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Fuzzyfication of the Bloch Ball

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Abstract

By the concept of the Bloch ball we mean a three-dimensional shape that includes the sphere as well as all of the interior points (3-ball enclosed by 2-sphere). The Bloch ball is a geometric representation of qubit states. In this article we suggest a new sight on the Bloch ball considering this one via fuzzy set. A C++ style algorithm computes the value of a hash function using the von Neumann entropy of a density matrix. This value is transformed into the value of the linguistic variable. We get the possibility to express our knowledge on quantum states in linguistic terms.

Keywords: *Qubit, the Bloch ball, density matrix, von Neumann entropy, fuzzy set, linguistic variable.*

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I. QUBIT AND THE BLOCH BALL

Articles of P. Benioff [1], R. Feynman [2] and D. Deutsch [3] must be regarded as the origins of quantum computers. R. Feynman asked: "The first question is, What kind of computer are we going to use to simulate physics?" After analyzing the physical reality he answered: "…if you want to make a simulation of nature, you'd better make it quantum mechanical…" (see [2]).

The well-known textbook by Michael A. Nielsen and Isaac L. Chuang [4] is one of the most cited books in Quantum Computation and Quantum Information. There are many publications (see, for example [5], [6] and bibliography in nominated books) also devoted to the field of science mentioned above.

Counterpart of bit (classical unit of information) in quantum computation and quantum information is qubit (or quantum bit) - the simplest quantum state system.

Formally, a qubit is described by the unit vector of 2-dimensional complex Hilbert space using the Dirac's bra-ket notation:

$$|\psi\rangle = a|0\rangle + b|1\rangle = a(1,0)^t + b(0,1)^t = (a, b)^t$$

where a and b are probability amplitudes - complex numbers satisfying the normalization constraint

$$|a|^2 + |b|^2 = 1 \quad (1)$$

$|0\rangle = (1,0)^t$ and $|1\rangle = (0,1)^t$ are usually chosen as an orthonormal computational basis states. This notation is convenient because it labels the basis vectors explicitly and gives the possibility to write out the physical quantities. Therefore, in distinction from a bit, the qubit can exist in a superposition or linear combination of the basis states. This fact is fundamental to quantum computing. But we can't determine a and b. The quantum state isn't directly observable. Measurement of a state transforms the state into one of the basis vectors that are associated with the measuring device. The outcome of the measurement is the new state of the quantum system. Measurement of $|\psi\rangle = a|0\rangle + b|1\rangle$ gives us $|0\rangle$ with the probability $|a|^2$ or $|1\rangle$ with the probability $|b|^2$. It is impossible to get any additional information on a and βb. By measurement of the qubit we get exactly one bit of information.

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Operation on a qubit must preserve the property (1) and thus are described by 2×2 unitary matrices. Most important among these are the Pauli matrices (or operators).

There are several actual physical systems that can be used to realize the qubit:

- |0› corresponds to the horizontally polarized photons |↔›; |1› corresponds to the vertically polarized photons |↕›.
- Basis states of a qubit may be represented by the ground state and the first excited state of atomic energy levels.
- Spin-1/2 $\frac{1}{2}$ particle such an electron, whose spin can have values $+\hbar/2$ or $-\hbar/2$, where ħ is the reduced Plank constant (spin-up and spin-down as |0› and |1› accordingly).
- Quantum dots – tiny semiconductor crystals – 2 to 10 nanometers in size.

The Bloch ball (named after Swiss physicist Felix Bloch (1905-1983)) is a geometric representation of qubit states. The normalization constraint allows to specify a qubit in a manner given below:

$$|\psi\rangle = e^{i\gamma} ((\cos \theta/2)\, |0\rangle + e^{i\varphi} (\sin \theta/2)\, |1\rangle), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi.$$

θ and φ are polar and azimuthal angles; $e^{i\gamma}$ is a global phase factor that does not have any physical meaning (or observable effects). Qubit states with different values of γ define the same point on the Bloch sphere. Therefore, $e^{i\gamma}$ may be ignored and we can write

$$|\psi\rangle = (\cos \theta/2,\ e^{i\varphi} \sin \theta/2)^t$$

II. PURE AND MIXED STATES

The state of a quantum system represented by the unit vector $|\psi\rangle$ is called a pure state. It is also possible for a system to be in a statistical (or classical) ensemble of state vectors

$$p\, |0\rangle + (1 - p)\, |1\rangle$$

where p is a probability that the state vector is |0› and $(1- p)$ is a probability that the state vector is |1›. This system would be in a mixed state. A pure state is a special case of a mixed state in which $p = 0$ or $p = 1$. A mixed state is different from a quantum superposition of |0› and |1›. Quantum superposition of pure states is another pure state.

Any real three-dimensional vector $\boldsymbol{r}$ with $\| \boldsymbol{r} \| = r \le 1$ corresponds to a valid qubit state. Such a vector (the Bloch vector) is given by coordinates

$$\boldsymbol{r} = (r \sin \theta \cos \varphi,\ r \sin \theta \sin \varphi, r \cos \theta).$$

Points on the surface ($r = 1$) represent the pure qubit states. Interior points ($r < 1$) correspond to mixed states. The origin ($r = 0$) represents the fully (maximally) mixed state.

The ensemble of quantum states can be described by the density matrix (or the density operator) as a weighted sum of outer (tensor) products

$$\rho = p_1 |\psi_1\rangle \langle\psi_1| + \ldots + p_n |\psi_n\rangle \langle\psi_n|,$$

$$0 \le p_i \le 1,\ i = 1, \ldots n, \quad p_1 + \ldots + p_n = 1$$

where p_i is the probability of the pure state $|\psi_i\rangle$. It is known that density matrix for a single qubit can be expressed as the sum of the identity matrix and Pauli matrices with real coefficients b_x, b_y, b_z :

$$\rho = (1/2)(I + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z)$$

The density matrix must have a positive determinant. This condition involves an inequality

$$\| \boldsymbol{b} \|^2 = b_x^2 + b_y^2 + b_z^2 \le 1$$

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where a vector $\boldsymbol{b} = (b_x, b_y, b_z)$ is the Bloch vector for the state ρ.

The von Neumann entropy of a density matrix ρ is defined as

$$S(\rho) = -\, tr\,(\rho \log \rho) = -\,\eta_1 \log \eta_1 - \ldots - \eta_n \log \eta_n\,, \qquad (2)$$

where tr denotes the trace; η_j are eigenvalues of a matrix ρ (see [4] p. 510). Eigenvalues can be calculated using a built-in MATLAB function eig( ).

The von Neumann entropy has properties:

- $S(\rho) \geq 0$ and $S(\rho)$ is zero if and only if ρ represents a pure state. (3)
- $S(\rho)$ is maximal and equal to $\log d$ for a maximally mixed state, where d is the dimension of the Hilbert space. (4)

Properties mentioned above gives us the ability to use the von Neumann entropy to characterize a mixedness of a qubit states. Another measure is so-called linear entropy that is defined as follows

$$S_L(\rho) = (d\,(d-1))\,(1 - tr\,(\rho^2)\,)$$

III. ENTANGLEMENT

Entanglement – the essence of quantum formalism - is a strange phenomenon that play an important role in the quantum information processing and is widely used in various branches of quantum theory. It was first discussed in works of Erwin Schrödinger [7], Albert Einstein, Boris Podolsky and Nathan Rosen [8]. Entanglement characterize a compound quantum system.

The Hilbert space H associated with a bipartite system is a tensor product of the spaces associated with components $H = H_1 \otimes H_2$. A basis for the space H can be generated by the tensor products of basis vectors in H_1 and H_2. So, a general state of the two-qubit system is a linear combination

$$|\psi\rangle = \alpha_0\,|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

where $\alpha_0, \ldots, \alpha_3$ are complex numbers, $\alpha_0^2 + \ldots + \alpha_3^2 = 1$,

$$|00\rangle = |0\rangle \otimes |0\rangle = (1, 0, 0, 0)^t\,, \quad |01\rangle = |0\rangle \otimes |1\rangle = (0, 1, 0, 0)^t\,,$$

$$|10\rangle = |1\rangle \otimes |0\rangle = (0, 0, 1, 0)^t\,, \quad |11\rangle = |1\rangle \otimes |1\rangle = (0, 0, 0, 1)^t.$$

By definition, a state $|\psi\rangle$ in H is said to be entangled or non-separable, if it cannot be written as a simple tensor product of a state $|\alpha\rangle$ belonging to H_1 and a state $|\beta\rangle$ belonging to H_2. On the contrary, if we can write $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, we say that the state $|\psi\rangle$ is separable. For simplicity's sake we shall consider below only the case of a two-qubit system. The Bloch sphere model for two qubit pure states is given in paper by Chu-Ryang Wie [9].

Let $|\psi\rangle_{AB} \in H_A \otimes H_B$ be a pure state of two qubits. If it is entangled, it may not be possible to factor out the state vector $|\psi\rangle_A \in H_A$ for the state of first qubit. However, it can be computed the reduced density operator ρ^A on H_A, that describes the state $|\psi\rangle_A$ as a mixed state (see [4] pp. 105-106). According to the Schmidt decomposition theorem $|\psi\rangle_{AB}$ can be written as $\sum_j \lambda_j\, |\psi_j\rangle_A\, |\psi_j\rangle_B$, where states are orthonormal, $\lambda_j > 0$, $\sum_j \lambda_j^2 = 1$, the λ_j^2 (squared Schmidt coefficients) are the eigenvalues of the ρ^A (see [4] pp. 109-111).

The von Neumann entropy of reduced density matrix

$$e = S(\rho^A) = -\, tr\,(\rho^A \log \rho^A) = -\, tr\,(\rho^B \log \rho^B) = S(\rho^B) = -\sum_j \lambda_j^2 \log \lambda_j^2 \qquad (5)$$

is a suitable measure of entanglement. When $e = 0$, the state $|\psi\rangle_{AB}$ is separable (nonentangled); the maximum value of entropy $e = \log 2$ we get, when eigenvalues are {1/2, 1/2} and the state in this case is maximally entangled.

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IV. FUZZY SETS AND LINGUISTIC VARIABLES

In ordinary (classical, crisp) set theory any object belongs or does not belong to a given set - "*tertium non datur*" (logical law of excluded middle).

In 1965 year Lotfi Aliasker Zadeh published a pioneering work on fuzzy sets [10]. He wrote: "More often than not the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". Here are some examples such classes (by Zadeh):

- The class of all real numbers which are much greater than 1.
- The class of beautiful women.
- The class of tall men.

The origins of fuzzy sets are described in the book by Witold Pedrycz and Fernando Gomide [11]. This book is a self-contained treatise on fuzzy sets.

Formally, a fuzzy set *A* is described by a membership function $\mu_A(x)$ mapping the elements of universe *X* to the unit interval [0, 1].

$$\mu_A : X \rightarrow [0, 1].$$

The membership functions are therefore synonymous of fuzzy sets.

The Bloch ball as a generalized set of "quantum truth values" is considered by Mirco A. Mannucci [12]. A quantum fuzzy subset of a set *X* is defined as a point in the space $C^{P(X)}$, where *P(X)* is $2^{|X|}$. A generic fuzzy set is seen as a (possibly entangled) superposition of many fuzzy sets at once, offering new opportunities for modeling ucertainty.

Fuzzyfication of sets or notions is available using linguistic variables. By a linguistic variable we mean a variable whose values are words or sentences in a natural or artificial language [13], [14]. For example, possible values of a linguistic variable *Age* are: very young, not very young, young, middle-aged, etc.

V. LINGUSTIC VARIABLE FOR MIXEDNESS LEVELS

We define a radius of the Bloch vector using (2) based on the properties (3) and (4) of the von Neumann entropy $S(\rho)$ as follows

$$r = 1 - S(\rho) = 1 + \sum_j \eta_j \log \eta_j \tag{6}$$

The [0, 1] interval is divided into 7 parts:

$$[(i-1)/7, i/7), \; i = 1, 2, \dots 6, \; [6/7, 1] \tag{7}$$

Let us introduce the linguistic variable *qubit-state*. We link defined above subintervals to the values of the variable *qubit-state* (accordingly):

1. fully or almost fully mixed state
2. near fully mixed state
3. strongly mixed state
4. moderately mixed state
5. weakly mixed state
6. near pure state
7. pure or almost pure state

A hash function is a mathematical function that maps keys into integers. Below we define the Hash function, that for any value of radius *r* returns the label of the suitable subinterval.

```
int Hash (double r)    // r is calculated using (6)
  {
    if  (r < 0 | | r > 1)
        return −1;    // indicates error
    for ( int i = 1;  i ≤ 6;  i ++ )
```

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```
        if ((i −1) / 7 ≤ r < i / 7 )
            return i;
    return 7;
  }
```

We will use the return value of Hash function in C++ switch statement.

The function that converts double data type into string data type

```
string conDtoS (double d)
  {
    stringstream ss;
    ss << d;
    string degs = ss.str();
    return degs;
  }
```

Before calling this function in C++ program must be included header files <iostream>, <sstream>, <string>, using namespace std. conDtoS( ) procedure is needed to include the degree of membership into the value of linguistic variable using a string concatenation.

The transformation of Hash function return values into values of the linguistic variable

```
int h = Hash(r);
string qubit-state, qubit-state1;
string st = “qubit state is”;
string st1 = “of degree”;
int  k = 7π / 2 ;        // shrinkage multiplier for cos(x)
double  deg;
switch (h) {
        case 1:
              qubit-state = “fully or almost fully mixed”;
              deg =  cos (kr);
              degstr = conDtoS(deg);
              qubit-state1 = st+qubit-state + st1 + degstr;
              break;
        case 2:
              qubit-state = “near fully  mixed”;
              deg = cos k(r – 3/14);   // shrinkage and displacement
              degstr = conDtoS(deg);
              qubit-state1 =st+ qubit-state + st1+ degstr;
              break;
        case 3:
              qubit-state = “strongly mixed”;
              deg =   cos k(r – 5/14);
```

$\cos k\left(r - \frac{5}{14}\right)$

```
              degstr = conDtoS(deg);
              qubit-state1 =st+qubit-state +st1 + degstr;
              break;
        case 4:
              qubit-state = “moderately mixed”;
              deg =  cos k(r – 1/2);
              degstr = conDtoS(deg);
```

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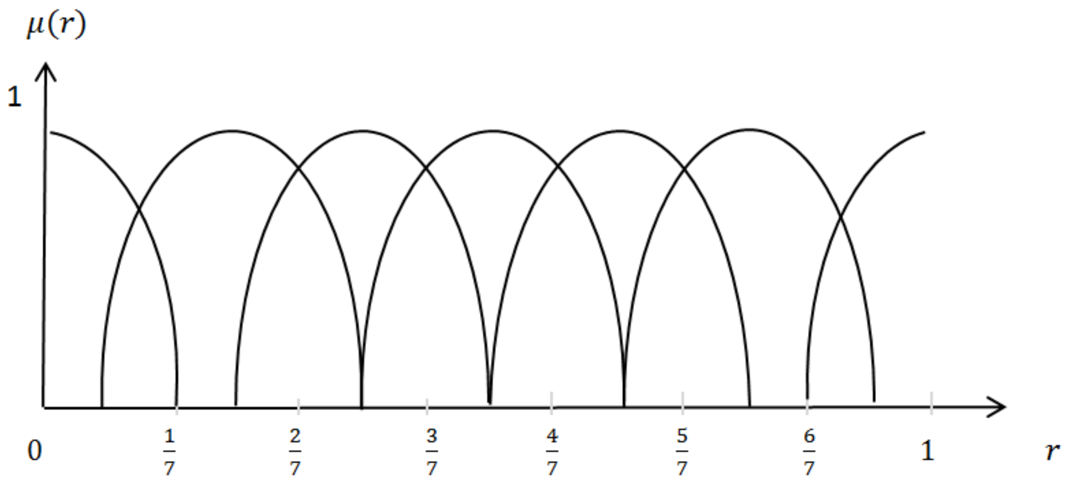
```
            qubit-state1 =st+qubit-state +st1+ degstr;
            break;
    case 5:
            qubit-state = "weakly mixed";
            deg =  cos k(r – 9/14);
            degstr = conDtoS(deg);
            qubit-state1 =st+ qubit-state + st1+ degstr;
            break;
    case 6:
            qubit-state = "near pure state";
            deg =  cos k(r – 11/14);
            degstr = conDtoS(deg);
            qubit-state1 =st+ qubit-state + st1+ degstr;
            break;
     case 7:
            qubit-state = "pure or almost pure";
            deg = cos k(r – 1);
            degstr = conDtoS(deg);
            qubit-state1 = st+qubit-state + st1+ degstr;
            break;
}
```

For example, the possible value of the linguistic variable *qubit-state1* may be: "qubit state is strongly mixed of degree 0.85".



Draft graphs of membership functions that correspond to the values of linguistic variable

VI. LINGUISTIC VARIABLE FOR ENTANGLEMENT LEVELS

To construct the linguistic variable for entanglement levels of the two qubit system pure states we shall use the same scheme as in previous section with some changes.

As a parameter in Hash function we take a value of a variable e, that is calculated according to (5).

We link subintervals (7) to the values of the linguistic variable *two-qubit-state* (accordingly):

1. separable or almost separable
2. near separable
3. weakly entangled
4. moderately entangled
5. strongly entangled

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6. near maximally entangled
7. maximally or almost maximally entangled.

VII. CONCLUSION

Peter W. Shor mentioned: "…my belief is that any new techniques have the potential to be great value in further exploration of quantum algorithms…" [15].

A discussion on quantum events often has the probabilistic nature. When we cannot describe precisely the state before a measurement or an outcome of a measurement, it is justified to use a fuzzy reasoning. The linguistic variables suggested in this article can be used in such cases.

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