2(-0.6) = -1.2,$

$d_G(NO) = (d_G^+(NO), d_G^-(NO)) = (0.8, -1.2).$

5. $d_G^+(OK) = d_G^+(O) + d_G^+(K) - 2\mu_F^+(OK) = 0.9 + 0.6 - 2(0.5) = 0.5,$

$d_G^-(OK) = d_G^-(O) + d_G^-(K) - 2\mu_F^-(OK) = -1.3 - 1.0 - 2(-0.7) = -0.9,$

$d_G(OK) = (d_G^+(OK), d_G^-(OK)) = (0.5, -0.9).$

Further the total degrees of the edges are as follows:

1. $td_G^+(KL) = d_G^+(K) + d_G^+(L) - \mu_F^+(KL) = 0.6 + 0.3 - (0.1) = 0.8,$

$td_G^-(KL) = d_G^-(K) + d_G^-(L) - \mu_F^-(KL) = -1.0 - 0.7 - (-0.3) = -1.4,$

$td_G(KL) = (td_G^+(KL), td_G^-(KL)) = (0.8, -1.4).$

2. $td_G^+(LM) = d_G^+(L) + d_G^+(M) - \mu_F^+(LM) = 0.3 + 0.5 - (0.2) = 0.6,$

$td_G^-(LM) = d_G^-(L) + d_G^-(M) - \mu_G^-(LM) = -0.7 - 0.9 - (-0.4) = -1.2,$

$td_G(LM) = (td_G^+(LM), td_G^-(LM)) = (0.6, -1.2).$

3. $td_G^+(MN) = d_G^+(M) + d_G^+(N) - \mu_F^+(MN) = 0.5 + 0.7 - (0.3) = 0.9,$

$td_G^-(MN) = d_G^-(M) + d_G^-(N) - \mu_F^-(MN) = -0.9 - 1.1 - (-0.5) = -1.5,$

$td_G(MN) = (td_G^+(MN), td_G^-(MN)) = (0.9, -1.5).$

4. $td_G^+(NO) = d_G^+(N) + d_G^+(O) - \mu_F^+(NO) = 0.7 + 0.9 - (0.4) = 1.2,$

$td_G^-(NO) = d_G^-(N) + d_G^-(O) - \mu_F^-(NO) = -1.1 - 1.3 - (-0.6) = -1.8,$

$td_G(NO) = (td_G^+(NO), td_G^-(NO)) = (1.2, -1.8).$

5. $td_G^+(OK) = d_G^+(O) + d_G^+(K) - \mu_F^+(OK) = 0.9 + 0.6 - (0.5) = 1,$

$td_G^-(OK) = d_G^-(O) + d_G^-(K) - \mu_G^-(OK) = -1.3 - 1.0 - (-0.7) = -1.6,$

$td_G(OK) = (td_G^+(OK), td_G^-(OK)) = (1, -1.6).$

Definition 3.3 If every vertex in a bipolar fuzzy graph $G = (W, F)$ has the same degree (l_1, l_2) then $G = (W, F)$ is called regular bipolar fuzzy graph or bipolar fuzzy graph of degree (l_1, l_2).

Definition 3.4 If every edge in a bipolar fuzzy graph $G = (W, F)$ has the same open neighborhood degree (l_1, l_2) then $G = (W, F)$ is called $s = (l_1, l_2)-$edge regular bipolar fuzzy graph.

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Definition 3.5 If every edge in a bipolar fuzzy graph $G=(W,F)$ has the same total open neighborhood degree (l_1,l_2) then $G=(W,F)$ is called a totally $(l_1,l_2)-$ edge regular bipolar fuzzy graph.

Example 3.2

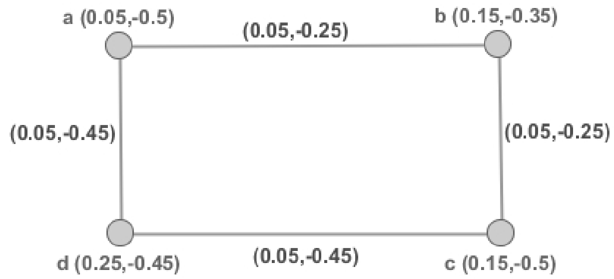

Figure 2: Edge regular bipolar fuzzy graph.

Consider the bipolar fuzzy graph $G=(W,F)$ of $G^*=(V,E)$ as shown in Figure 2. Then $d_G(ab)=d_G(bc)=d_G(cd)=d_G(da)=(0.1,-0.7)$.

Theorem 3.1 Let $G=(W,F)$ be an edge regular bipolar fuzzy graph on a cycle $G^*=(V,E)$. Then $\sum_{w_i\in V} d_G(w_i)=\sum_{w_iw_{i+1}\in E} d_G(w_iw_{i+1})$.

Proof Suppose that $G=(W,F)$ is an edge regular bipolar fuzzy graph and G^* be a cycle $w_1w_2w_3...w_nw_1$. Then $\sum_{i=1}^{n} d_G(w_iw_{i+1})=\left(\sum_{i=1}^{n} d_G^+(w_iw_{i+1}),\sum_{i=1}^{n} d_G^-(w_iw_{i+1})\right)$.

Now, we get $\sum_{i=1}^{n} d_G^+(w_iw_{i+1}) = d_G^+(w_1w_2)+d_G^+(w_2w_3)+...+d_G^+(w_nw_1)$, where $w_{n+1}=w_1$

$$= d_G^+(w_1)+d_G^+(w_2)-2\mu_F^+(w_1w_2)+d_G^+(w_2)+d_G^+(w_3)-2\mu_F^+(w_2w_3)+...+ d_G^+(w_n)+d_G^+(w_1)-2\mu_F^+(w_nw_1)$$

$$=2d_G^+(w_1)+2d_G^+(w_2)+...+2d_G^+(w_n)-2\left(\mu_F^+(w_1w_2)+\mu_F^+(w_2w_3)+...+\mu_F^+(w_nw_1)\right)$$

$$=2\sum_{w_i\in V} d_G^+(w_i)-2\sum_{i=1}^{n}\mu_F^+(w_iw_{i+1})$$

$$=\sum_{w_i\in V} d_G^+(w_i)+2\sum_{i=1}^{n}\mu_F^+(w_iw_{i+1})-2\sum_{i=1}^{n}\mu_F^+(w_iw_{i+1})$$

$$=\sum_{w_i\in V} d_G^+(w_i)$$

Similarly, $\sum_{i=1}^{n} d_G^-(w_iw_{i+1})=\sum_{w_i\in V} d_G^-(w_i)$.

So $\sum_{i=1}^{n} d_G(w_iw_{i+1})=\left(\sum_{w_i\in V} d_G^+(w_i),\sum_{w_i\in V} d_G^-(w_i)\right)=\left(\sum_{w_i\in V} d_G(w_i)\right)$.

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Remark 3.1 Let $G=(W,F)$ be an edge regular bipolar fuzzy graph on a crisp graph G^*. Then

$$\sum_{w_iw_j\in E} d_G(w_iw_j)=\left(\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^+(w_iw_j),\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^-(w_iw_j)\right),$$

where $d_{G^*}(w_iw_j)=d_{G^*}(w_i)+d_{G^*}(w_j)-2$ for all $w_iw_j\in E.$

Theorem 3.2 Let $G=(W,F)$ be an edge regular bipolar fuzzy graph on a c-regular crisp graph G^*. Then $\sum_{w_iw_j\in E} d_G(w_iw_j)=\left((c-1)\sum_{w_i\in V} d_G^+(w_i),(c-1)\sum_{w_i\in V} d_G^-(w_i)\right).$

Proof From remark 3.1, we get

$$\sum_{w_iw_j\in E} d_G(w_iw_j)=\left(\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^+(w_iw_j),\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^-(w_iw_j)\right)$$

$$=\left(\sum_{w_iw_j\in E}(d_{G^*}(w_i)+d_{G^*}(w_j)-2)\mu_F^+(w_iw_j),\sum_{w_iw_j\in E}\left(d_{G^*}(w_i)+d_{G^*}(w_j)-2\right)\mu_F^-(w_iw_j)\right).$$ Since G^* is a regular crisp graph, we have the degree of every vertex in G^* is c

i.e. $d_{G^*}(w_i)=c,$ so $\sum_{w_iw_j\in E} d_G(w_iw_j)=\left((c+c-2)\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),(c+c-2)\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right),$

$$\sum_{w_iw_j\in E} d_G(w_iw_j)=\left(2(c-1)\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),\ 2(c-1)\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right),$$

$$\sum_{w_iw_j\in E} d_G(w_iw_j)=\left((c-1)\sum_{w_i\in V} d_G^+(w_i),\ (c-1)\sum_{w_i\in V} d_G^-(w_i)\right).$$

Theorem 3.3 Let $G=(W,F)$ be an edge regular bipolar fuzzy graph on a crisp graph G^*. Then

$$\sum_{w_iw_j\in E} td_G(w_iw_j)=\left(\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^+(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),\ \sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^-(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right).$$

Proof From the definition of total open neighborhood edge degree of G, we get

$$\sum_{w_iw_j\in E} td_G(w_iw_j)=\left(\sum_{w_iw_j\in E} td_G^+(w_iw_j),\ \sum_{w_iw_j\in E} td_G^-(w_iw_j)\right)$$

$$=\left(\sum_{w_iw_j\in E}\left(d_G^+(w_iw_j)+\mu_F^+(w_iw_j)\right),\sum_{w_iw_j\in E}\left(d_G^-(w_iw_j)+\mu_F^-(w_iw_j)\right)\right)$$

$$=\left(\sum_{w_iw_j\in E}\left(d_G^+(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^+(w_iw_j)\right),\sum_{w_iw_j\in E}\left(d_G^-(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right)\right)$$

From remark 3.1, we have

$$\sum_{w_iw_j\in E} td_G(w_iw_j)=\left(\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^+(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),\ \sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^-(w_iw_j)+\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right).$$

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Theorem 3.4 Let $G=(W,F)$ be a bipolar fuzzy graph. Then (μ_F^+,μ_F^-) is a constant function if and only if the next conditions are equivalent.

(i) G is an edge regular bipolar fuzzy graph.
(ii) G is a totally edge regular bipolar fuzzy graph.

Proof Assume that (μ_F^+,μ_F^-) is a constant function. Then $\mu_F^+(w_iw_j)=k_1$ and $\mu_F^-(w_iw_j)=k_2$, $\forall\, w_iw_j \in E$, where k_1 and k_2 are constants. Let G be an (r_1,r_2) edge regular bipolar fuzzy graph. Then for all $w_iw_j \in E, d_G(w_iw_j)=(r_1,r_2)$.

Now $td_G(w_iw_j)=\left(d_G^+(w_iw_j)+\mu_F^+(w_iw_j),d_G^-(w_iw_j)+\mu_F^-(w_iw_j)\right)=$ (r_1+k_1,r_2+k_2) $\forall\, w_iw_j \in E.$ Then G is totally edge regular. Now, let G be a (h_1,h_2)- totally edge regular bipolar fuzzy graph. Then $td_G(w_iw_j)=(h_1,h_2),$ for all $w_iw_j \in E.$

So, we have $td_G(w_iw_j)=\left(d_G^+(w_iw_j)+\mu_F^+(w_iw_j),d_G^-(w_iw_j)+\mu_F^-(w_iw_j)\right)=(h_1,h_2)$. Hence, $\left(d_G^+(w_iw_j),d_G^-(w_iw_j)\right)=\left(h_1-\mu_F^+(w_iw_j),h_2-\mu_F^-(w_iw_j)\right)=(h_1-k_1,h_2-k_2)$. Then, G is an (h_1-k_1,h_2-k_2) -edge regular bipolar fuzzy graph. Conversely suppose that conditions (i) and (ii) are equivalent. We have to prove that the function (μ_F^+,μ_F^-) is a constant function. In a contrary way we suppose that (μ_F^+,μ_F^-) is not a constant function. Then $\mu_F^+(w_iw_j)\neq\mu_F^+(w_rw_s)$ and $\mu_F^-(w_iw_j)\neq\mu_F^-(w_rw_s)$ for at least one pair of edges $w_iw_j, w_rw_s \in E$. Let G be an (r_1,r_2) edge regular bipolar fuzzy graph. Then, $d_G(w_iw_j)=d_G(w_rw_s)=(r_1,r_2)$. So for all $w_iw_j \in E$ and for all $w_rw_s \in E.$

$$td_G(w_iw_j)=\left(d_G^+(w_iw_j)+\mu_F^+(w_iw_j),d_G^-(w_iw_j)+\mu_F^-(w_iw_j)\right)=\left(r_1+\mu_F^+(w_iw_j),r_2+\mu_F^-(w_iw_j)\right)$$
$$td_G(w_rw_s)=\left(d_G^+(w_rw_s)+\mu_F^+(w_rw_s),d_G^-(w_rw_s)+\mu_F^-(w_rw_s)\right)=\left(r_1+\mu_F^+(w_rw_s),r_2+\mu_F^-(w_rw_s)\right).$$

Since $\mu_F^+(w_iw_j)\neq\mu_F^+(w_rw_s)$ and $\mu_F^-(w_iw_j)\neq\mu_F^-(w_rw_s)$, we have $td_G(w_iw_j)\neq td_G(w_rw_s)$. Hence, G is not a totally edge regular bipolar fuzzy graph. This is a contradiction to our assumption. Hence (μ_F^+,μ_F^-) is a constant function. In the same way, we can prove that (μ_F^+,μ_F^-) is a constant function, when G is a totally edge regular bipolar fuzzy graph.

Theorem 3.5 Let G^* be a $h-$ regular crisp graph and $G=(W,F)$ be a bipolar fuzzy graph on G^*. Then, (μ_F^+,μ_F^-) is a constant function if and only if G is both regular bipolar fuzzy graph and totally edge regular bipolar fuzzy graph.

Proof Let $G=(W,F)$ be a bipolar fuzzy graph of G^* and let G^* be a $h-$ regular crisp graph. Assume that (μ_F^+,μ_F^-) is a constant function. i.e. $\mu_F^+(w_iw_j)=k_1$ and $\mu_F^-(w_iw_j)=k_2$ for all $w_iw_j \in E$ where k_1,k_2 are constants. From definition of degree of a vertex,

We get $d_G(w_i)=\left(d_G^+(w_i),d_G^-(w_j)\right)=\left(\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),\sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right)$

$$=\left(\sum_{w_iw_j\in E}k_1,\sum_{w_iw_j\in E}k_2\right)=\left(hk_1,hk_2\right),$$

for every $w_i\in V$. So, $d_G(w_i)=(hk_1,hk_2)$. Therefore G is regular bipolar fuzzy graph.

Now, $td_G(w_iw_j)=\left(td_G^+\left(w_iw_j\right),td_G^-\left(w_iw_j\right)\right)$,

where $td_G^+\left(w_iw_j\right)=\sum\mu_F^+\left(w_iw_k\right)+\sum\mu_F^+\left(w_kw_j\right)+\mu_F^+\left(w_iw_j\right)$

$$=\sum_{\substack{w_iw_k\in E\\ j\neq k}}k_1+\sum_{\substack{w_kw_j\in E\\ k\neq l}}k_1+k_1=k_1(h-1)+k_1(h-1)+k_1=k_1(2h-1).$$

Similarly, $td_G^-\left(w_iw_j\right)=k_2(2h-1)$, for all $w_iw_j\in E.$

Hence G is also a totally edge regular bipolar fuzzy graph.

Conversely, assume that G is both regular and edge regular bipolar fuzzy graph. Now we have to prove that $\left(\mu_F^+,\mu_F^-\right)$ is a constant function. Since G is regular, $d_G(w_i)=(k_1,k_2)$, for all $w_i\in V$. As G is totally edge regular, we have $td_G\left(w_i,w_j\right)=(h_1,h_2)$, for all $w_iw_j\in E$. From the definition of totally edge degree we get $td_G\left(w_iw_j\right)=\left(td_G^+\left(w_iw_j\right),td_G^-\left(w_iw_j\right)\right)$, where $td_G^+\left(w_iw_j\right)=d_G^+\left(w_i\right)+d_G^+\left(w_j\right)-\mu_F^+\left(w_iw_j\right)$ for all $w_iw_j\in E,\ h_1=k_1+k_1-\mu_F^+\left(w_iw_j\right).$

So, $\mu_F^+\left(w_iw_j\right)=2k_1-h_1.$ Similarly, we have $\mu_F^-\left(w_iw_j\right)=2k_2-h_2,$ for all $w_iw_j\in E.$ Hence, $\left(\mu_F^+,\mu_F^-\right)$ is a constant function.

Definition 3.6 A bipolar fuzzy graph $G=\left(W,F\right)$ of $G^*=\left(V=\left\{w_1,\ w_2,\ \cdots,\ w_n\right\},\ E\right)$ said to be strongly regular, if:

(i) G is $\left(h_1,h_2\right)$-regular bipolar fuzzy graph.

(ii) The sum of membership values and non-membership values of the common neighborhood vertices of any pair of adjacent vertices and non-adjacent vertices w_i,w_j of G has the same weight and is denoted by $\lambda=\left(\lambda_1,\lambda_2\right),\delta=\left(\delta_1,\delta_2\right)$ respectively.

Any strongly regular bipolar fuzzy graph G is denoted by $G=\left(n,h,\lambda,\delta\right)$.

Theorem 3.6 If $G=\left(W,F\right)$ is a complete bipolar fuzzy graph with $\left(\mu_W^+,\mu_W^-\right)$ and $\left(\mu_F^+,\mu_F^-\right)$ as constant functions, then G is a strongly regular bipolar fuzzy graph.

Proof Let $G=\left(W,F\right)$ be a complete bipolar fuzzy graph of $G^*=\left(V,E\right)$, where $V=\left\{w_1,\ w_2,\ \cdots,\ w_n\right\}.$ Since μ_W^+,μ_W^- and μ_F^+,μ_F^- are constant functions, we have $\mu_W^+\left(w_i\right)=l,\mu_W^-\left(w_i\right)=m,$ for all $w_i\in V$ and $\mu_F^+\left(w_iw_j\right)=k_1,$ $\mu_F^-\left(w_iw_j\right)=k_2$ for all $w_iw_j\in E$ where l,m,k_1,k_2 are constants. Now we have to show that G is a strongly regular bipolar fuzzy graph. Now since is G complete, we have

$$d_G\left(w_i\right)=\left(d_G^+\left(w_i\right),\ d_G^-\left(w_i\right)\right)=\left(\sum_{w_iw_j\in E}\mu_F^+\left(w_iw_j\right),\ \sum_{w_iw_j\in E}\mu_F^-\left(w_iw_j\right)\right)=\left(\left(n-1\right)k_1,\ \left(n-1\right)k_2\right).$$

Hence, G is $\left(\left(n-1\right)k_1,\left(n-1\right)k_2\right)$ regular bipolar fuzzy graph.

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Now here G is complete bipolar fuzzy graph the sum of membership values and non-membership values of common neighborhood vertices of any pair of adjacent vertices $\lambda = ((n-2)l,(n-2)m)$ are the equal and the sum of membership values and non-membership values of common neighborhood vertices of any pair of non-adjacent vertices $\delta = (0,\ 0)$ are same.

Definition 3.7 A bipolar fuzzy graph $G = (W, F)$ is said to be a biregular bipolar fuzzy graph if

(i) G is a (h_1, h_2)-regular bipolar fuzzy graph,

(ii) $V = V_1 \cup V_2$ be the bipartition of V and every vertex in V_1 has the same neighborhood degree $P = (P_1, P_2)$ and every vertex in V_2 has the same neighborhood degree $Q = (Q_1, Q_2)$, where P and Q are constants.

Example 3.3

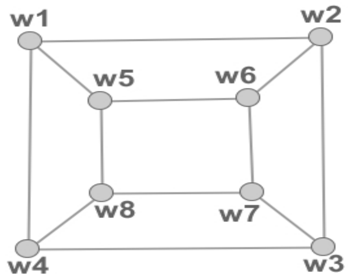

Figure 3: Biregular bipolar fuzzy graph G

Consider a bipolar fuzzy graph $G = (W, F)$ shown in Figure 3. Table 1. and Table 2. gives the membership values of vertices and edges and also neighborhood degree of vertices.

Vertices	Membership Values	Neighborhood Degree
w_1	(0.4,-0.6)	(1.5,-2.1)
w_2	(0.5,-0.7)	(1.2,-1.8)
w_3	(0.4,-0.6)	(1.5,-2.1)
w_4	(0.5,-0.7)	(1.2,-1.8)
w_5	(0.5,-0.7)	(1.2,-1.8)
w_6	(0.4,-0.6)	(1.5,-2.1)
w_7	(0.5,-0.7)	(1.2,-1.8)
w_8	(0.4,-0.6)	(1.5,-2.1)

Table 1. Vertex membership values along with neighborhood degree.

Edges	Membership Values
w_1w_2	(0.2,-0.5)
w_1w_4	(0.3,-0.6)
w_1w_5	(0.4,-0.5)
w_2w_6	(0.4,-0.5)
w_2w_3	(0.3,-0.6)
w_3w_4	(0.2,-0.5)
w_3w_7	(0.4,-0.5)
w_4w_8	(0.4,-0.5)
w_5w_6	(0.3,-0.6)
w_5w_8	(0.2,-0.5)
w_6w_7	(0.2,-0.5)
w_7w_8	(0.3,-0.6)

Table 2. Edge membership values

Then, $n = 8,\ h = (h_1,\ h_2) = (0.9,\ -1.6),\ V_1 = \{w_1,\ w_3,\ w_6,\ w_8\},\ V_2 = \{w_2,\ w_4,\ w_5,\ w_7\},$

$P = (P_1, P_2) = (1.5, -2.1)$ *and* $Q = (Q_1, Q_2) = (1.2, -1.8)$.

Theorem 3.7 If $G=(W,F)$ is a strongly regular bipolar fuzzy graph which is strong then $\overline{G}$ is a (h_1,h_2)-regular.

Proof Let $G=(W,F)$ be a strongly regular bipolar fuzzy graph. Then by definition G is $(h_1,h_2)-$regular. Since G is strong, we have

$$\overline{\mu_F^+}(w_iw_j)=\begin{cases}0 & \text{for all } w_iw_j\in E\\ \min\{\mu_W^+(w_i),\mu_W^+(w_j)\} & \text{for all } w_iw_j\notin E\end{cases}$$

$$\overline{\mu_F^-}(w_iw_j)=\begin{cases}0 & \text{for all } w_iw_j\in E\\ \max\{\mu_W^-(w_i),\mu_W^-(w_j)\} & \text{for all } w_iw_j\notin E.\end{cases}$$

Now, since G is strong, the degree of a vertex w_i in G is $d_{\overline{G}}(w_i)=\left(d^+_{\overline{G}}(w_i),d^-_{\overline{G}}(w_i)\right)$, where

$$d^+_{\overline{G}}(w_i)=\sum_{w_i\neq w_j}\overline{\mu_F^+}(w_iw_j)=\sum_{w_i\neq w_j}\overline{\mu_W^+}(w_i)\wedge\overline{\mu_W^+}(w_j)=h_1$$

$d^-_{\overline{G}}(w_i)=\sum_{w_i\neq w_j}\overline{\mu_F^-}(w_iw_j)=\sum_{w_i\neq w_j}\overline{\mu_W^-}(w_i)\vee\overline{\mu_W^-}(w_i)=h_2$. Hence $d_{\overline{G}}(w_i)=(h_1,h_2)$ for all $w_i\in V$. So $\overline{G}$ is a (h_1,h_2) -regular bipolar fuzzy graph.

Theorem 3.8 Let $G=(W,F)$ be a strong bipolar fuzzy graph. Then G is a strongly regular graph if and only if $\overline{G}$ is a strongly regular.

Proof Suppose that $G=(W,F)$ is a strongly bipolar fuzzy graph. Then we have to prove that $\overline{G}$ is a strongly regular bipolar fuzzy graph. If G is strongly regular bipolar fuzzy graph and which is strong then $\overline{G}$ is a (h_1,h_2) regular bipolar fuzzy graph by theorem 3.7. Next, let D_1 and D_2 be the sets of all adjacent vertices and non-adjacent vertices of G. That is $S_1=\{w_i,\, w_j\,/\,w_iw_j\in E\}$ where w_i and w_j have same common neighborhood $\lambda=(\lambda_1,\lambda_2)$ and $S_2=\{w_i,\, w_j\,/\,w_iw_j\notin E\}$ where w_i and w_j have same common neighborhood $\delta=(\delta_1,\delta_2)$. Then $\overline{S_1}=\left\{w_i,\, w_j\,/\,w_iw_j\in\overline{E}\right\}$ where w_i and w_j have same common neighborhood $\delta=(\delta_1,\delta_2)$ and $\overline{S_2}=\left\{w_i,\, w_j\,/\,w_iw_j\notin\overline{E}\right\}$, where w_i and w_j have same common neighborhood $\lambda=(\lambda_1,\lambda_2)$. Which implies $\overline{G}$ is a strongly regular. Similarly we can prove the converse.

Theorem 3.9 A strongly regular bipolar fuzzy graph G is a biregular bipolar fuzzy graph if the adjacent vertices have the same common neighborhood $\lambda=(\lambda_1,\lambda_2)\neq 0$ and the non-adjacent vertices have the same common neighborhood $\delta=(\delta_1,\delta_2)\neq 0$.

Proof Let $G=(W,F)$ be a strongly regular bipolar fuzzy graph. Then we have $d_G(w_i)=(h_1,\, h_2)$ for all $w_i\in V$. Assume that the adjacent vertices have the same common neighborhood $\delta=(\delta_1,\delta_2)\neq 0$. Let S be the set of all non-adjacent vertices.

That is $S=\{w_i, w_j / w_i \text{ is not adjacent to } w_j,\ i\neq j,\ w_i, w_j \in V\}$. Now the vertex partition of G is $V_1=\{w_i \mid w_i\in S\}$ and $V_2=\{w_j \mid w_j\in S\}$. Then V_1 and V_2 have the same neighborhood degree, since G is a strongly regular. Hence G is a biregular bipolar fuzzy graph.

Definition 3.8 (i) If the underlying graph G^* is an edge regular graph then G is said to be a partially edge regular bipolar fuzzy graph.

(ii) If G is both edge regular and partially edge regular bipolar fuzzy graph, then G is said to be a full edge regular bipolar fuzzy graph.

Theorem 3.10 Let G be a bipolar fuzzy graph of G^* such that (μ_F^+, μ_F^-) is a constant function. If G is full regular, then G is full edge regular bipolar fuzzy graph.

Proof Let (μ_F^+, μ_F^-) be a constant function. Then, $\mu_F^+(w_iw_j)=c_1$ and $\mu_F^-(w_iw_j)=c_2$ for every $w_iw_j\in E$, where c_1 and c_2 are constants. Suppose that G is full regular bipolar fuzzy graph then $d_G(w_i)=(k_1, k_2)=k$ and $d_{G^*}(w_i)=r$, for all $w_i\in V$, where k and r are constants.

$d_{G^*}(w_iw_j)=d_{G^*}(w_i)+d_{G^*}(w_j)-2=2r-2=$ constant. Hence G^* is an edge regular graph. Since G is regular, $d_G(w_iw_j)=(d_G^+(w_iw_j), d_G^-(w_iw_j))$, for all $w_iw_j\in E$, where $d_G^+(w_iw_j)=d_G^+(w_i)+d_G^+(w_j)-2\mu_F^+(w_iw_j)=k_1+k_1-2c_1=2k_1-2c_1=$ constant. Similarly, for all $w_iw_j\in E, d_G^-(w_iw_j)=2k_2-2c_2=$ constant. Hence, G is an edge regular bipolar fuzzy graph. Therefore, G is full edge regular bipolar fuzzy graph.

Theorem 3.11 Let G be a t-totally edge regular bipolar fuzzy graph and t_1-partially edge regular bipolar fuzzy graph. Then $S(G)=\dfrac{nt}{1+t_1}$, where $n=|E|$.

Proof The size of bipolar fuzzy graph G is $S(G)=\left(\sum_{w_iw_j\in E}\mu_F^+(w_iw_j),\ \sum_{w_iw_j\in E}\mu_F^-(w_iw_j)\right)$. Since G is a t-totally edge regular bipolar fuzzy graph, i.e. $td_G(w_iw_j)=t$. Again since G^* is t_1-partially edge regular bipolar fuzzy graph, i.e. $d_{G^*}(w_iw_j)=t_1$.

Thus $\sum td_G(w_iw_j)=\left(\sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^+(w_iw_j),\ \sum_{w_iw_j\in E} d_{G^*}(w_iw_j)\mu_F^-(w_iw_j)\right)+S(G)$

$qt=t_1S(G)+S(G)$. Hence, $S(G)=\dfrac{nt}{1+t_1}$.

Conclusion

In this article, we introduced edge regular, strongly regular and biregular bipolar fuzzy graph and studied some of its properties. The new method developed in this paper based on the pattern of unique cases help us to make a better choice in contrast to the established fuzzy graph solutions.

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