ISSN 1512-1461

"HYDRINO" STATES IN RELATIVISTIC EQUATIONS

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Abstract: *The problems of so-called "Hydrino" additional solutions are studied in relativistic equations. Dirac Fermions as well as higher spin particles are considered. In one- and two-body Klein-Gordon and quasipotential equations for Coulomb potential the self-adjoint extension procedure is carried out.*

***Keywords**: "hydrino" , singular potentials, self-adjoint extension.*

1. Introduction

Last years much attention a widespread attention was invoked to the existence of additional solutions in various relativistic equations [1-6], where the Coulomb potential 1/r is also critical. It is interesting that the relativistic equations may have such "additional" solutions, that have no analogue in non-relativistic case. In [1-6] while these additional states were found, no attention is payed to the self-adjoint extansion, which on the own side can cause to lose of physical solutions.

This article is organized as follows: First of all we describe the appearance of "hydrino" states in various spinless relativistic equations . Then we study the same problems in quasipotential-like equation of Crater. At least, the Proca equation is also considered and the obtained results are summarized.

2.1 Hydrino states and a self-adjoint extension in the Klein-Gordon Equations

As we discussed in the introduction, the probem was considered in papers [1-6]. In particular, in [1] the Klein-Gordon equation was studied for attractive Coulomb $V=-\alpha/r \ (\alpha>0)$ potential

$$R''+\frac{2}{r}R'+\left[E^2-m^2-\frac{l(l+1)}{r^2}+\frac{2E\alpha}{r}+\frac{\alpha^2}{r^2}\right]R=0 \qquad (2.1)$$

The author mentioned that there must be an additional states. Let survey this problem in more detail. First of all let us note that this equation coincides formally to the Schrodinger equation in the valence electron problem, considered in [7] precisely

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$$\left(\frac{d^2}{d\rho^2}+\frac{2}{\rho}\frac{d}{d\rho}-\frac{P^2-1/4}{\rho^2}+\frac{\lambda}{\rho}-\frac{1}{4}\right)R=0 \tag{2.2}$$

But now

$$\rho=2\sqrt{m^2-E^2}r; \quad \lambda=\frac{E\alpha}{\sqrt{m^2-E^2}}; \quad P=\sqrt{(l+1/2)^2-\alpha^2}>0 \tag{2.3}$$

For the existence of bound states we have to require that $m^2>E^2$. Because of mentioned similarity one can to use the results of [7] accounting the relations (2.3). Particularly, the general solution of (2.2) is

$$R=C_1\rho^{-1/2+P}e^{-\frac{\rho}{2}}F(1/2+P-\lambda,1+2P;\rho)+C_2\rho^{-1/2-P}e^{-\frac{\rho}{2}}F(1/2-P-\lambda,1-2P;\rho) \tag{2.4}$$

The self-adjoint parameter τ is defined in the following manner

$$\tau=\frac{C_2}{C_1}\frac{1}{\left(2\sqrt{m^2-E^2}\right)^P}, \tag{2.5}$$

And the equation for eigenvalues looks like

$$\frac{\Gamma(1/2-\lambda-P)}{\Gamma(1/2-\lambda+P)}=-\tau\left(2\sqrt{m^2-E^2}\right)^P\frac{\Gamma(1-2P)}{\Gamma(1+2P)} \tag{2.6}$$

It must be underlined that (2.6) is a new equation, because the self-adjoint extension did not considered by above mentioned authors.

As we see (2.6) is a complicated transcendental equation for eigenvalues E, which depends on the τ. Only in two cases is possible the analytic solution of the Eq. (2.6):

- $\tau=0$ - only the standard levels. They can be deduced as the poles of $\Gamma(1/2-\lambda+P)$:

  $$1/2-\lambda+P=-n_r \qquad n_r=0,1,2... \tag{2.7}$$

Which gives,

$$E_{st}=\frac{m}{\sqrt{1+\frac{\alpha^2}{(1/2+n_r+P)^2}}}; \qquad n_r=0,1,2... \tag{2.8}$$

And the corresponding wave function is

$$R_{st}=C_1\rho^{-1/2+P}e^{-\frac{\rho}{2}}F(1/2+P-\lambda,1+2P;\rho) \tag{2.9}$$

- $\tau=\pm\infty$. In this case we have only additional states, determined from the poles of $\Gamma(1/2-\lambda-P)$ and they are

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$$1/2-\lambda-P=-n_r \qquad (n_r=0,1,2...) \tag{2.10}$$

From which we find

$$E_{add}=\frac{m}{\sqrt{1+\frac{\alpha^2}{(1/2+n_r-P)^2}}}; \qquad (n_r=0,1,2...) \tag{2.11}$$

And the corresponding wave function is

$$R_{add}=C_1\rho^{-1/2-P}e^{-\frac{\rho}{2}}F(1/2-P-\lambda,1-2P;\rho) \tag{2.12}$$

By analogy of [7] the function (2.4) may be written in the form of a single function

$$R(r)=C_1\Gamma(1+2P)\Gamma(1/2-P-\lambda)\frac{\sin\pi(1+2P)}{\pi\rho}e^{-\frac{\rho}{2}}\rho^{\frac{1}{2}-P}\Psi\left(\frac{1}{2}-\lambda-P,\,1-2P;\,\rho\right) \tag{2.13}$$

Exactly the states (2.12) are called in [1-6] as "hydrino" states. There is one difference between our and [1-6] approaches. In valence electron model the limiting procedure $V_0\to 0$ can be carried out and derive the hydrogen atom problem. But in (2.1) analogous constants $(\alpha \ and \ \alpha^2)$ are mutually dependent and in this limit we turn to free particle problem instead of Coulomb's one. The self-adjoint procedure is not performed in cited [1-6] papers. In [4] the authers think that the exclusion of hydrino states is possible by requirement of orthogonality. But the detailed consideration shows that these additional solution must be retained [7].

The difference between standard and hydrino states manifests itself in the process of non-relativistic limit, which must be realized be safety. Detailed consideration in [7] gives the restriction

$$l(l+1)<\alpha^2 \tag{2.14}$$

It follows that the l=0 states tending to non-relativistic limit is impossible, because the standard states remain, but the hydrino states disappear. Therefore we have to consider only l=0 states.

Indeed, it follows from (2.8-11) that for basic state ($n_r=l=0$):

$$E_{st}^{(0)}=\frac{m}{\sqrt{2}}\sqrt{1+\sqrt{1-4\alpha^2}} \tag{2.15}$$

$$E_{HYD}^{(0)}\equiv E_{add}^{(0)}=\frac{m}{\sqrt{2}}\sqrt{1-\sqrt{1-4\alpha^2}} \tag{2.16}$$

The expansion in powers of α gives

$$E_{st}^{(0)}=m\left(1-\frac{\alpha^2}{2}-\frac{\alpha^4}{8}\right) \tag{2.17}$$

$$E_{HYD}^{(0)}=m(\alpha+\alpha^3/2) \tag{2.18}$$

The standard solution does not feel the sign of α, whereas hydrino does.

Let us mention that expanding to order α^2 we derive

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$$E_{st}^{(0)} = m\left(1 - \frac{\alpha^2}{2(n_r+1)^2}\right) \tag{2.19}$$

$$E_{HYD}^{(0)} = m\left(1 - \frac{\alpha^2}{2(n_r)^2}\right) \tag{2.20}$$

Comparing these two results we see that there is some degeneracy between $n_r + 1$ and n_r levels, but this degeneracy disappear in the higher orders.

Now let us study the influence of scalar potential in case of attractive Coulomb's one

$$S = -\frac{\beta}{r};\ \beta > 0 \tag{2.21}$$

In this case we will derive the Eq. (2.2) but now

$$\rho = 2\sqrt{m^2 - E^2}\,r; \quad \lambda = \frac{E\alpha + m\beta}{\sqrt{m^2 - E^2}}; \quad P = \sqrt{(l+1/2)^2 + \beta^2 - \alpha^2} > 0 \tag{2.22}$$

The equations (2.4-12) are valid again. They take the form now

$$E_{st} = m\left\{\frac{\sqrt{1 + \frac{\alpha^2 - \beta^2}{n_{st}^2} - \frac{2\alpha\beta}{n_{st}^2}}}{1 + \frac{\alpha^2}{n_{st}^2}}\right\}, \quad n_{st} = \frac{1}{2} + n_r + P; \quad n_r = 0,1,2... \tag{2.23}$$

$$E_{add} = m\left\{\frac{\sqrt{1 + \frac{\alpha^2 - \beta^2}{n_{add}^2} - \frac{2\alpha\beta}{n_{add}^2}}}{1 + \frac{\alpha^2}{n_{add}^2}}\right\}, \quad n_{add} = \frac{1}{2} + n_r - P; \quad n_r = 0,1,2... \tag{2.24}$$

Here the existence of $E_{st} = 0$ and $E_{add} = 0$ levels is possible even for limiting $\tau = 0, \pm\infty$ cases if

$$\beta = \frac{1}{2n_r + 1}\left\{n_r(n_r+1) - l(l+1) + \alpha^2\right\} \tag{2.25}$$

As well as for

$$\tau_0 = -\frac{1}{2m^P}\frac{\Gamma(1/2 - \lambda - P)}{\Gamma(1/2 - \lambda + P)}\frac{\Gamma(1+2P)}{\Gamma(1-2P)}, \tag{2.26}$$

when E=0. In this case (2.14) is replaced by

$$l(l+1) < \alpha^2 - \beta^2 \tag{2.27}$$

Which says that hydrino states exist when

$$\alpha^2 > \beta^2 \tag{2.28}$$

i.e. the vector potential must be more strong than the scalar one, the inclusion of scalar potential hinders the appearance of hydrino states.

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2.2 Self-adjoint procedure in two-body Klein-Gordon equation with equal masses

Consider [8,9] the Klein-Gordon two-body equation for equal mass particles

$$\left\{\vec{p}^{\,2}+\left(m+\frac{S(\vec{r})}{2}\right)^2\right\}\psi(\vec{r})=\frac{1}{4}(M-V(\vec{r}))^2\psi(\vec{r}), \tag{2.29}$$

In case of central potentials the radial equation looks like

$$R''+\frac{2}{r}R'+\left[\frac{M^2}{4}-m^2-\frac{l(l+1)}{r^2}-\frac{MV}{2}-mS+\frac{V^2-S^2}{4}\right]R=0, \tag{2.30}$$

Which for attractive Coulomb potential

$$V=-\frac{\alpha}{r};\ \alpha>0 \quad \text{and} \quad S=-\frac{\beta}{r};\ \beta>0 \tag{2.31}$$

takes the form

$$R''+\frac{2}{r}R'+\left[\frac{M^2}{4}-m^2-\frac{l(l+1)}{r^2}+\frac{M\alpha}{2r}+\frac{m\beta}{r}+\frac{\alpha^2-\beta^2}{4r^2}\right]R=0 \tag{2.32}$$

Here $M=2m+\varepsilon$ - M is a mass of a bound state, while ε is a bound state energy. It follows the analogous to (2.2) equation, but we must require $4m^2>M^2$.

The equations (2.4 – 2.12) are still valid, but now

$$\rho=\sqrt{4m^2-M^2}\,r;\quad \lambda=\frac{\frac{M\alpha}{2}+m\beta}{\sqrt{4m^2-M^2}};\quad P=\sqrt{(l+1/2)^2+\frac{\beta^2-\alpha^2}{4}}>0 \tag{2.33}$$

And therefore

$$\frac{M_{st}}{2m}=\frac{\sqrt{1+\frac{\alpha^2-\beta^2}{4n_{st}^2}}-\frac{\alpha\beta}{4n_{st}^2}}{1+\frac{\alpha^2}{4n_{st}^2}};\quad n_{st}=n_r+1/2+P;\quad n_r=0,1,2... \tag{2.34}$$

$$\frac{M_{add}}{2m}=\frac{\sqrt{1+\frac{\alpha^2-\beta^2}{4n_{add}^2}}-\frac{\alpha\beta}{4n_{add}^2}}{1+\frac{\alpha^2}{4n_{add}^2}};\ n_{add}=n_r+1/2-P;\quad n_r=0,1,2... \tag{2.35}$$

Here the Eq. (2.14)-like condition would be

$$4l(l+1)+\beta^2<\alpha^2 \tag{2.36}$$

And the necessary condition for hydrino states is

$$\alpha^2>\beta^2 \tag{2.37}$$

Let us make some comments:

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(1) When $\beta > \alpha$, P is a real number and there is no "falling" of particles. For very strong value of scalar potential ($\beta \to \infty$), it follows

$$M \to 2m\sqrt{1 - \frac{\beta^2}{4}\frac{4}{\beta^2}} = 0 \qquad (2.38)$$

So we have no falling independently of magnitude of β. It is one of non-trivial manifestation of relativity.

(2) When $\alpha > \beta$, then P becomas a complex number if

$$\alpha^2 > 4l(l+1) + \beta^2 + 1 \qquad (2.39)$$

In this case for (2.35) we obtain the following restriction

$$4l(l+1) + \beta^2 < \alpha^2 < 4l(l+1) + \beta^2 + 1 \qquad (2.40)$$

But for standard masses (2.34) only upper restriction follows

$$\alpha^2 < 4l(l+1) + \beta^2 + 1 \qquad (2.41)$$

If this restriction is fulfilled, then wave functions corresponding to (2.34) and (2.35) oscillates very fastly near origin and have no definite limit. We consider it as a falling of particles on each others. This falling takes place even for finite values of parameters α, β.

(3) The equal portion of vector and scalar potentials, $\alpha = \beta$. The Eq. (2.31) reduces to Schrodinger like equation an (2.35) simplifies

$$\frac{M_{st}}{2m} = \frac{4n^2 - \alpha^2}{4n^2 + \alpha^2}; \qquad n = n_r + l + 1 \qquad (2.42)$$

(4) For pure vector potential ($\beta = 0$), because of $n_{add} < n_{st}$, the standard level is belower ,, than the hidrino level, $M_{st} < M_{add}$.

(5) Remark that for large $n_r >> 1$, it folows $n_{st} \approx n_{add}$, therefore $M_{st} \approx M_{add}$. Remember that the existence of additional states is possible only if P<1/2. It follows that quasiclassical reagion does not distinguish between standard and hidrino states.

(6) In case of pure vector potential ($\beta = 0$) it follows

$$M_{st} = 2m - \frac{m\alpha^2}{4} - \frac{5m\alpha^4}{64}; \quad n_r = l = 0 \qquad (2.43)$$

$$M_{add} = m\alpha + \frac{m\alpha^3}{8}; \qquad n_r = l = 0 \qquad (2.44)$$

$$M_{st} = 2m - \frac{m\alpha^2}{4(n_r+1)^2}; \qquad l = 0; n_r \neq 0 \qquad (2.44.1)$$

In (2.1) and (2.31)

$$M_{add} = 2m - \frac{m\alpha^2}{4n_r^2}; \qquad l = 0; n_r \neq 0 \qquad (2.45)$$

It is seen from these relations, that we have the similar situation as in one-particle Klein-Gordon equation.

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It is interesting to know if two equal masses particle can form a massless bound state, $M=0$. It is easy to see that in case of scalar potential such a situation can take place for all values of τ, including the extreme cases $\tau=0,\pm\infty$ for

$$\beta=\frac{1}{2n_r+1}\left\{n_r\left(n_r+1\right)-l(l+1)+\alpha^2\right\}. \tag{2.46}$$

Now we can formulate the necessary conditions for the existing of hidrino states. Let us consider the attractive vector and scalar potentials with the following behavior at the origin:

$$\lim_{r\to 0} rV\left(r\right)=-\alpha;\ \alpha>0\,, \tag{2.47}$$

$$\lim_{r\to 0} rS\left(r\right)=-\beta;\ \beta>0\,. \tag{2.48}$$

Remark, that these potentials are "critical" in relativistic equations in the sense of falling on the center. We have derived that for such potentials in equations (2.1) and (2.31) the hidrino states appear in area, where the falling onto the center begins.

2.3 Hydrino states in other two-body Klein-Gordon equations

For non-equal masses the Klein-Gordon two-body equation has the form

$$R''+\frac{2}{r}R'+\left[\begin{array}{l}\frac{M^2}{4}-m^2-\frac{l(l+1)}{r^2}-\frac{MV}{2}-\frac{m_0}{2}S+\frac{V^2-S^2}{4}+\\ +\frac{V}{2}\left(M-\frac{V}{2}\right)\frac{m_0^2\delta^2}{M^2(M-V)^2}+\frac{S}{2}\left(m_0+\frac{S}{2}\right)\frac{\delta^2}{(M-V)^2}\end{array}\right]R=0\,, \tag{2.49}$$

Where

$$m_0=m_1+m_2\,; \qquad \delta=m_2-m_1>0 \tag{2.50}$$

Let us study various cases for relations of vector and scalar potentials:

a) Pure scalar potential - $V=0;\ S\neq 0$. Then the Eq. (2.49) simplifies

$$R''+\frac{2}{r}R'+\left[\frac{M^2}{4}-m^2-\frac{l(l+1)}{r^2}-\frac{m_0}{2}S-\frac{S^2}{4}+\frac{S}{2}\left(m_0+\frac{S}{2}\right)\frac{\delta^2}{M^2}\right]R=0 \tag{2.51}$$

The hidrino states are expected in cases, when

$$\lim_{r\to 0} rS\left(r\right)=S_0 \tag{2.52}$$

At short distances (2.51) becomes

$$R''+\frac{2}{r}R'-\frac{P^2-\frac{1}{4}}{r^2}R=0\,, \tag{2.53}$$

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where

$$P = \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{S_0^2}{4}\left(1 - \frac{\delta^2}{M^2}\right)}, \tag{2.54}$$

And require $P < 1/2$, we derive the condition of existence hidrino states

$$4l(l+1) + S_0^2\left(1 - \frac{\delta^2}{M^2}\right) < 0 \tag{2.55}$$

This condition is not fulfilled in equal mass case. Now if we have

$$\Delta = 1 - \frac{\delta^2}{M^2} < 0 \tag{2.56}$$

there is a chance to fulfill (2.55).

Let, for simplicity, consider the case l=0. Remember that

$$M_n = m_1 + m_2 + \varepsilon_n \tag{2.57}$$

We have

$$(2m_2 + \varepsilon_n)(2m_1 + \varepsilon_n) + \frac{4l(l+1)}{S_0^2}M^2 < 0 \tag{2.58}$$

$$\Delta = \frac{(2m_2 + \varepsilon_n)(2m_1 + \varepsilon_n)}{(m_1 + m_2 + \varepsilon_n)^2} < 0 \tag{2.58.1}$$

It's clear that this condition be satisfied if

$$-2m_2 < \varepsilon_n < -2m_1, \tag{2.59}$$

This condition does not contradict to appearance of bound states for non-confining potentials, when the bound state energy may be positive.

We think that it is a kinematical (but not dynamical) effect.

In case of $l \neq 0$ it follows from (2.56) –(2.57) that the hidrino states may occur in cases, which satisfies to inequality

$$(2m_2 + \varepsilon_n)(2m_1 + \varepsilon_n) + \frac{4l(l+1)}{S_0^2}M^2 < 0 \tag{2.60}$$

b) Pure vector potential $V \neq 0;\ S = 0$.

In this case the equation takes the following form

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$$R'' + \frac{2}{r}R' + \left[\frac{M^2}{4} - m^2 - \frac{l(l+1)}{r^2} - \frac{MV}{2} + \frac{V^2}{4} + \frac{V}{2}\left(M - \frac{V}{2}\right)\frac{m_0^2\delta^2}{M^2(M-V)^2}\right]R = 0 \qquad (2.61)$$

It is easy to see that the condition of hydrino solution appearance coincides to that of equal masses case. Therefore results are the same as in equal mass case.

c) Let us now suppose that potentials have the following behaviors at the origin

$$\lim_{r\to 0} V(r) = \frac{V_0}{r^{n_1}}, \qquad \lim_{r\to 0} S(r) = \frac{S_0}{r^{n_2}}, \qquad n_1, n_2 > 0 \qquad (2.62)$$

Then at low distances we get

$$R'' + \frac{2}{r}R' + \left[\frac{V_0^2}{4r^{2n_1}} - \frac{S_0^2}{4r^{2n_2}} + \frac{S_0^2\delta^2}{4V_0^2}r^{2(n_1-n_2)} - \frac{l(l+1)}{r^2}\right]R = 0 \qquad (2.63)$$

It is evident, that $n_1 \le 1$ and $n_2 \le 1$, otherwise the falling onto the canter occurs. The only interesting case is $n_1 - n_2 = -1$. Let $n_1 = 0,\ n_2 = 1$, then

$$P = \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{S_0^2}{4} - \frac{S_0^2\delta}{4V_0^2}}, \qquad (2.64)$$

It follows (from P>1/2) that

$$4l(l+1) + S_0^2 < \left(\frac{S_0}{V_0}\right)^2\delta^2 \qquad (2.65)$$

Then for $l = 0$

$$(m_2 - m_1)^2 > V_0^2 \qquad (2.65.1)$$

And for $l \neq 0$, hidrino is expected as V_0 is small and δ is large.

2.4 Other spinless equations

1) The Todorov's equation [13]

It has a form

$$R'' + \frac{2}{r}R' + \left[V^2 - S^2 - 2m_M S - 2\varepsilon_M V - \frac{l(l+1)}{r^2}\right]R = 0, \qquad (2.66)$$

Where

$$m_M = \frac{m_1 m_2}{M}; \qquad \varepsilon_M = \frac{M^2 - m_1^2 - m_2^2}{2M} \qquad (2.67)$$

It is seen from (2.66) that the lesding term at origin should be $V^2 - S^2$. Therefore here the situation is analogous to that of Klein-Gordon two-body equation.

2) Likhtenberg equation [12]

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$$R'' + \frac{2}{r}R' + \left[\rho(V^2 - S^2) - 2\mu S - 2\sigma V - \frac{l(l+1)}{r^2}\right]R = 0 \qquad (2.68)$$

Where

$$\sigma = \frac{E_1 E_2}{M}; \quad E_i = \left(m_i^2 - 2\mu\varepsilon\right)^{\frac{1}{2}}; \ (i = 1,2); \quad \rho = \frac{m_1^2 + m_2^2}{M^2}; \ \mu = \frac{m_1 m_2}{m_1 + m_2}. \qquad (2.69)$$

The situation here is the same as in Todorov's equation.

3) Krolikovsky equation [14]

$$R'' + \frac{2}{r}R' + \left[\frac{M_1^2 + M_2^2}{2} - \frac{\left(M_1^2 - M_2^2\right)}{4(M - V)^2} - \frac{(M - V)^2}{4} - \frac{l(l+1)}{r^2}\right]R = 0 \qquad (2.70)$$

Where

$$M_1 = m_1 + \frac{S}{2}; \qquad M_2 = m_2 + \frac{S}{2} \qquad (2.71)$$

In difference of two above equations here the term $V^2 - S^2$ is not a leading one. Therefore more detailed analysis is necessary:

a) Pure scalar potential $V = 0; \quad S \neq 0$

Then we derive for potentials like (2.57)

$$P = \sqrt{(l + 1/2)^2 + S_0^2/4} \qquad (2.72)$$

And situation is analogous to (5.43) equation below: No hydrino states.

b) Pure vector potential $V \neq 0; \ S = 0$. NNow

$$P = \sqrt{(l + 1/2)^2 - V_0^2/4} \qquad (2.73)$$

determines the leading behavior and the situation is analogous to equal mass Klein-Gordon equation (2.30): when P<1/2, hidrino states occur.

c) $V \neq 0; \ S \neq 0$

Then for leading behavior we must take

$$P = \sqrt{(l + 1/2)^2 + \frac{S_0^2 - V_0^2}{4}} \qquad (2.74)$$

Now situation is analogous to equal mass Klein-Gordon equation, (2.30): when P<1/2 – hidrino states occur.

According to above mentioned discussion one concludes that only in different- masses Klein-Gordon equation appearance of hidrino states is possible and the other considered equations – Todorov's, Likhtenberg's, Krolikovsky – have the same situation, which coinsides to two-body Klein-Gordon equation with equal and different masses.

"Hydrino" states in fermionic equations

We already [15] have shown that hidrino states do not appear in one-particle Dirac equation.

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3.1 Crater's (quasipotential equation)

In several articles [5,6,16] Crater considered two-body Dirac equation, which is modified quasipotential equation – belongs to this klass of relativistic equations. We analyze thE last paper [16],in which He used his version of three-dimensional equations and for the S_0^1 - state his equation looks like the Schrodinger equation

$$R'' + \frac{2}{r}R' + \left[b^2 + \frac{2\varepsilon_w \alpha}{r} + \frac{\alpha^2}{r^2}\right]R = 0 \qquad (3.1)$$

Where $A(r) = -\frac{\alpha}{r}$, and α - fine structure constant, w – total energy in CM system, $b^2 = \varepsilon_w^2 - m_w^2$ is two-body relativistic energy, $\varepsilon_w = \frac{w^2 - 2m^2}{2w}$ - invariant energy of relative motion, and $m_w = \frac{m^2}{\omega}$ is a reduced mass of relative motion.

Remark that the equation (3.1) formally coinsides to Schredinger equation for valence electron model [7], but now

$$\rho = 2\sqrt{m_w^2 - \varepsilon_w^2}\, r; \qquad \lambda = \frac{\varepsilon_w \alpha}{\sqrt{m_w^2 - \varepsilon_w^2}}; \qquad P = \sqrt{\frac{1}{4} - \alpha^2} > 0 \qquad (3.2)$$

For bound states to occur there must be $m_w^2 > \varepsilon_w^2$. One can to use the results of [7]. In particular:
The general solution

$$R = C_1 \rho^{-1/2+P} e^{-\frac{\rho}{2}} F(1/2 + P - \lambda, 1 + 2P; \rho) + C_2 \rho^{-1/2-P} e^{-\frac{\rho}{2}} F(1/2 - P - \lambda, 1 - 2P; \rho) \qquad (3.3)$$

The self-adjoint parameter in this case looks like

$$\tau = \frac{C_2}{C_1} \frac{1}{\left(2\sqrt{m_w^2 - \varepsilon_w^2}\right)^P}, \qquad (3.4)$$

And the equation of eigenvalues is

$$\frac{\Gamma(1/2 - \lambda - P)}{\Gamma(1/2 - \lambda + P)} = -\tau \left(2\sqrt{m_w^2 - \varepsilon_w^2}\right)^P \frac{\Gamma(1 - 2P)}{\Gamma(1 + 2P)} \qquad (3.5)$$

It must be noted that (35) is a new relation, as Crater does not consider a self-adjoint extension in his equation (3.1).

As follows, Eq.(3.5) is complicated transcendental equation. Only in two cases is possible to extract analytic solutions:

1) $\tau = 0$

We get a solution from the pole of $\Gamma(1/2 - \lambda + P)$, namely

$$1/2 - \lambda + P = -n_r; \qquad n_r = 0,1,2... \qquad (3.6)$$

From which we find

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$$w_{st} = m\sqrt{2+\frac{2}{\sqrt{1+\frac{\alpha^2}{\left(n+\sqrt{1/4-\alpha^2}-1/2\right)^2}}}}; \qquad (3.7)$$

Where $n = n_r + 1$ is a principal quantum number ($l = 0$). The corresponding wave function has a form

$$R_{st} = C_1\rho^{-1/2+P}e^{-\frac{\rho}{2}}F(1/2+P-\lambda,1+2P;\rho) \qquad (3.8)$$

2) $\tau = \pm\infty$

In this case we have only additional levels, which are found from poles of $\Gamma(1/2-\lambda-P)$ and give

$$1/2-\lambda-P = -n_r \qquad (n_r = 0,1,2...) \qquad (3.9)$$

And

$$w_{add} = m\sqrt{2+\frac{2}{\sqrt{1+\frac{\alpha^2}{\left(n-\sqrt{1/4-\alpha^2}-1/2\right)^2}}}}; \quad n = 0,1,2,... \qquad (3.10)$$

Corresponding wave function has a form

$$R_{add} = C_1\rho^{-1/2-P}e^{-\frac{\rho}{2}}F(1/2-P-\lambda,1-2P;\rho) \qquad (3.11)$$

One can rewrite the wave function in a Unified form as in [7]

$$R(r) = C_1\Gamma(1+2P)\Gamma(1/2-P-\lambda)\frac{\sin\pi(1+2P)}{\pi\rho}e^{-\frac{\rho}{2}}\rho^{\frac{1}{2}-P}\Psi\left(\frac{1}{2}-\lambda-P,\, 1-2P;\ \rho\right) \qquad (3.12)$$

Exactly these states are called “hydrino”. Crated called them *peculiar* states. Distinction from the standard states manifests in passing to non-relativistic (small coupling) limit $\alpha \to 0$. Indeed, the relation (3.7) gives the form

$$w_{st} = 2m - \frac{m\alpha^2}{4n^2} - \frac{m\alpha^4}{2n^3\left(1-\frac{11}{32}n\right)} + O(\alpha^6); \quad n = 1,2,3... \qquad (3.13)$$

And (3.1) gives for the ground state of hidrino relation (39) from [5]

$$w_{add,1} = \sqrt{2}m\sqrt{1+\alpha} \qquad (3.14)$$

Which is a tightly bound state. At the same time from (3.10) we find for excited states (n>1):

$$w_{st} = 2m - \frac{m\alpha^2}{4n^2} - \frac{m\alpha^4}{2n^3\left(1-\frac{11}{32}n\right)} + O(\alpha^6); \quad n = 1,2,3... \qquad (3.15)$$

As well as in case of one dimensional Klein-Gordon equation we have a degeneracy between the hidrino states with levels for $n_r + 1$ and n_r nodes. This degeneracy disappears in higher order

$$w_{st,n} - w_{add,n+1} = -\frac{m\alpha^4}{n^3} \qquad (3.16)$$

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Therefore in case of different masses there are no hidrino states in two-body Crater equation and for making one of particle heavier, derived one particle Dirac equation also does not have hidrino states. Hidrinos appear only in equal masses two-body fermionic equation.

4. Hidrino states in higher spin equations (Proca equation)

For particles with spin, as is remarked in [17,18] there are two types of states for total momentum J:

a) States with l=J or, for which orbital momentum l is an integral of motion. In this case the Proka equation reduces to one-body Klein-Gordon equation and according to our investigation above the Proka equation should have the hidrino states an the self-adjoint extension is necessary.
b) The states with $l = J \pm 1; J \geq 1$ belongs to the second type of states. At the low distances radial Proka equation has a form [17,18]

$$\chi''_{1,2} + f_{1,2}(r)\chi_{1,2} = 0 \tag{4.1}$$

Where

$$f_{1,2} = \mp \frac{\sqrt{J(J+1)}}{r} V'(r) - \frac{J(J+1)}{r^2} \tag{4.2}$$

Here χ_1 corresponds to l=J+1, and χ_2 - to l=J-1. It is seen from (4.2) that for $V = gr^n;\ n < 0$ potential we should have a fall onto the center, therefore in this case only physically interesting potential may be logarithmic one (formally, n=0)

$$V = V_0 \ln r;\ V_0 > 0 \tag{4.3}$$

For which from Eq. (4.1) folows

$$\chi''_1 - \frac{\sqrt{J(J+1)}V_0 + J(J+1)}{r^2}\chi_1 = 0 \tag{4.4}$$

Ffrom which we find

$$P = \sqrt{\left(J + \frac{1}{2}\right)^2 + J(J+1)V_0} \tag{4.5}$$

So, P>1/2 and we have no additional levels.

But when l=J-1, it follows from (4.1), that

$$\chi''_2 - \frac{J(J+1) - \sqrt{J(J+1)}V_0}{r^2}\chi_2 = 0 \tag{4.6}$$

and

$$P = \sqrt{\left(J + \frac{1}{2}\right)^2 - J(J+1)V_0} \tag{4.7}$$

Therefore from P<1/2 we derive the following inequality for the existence of additional states

$$V_0 > \sqrt{J(J+1)} \tag{4.8}$$

Hence the self-adjoint extension procedure is necessary.

ISSN 1512-1461

Conclusions

The obtained above results may be summarized as follows:

1) For spinless particles with equal and unequal masses there are hydrino states, as well as in case when one of the masses becomes infinitely higher.
2) For fermions one particle Dirac equation has not a hydrino states. But the Crater's equation does not have hydrino states for unequal masses. But it has for equal masses, also we do not have hydrino in case of tending one of mass to infinity, $m_2 \gg m_1$.
3) The Proka equation has hydrino in some restricted case.

Acknowledgement:

This work was supported by Shota Rustaveli National Science Foundation (SRNSF) [grant number № DI-2016-26, Project Title: ''Three-particle problem in a box and in the continuum''].

References

1. J.Naudts , arXiv: Physics /0507193
2. P. Giri arXiv: cond-mat.mtrl-sci/0808.3309
3. N. Dombey, Phys.Lett.A. (2006) 360, 62
4. A. De Castro,Phys.Lett.A. (2007) , 369, 380.
5. H.Crater,C.Wong C, Phys. Rev. D(2012) 85, 116005
6. H.Crater,C.Wong C.arXiv:1406.7268
7. A. Khelashvili and T. Nadareishvili.Physics of Particles and Nuclei Letters. (2015) Vol 12.No1. pp 11-25.
8. J.S.Kang, H.J.Schitzer.Phys.Rev. , (1975) D12; 841
9. D.B. Lichtenberg et al. Phys.Rev.Lett (1982) 48, 1653 .
10. L.Landau;E.Lifchiz."Quantum Mechanics". "Nauka". 2002, Moscow. (Russian).
11. D.B.Lichtenberg,E.Predazzi,C.Rosseti.Z.Phys.C. (1988),40,357.
12. D.B.Lichtenberg,W.Namgung E.Predazzi..Z.Phys.C. (1983). 19,19.
13. I.T.Todorov.Phys.Rev. (1971) D3,2351.
14. W.Krolikowski. Acta Phys. Pol.B11, 387 (1980); ibid (1981),B12,793.
15. A.Khelashvili and T.Nadareishvili. International Journal of Modern Physics E. (2017), Vol 26, No 7, 1750043
16. H. Crater. Possible New Positronium Bound State. Journal of Physics.548(2014)012004
17. Perelomov A M and V.S. Popov V S 1970 *Teor. Mat.Fiz* 4 48 – 65 (In Russian)
18. I.Tamm. Phys.Rev, (1940). 58, 952