ISSN 1512-1461

FINANCIAL MODELLING WITH PATH INTEGRALS

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ABSTRACT

The applications of path integrals are growing by the day. The intimate connection of these mathematical constructs with the theory of stochastic processes has enabled their adoption as versatile tools of mathematical finance. In this article, we further this program and revisit some seminal results relating to asset pricing using the path integral framework.

1. Introduction

Significant progress seems to have been achieved towards the convergence of physics and finance, at least, insofar as the adoption of the underlying apparatus of probability, group theory, gauge theory, path integral formalisms etc. for the solution of complex problems is concerned. The use of these analytic tools as well as empirical analysis have unmasked several radical similarities between financial phenomena of stock price patterns, critical crashes etc. [1-12]. The linkage between physics and finance has a chequered history with pioneering contributions from Pareto [13] and Batchlier [14]. The distribution of wealth was empirically shown to follow a power law by Pareto. Batchlier used geometric Brownian motion for capturing the randomness embedded in stock price patterns and, thereby, unleashed a strong mathematical formulation for modeling of market based pricing. However, the seminal work that gave unparalleled recognition to the role of physicists in finance was the Nobel prize winning contingent claims pricing model developed by Fischer Black & Myron Scholes [15] together with Robert Merton [16]. The problem was reduced to a PDE, converted to a diffusion equation and, thereafter solved for appropriate boundary conditions to closed form expressions for the pricing of contingent claims on financial assets.

2. The Path Integral Formalism

Path integrals constitute the contemporary framework of choice for exploring the theory of quantum fields, gauge theories in particle physics and string theory. Theories of economics and finance are also being premised on path integrals with methodologies being developed for the pricing of financial instruments in equilibrium and non-equilibrium market states.

This technique was developed by Wiener [17] and Kac [18] in context of stochastic processes and adapted by Richard Feynman [19, 20] for application in quantum field theory.

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Path integrals have percolated to the realms of finance with some fervor with adaptations of the underlying techniques for pricing of contingent claims being extensively investigated [21, 22]. Application of the path integral formulation for path dependent options [23] and fixed income securities [24] have also been reported. Variants of the approach with semigroup pricing kernels [25], Green's function [26, 27] and the basic Feynman-Kac formula [28] in financial modelling are also conspicuous.

3. Path Integrals in Financial Modelling: The Underlying Rationale

At this juncture, the appropriateness of the underlying rationality of using the path integral formulation in finance needs to be examined. The dynamics of a classical deterministic system are usually described by the Euler-Lagrange equations that emerge as the outcome of the minimization of the "action" functional. This constitutes the "Least Action Principle" of classical mechanics. It is both convenient and conventional to write the "action" functional as an integral over time of a "Lagrangian" function that captures the dynamics of the system.

The quantum mechanical version of this formulation involve an integration over all possible paths from the initial state of the system to its final state and hence, the name "path integration". The procedure, in essence, comprises of (i) evaluating the action functional on each path connecting the final state with the initial state of the system; (ii) defining an integration measure over the set of all possible paths as above (iii) exponentiating the negative of the classical action functional obtained in step (i) to get the weight of this path in the path integral; (iv) obtaining expectation values of the various dynamical attributes of the system that depend on paths by doing the integration over all possible paths. The path integration involves slicing of the time interval into equal partitions and thereafter taking limits as such partitions shrink to infinitesimals. The fact that the path integral formalism provides appropriate solutions of Schrodinger's equation which, in turn, is a transformed version of the diffusion equation akin to the Black Scholes equation of mathematical finance is an unmistakable pointer to the intimacy between quantum physics and mathematical finance, with "stochastic processes" being the "golden thread" intertwining the two disciplines. Stock price evolution over time has been shown, both empirically and by stochastic modeling, to represent a diffusion process obeying the Kolmogorov equation. Expectation values of quantum mechanical operators and/or stochastic variables of interest can be obtained by representing the dynamical evolution of the system as a time-dependent PDE and obtaining its Feynman–Kac solution.

A correspondence can be established between the physical systems whose dynamics are captured by the "principle of least action" and financial systems that are believed to obey the "no arbitrage principle" [29-32]. This equivalence enables us to identify quantities performing the functions of the action functional and the Lagrangian in context of asset pricing equations.

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As mentioned earlier, the underlying philosophy for modelling of financial processes has been by treating them as stochastic systems following some form of random time evolution. It immediately follows that expectations of various quantities contingent upon price paths would be obtainable by a framework akin to that for quantum systems. This analogy naturally builds up a case for adopting path integrals as a powerful technique for financial modelling. The intimacy is well vindicated by the fact that both the Schrodinger and Black-Scholes equations present themselves as the diffusion equation on appropriate algebraic transformations.

4. Path Integral Formalism: The Mathematics

We shall begin our development of the formalism of path integrals with the framework in the context of the diffusion equation [33] (of which the Schrodinger equation and the Black Scholes equation are special cases) and then carrying on to apply this techniques for the purposes of computing quantities of interest for financial systems. By way of illustrating the technique, we solve the problem of the pricing of contingent claims with the path integral method.

The general theory of path integrals is well documented [33]. We start from the diffusion equation,

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} D \nabla^2 \psi + U \psi \tag{1}$$

which can be written in the form,

$$\frac{\partial \psi}{\partial t} = (A + B)\psi \tag{2}$$

with the substitution $A \equiv \frac{1}{2} D \nabla^2$ and $B \equiv U$. Both A, B are simple operators, although their sum need not necessarily be a simple operator.

The solution of eq. (2) can be written as,

$$\psi(t) = \exp\left[t(A+B)\right]\psi(0) = \kappa(t)\psi(0) \tag{3}$$

Trotter's formula allows us to write,

$$\exp\left[t(A+B)\right] = \lim_{n\to\infty}\left[\exp(tA/n)\exp(tB/n)\right]^n \tag{4}$$

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Substituting expressions for A & B as $A \equiv \frac{1}{2} D \nabla^2$ and $B \equiv U$ respectively in eq. (4), we obtain, on confining ourselves to one space dimension,

$$\mathrm{K} \equiv \exp\left[t\left(\frac{1}{2} D \frac{d^2}{dx^2} + U \right) \right] = \lim_{n\to\infty} \left[\exp\left(\frac{t}{n} U \right) \exp\left(\frac{t}{n} \frac{D}{2} \frac{d^2}{dx^2} \right) \right]^n \tag{5}$$

To evaluate $\exp\left(\frac{t}{n} \frac{D}{2} \frac{d^2}{dx^2} \right)$, we start with the Gaussian integral,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 1 \tag{6}$$

and make the substitution $z = \sqrt{2a}\left(y - \frac{b}{2a} \right)$, $dz = \sqrt{2a}dy$ representing a translation together with a rescaling. Substitution in eq. (6) yields,

$$e^{\frac{b^2}{4a}} = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} dy e^{-ay^2 + by} \tag{7}$$

whence, by setting, $a = \frac{1}{2D}$, $b = \sqrt{\frac{t}{n}} \frac{d}{dx}$, we obtain,

$$\exp\left[\frac{t}{n} \frac{D}{2} \frac{d^2}{dx^2} \right] = \frac{1}{\sqrt{2\pi D}} \int dy \exp\left[-\frac{1}{2D} y^2 + \sqrt{\frac{t}{n}} y \frac{d}{dx} \right] \tag{8}$$

The maneuver using the Gaussian integral has enabled us to replace the second order derivative in the exponent of the integrand by a first order derivative, that operates as a translation generator, while adding a Gaussian integral.

Eq. (8) is of the form,

$$\int \omega p(\omega) d\omega = \langle \omega \rangle \tag{9}$$

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where $\omega = \exp\left(\sqrt{\frac{t}{n}}y\frac{d}{dx}\right)$ is distributed as a Gaussian with probability density,

$$p(\omega)d\omega = \frac{1}{\sqrt{2\pi D}}\exp\left(-\frac{y^2}{2D}\right)dy \tag{10}$$

so that,

$$\exp\left[\frac{t}{n}\frac{D}{2}\frac{d^2}{dx^2}\right] = \left\langle \exp\left(\sqrt{\frac{t}{n}}y\frac{d}{dx}\right)\right\rangle \tag{11}$$

Let us, now, return to the the expression $\left[\exp\left(\frac{t}{n}\frac{D}{2}\frac{d^2}{dx^2}\right)\right]^n$ of eq. (5). For each one of these n factors, $\exp\left(\frac{t}{n}\frac{D}{2}\frac{d^2}{dx^2}\right)$, we introduce an independent Gaussian random variable y_j with the corresponding probability density $p(\omega_j)d\omega_j = \frac{1}{\sqrt{2\pi D}}\exp\left(-\frac{y_j^2}{2D}\right)dy_j$ with $j = 1, 2, ..., n$. Making these substitutions in eq. (5), and using eq. (11), we obtain,

$$\mathrm{K} = \lim_{n\to\infty}\prod_{j=1}^{n}\left\langle \exp\left(\frac{t}{n}U\right)\exp\left(\sqrt{\frac{t}{n}}y_j\frac{d}{dx}\right)\right\rangle \tag{12}$$

The following observations on eq. (12) are appropriate:

(a) The expectation bracket stands for all n expectation values or Gaussian integrals;
(b) In each term of the product, the lower indexed y_j's are written to the right, by convention;
(c) The product in eq. (12) is an ordered product where a factor with $\frac{d}{dx}$ is followed by a factor with U(x), then followed by a factor with $\frac{d}{dx}$ etc.

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Since $\frac{d}{dx}$ is the generator of infinitesimal spatial translations i.e. $\exp\left(u\frac{d}{dx}\right)\exp\left[f(x)\right]=\exp\left[f(x+u)\right]\exp\left(u\frac{d}{dx}\right)$, we can use it to bring all the $\frac{d}{dx}$ terms in eq. (12) to the right. We have, on applying this identity to the terms $j=1,2$ in eq. (12),

$$
\begin{aligned}
&\exp\left[\frac{t}{n}U(x)\right]\exp\left[\sqrt{\frac{t}{n}}y_2\frac{d}{dx}\right]\exp\left[\frac{t}{n}U(x)\right]\exp\left[\sqrt{\frac{t}{n}}y_1\frac{d}{dx}\right]\\
&=\exp\left[\frac{t}{n}U(x)\right]\exp\left[\frac{t}{n}U\left(x+\sqrt{\frac{t}{n}}y_2\right)\right]\exp\left[\sqrt{\frac{t}{n}}y_2\frac{d}{dx}\right]\exp\left[\sqrt{\frac{t}{n}}y_1\frac{d}{dx}\right]\\
&=\exp\left[\frac{t}{n}U(x)+\frac{t}{n}U\left(x+\sqrt{\frac{t}{n}}y_2\right)\right]\exp\left[\left(\sqrt{\frac{t}{n}}y_2+\sqrt{\frac{t}{n}}y_1\right)\frac{d}{dx}\right]
\end{aligned}
\tag{13}
$$

Proceeding iteratively, we obtain:

$$
\mathrm{K}=\lim_{n\to\infty}\left\langle\exp\left(\frac{t}{n}\sum_{k=1}^{n}U\left(x+\sqrt{\frac{t}{n}}\sum_{j=k}^{n}y_{j+1}\right)\right)\exp\left(\sqrt{\frac{t}{n}}\sum_{k=1}^{n}y_k\frac{d}{dx}\right)\right\rangle \tag{14}
$$

with $y_{n+1}\equiv 0$.

Now, $z_j=\sqrt{2a}\left(y_j-\frac{b}{2a}\right)$, $a=\frac{1}{2D}$, $b=\sqrt{\frac{t}{n}}\frac{d}{dx}$ so that $y_j=z_j\sqrt{D}+D\sqrt{\frac{t}{n}}\frac{d}{dx}$ and each z_j is a standard Gaussian variate with mean zero and variance unity whence in the limit $n\to\infty$, $\left\langle y_j\right\rangle\to 0$ and $\left\langle y_j^2\right\rangle\to D$. $\sum_{k=1}^{n}y_k$ is a sum of n independent, identically distributed random variables, so that, by the Central Limit Theorem, it converges to the Gaussian distribution with mean zero and variance nD. Thus, the expression,

$$
W_k=-\sqrt{\frac{t}{n}}\sum_{j=1}^{k}y_j \tag{15}
$$

forms a Brownian motion, whence, for the Brownian motion increment $dW=W_k-W_{k-1}$, we have,

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$$\left\langle \left(dW\right)^2 \right\rangle = \left\langle \left(-\sqrt{\frac{t}{n}}y_k\right)^2 \right\rangle = \frac{t}{n}D = Ddt \tag{16}$$

In terms of W_k, we can write eq. (14) as,

$$\mathrm{K} = \lim_{n\to\infty}\left\langle \exp\left(\frac{t}{n}\sum_{k=1}^{n}U\left(x+W_k-W_n\right)\right)\exp\left(-W_n\frac{d}{dx}\right)\right\rangle \tag{17}$$

This is the operator form of the propagator. To obtain the expression for the kernel of this operator, i.e. the function $\mathrm{K}\left(x,t;y\right)$ that is a solution of eq. (1) in the form,

$$\psi_t\left(x\right) = \int dy\mathrm{K}\left(x,t;y\right)\psi_0\left(y\right) \tag{18}$$

We have,

$$\begin{aligned}
\psi_t\left(x\right) &= \left(\mathrm{K}\psi_0\right)\left(x\right)\\
&= \lim_{n\to\infty}\left\langle \exp\left[\frac{t}{n}\sum_{k=1}^{n}U\left(x+W_k-W_n\right)\right]\exp\left(-W_n\frac{d}{dx}\right)\psi_0\left(x\right)\right\rangle\\
&= \lim_{n\to\infty}\left\langle \exp\left[\frac{t}{n}\sum_{k=1}^{n}U\left(x+W_k-W_n\right)\right]\psi_0\left(x-W_n\right)\right\rangle\\
&= \int d\theta \lim_{n\to\infty}\left\langle \exp\left[\frac{t}{n}\sum_{k=1}^{n}U\left(\theta+W_k\right)\right]\delta\left(x-W_n-\theta\right)\right\rangle\psi_0\left(\theta\right) \text{ where } \theta = x-W_n
\end{aligned} \tag{19}$$

From eqs. (18) & (19), we get,

$$\mathrm{K}\left(x,t;y\right) = \lim_{n\to\infty}\left\langle \exp\left[\frac{t}{n}\sum_{k=1}^{n}U\left(y+W_k\right)\right]\delta\left(x-W_n-y\right)\right\rangle \tag{20}$$

In the continuum limit of Brownian motion, we can assume $W\left(s\right)$ as a Brownian motion path initiating at y i.e. $W\left(0\right)=y$ whence its value at $s=kt/n$ is $\left(y+W_k\right)$. Eq. (20), then, becomes,

$$\mathrm{K}\left(x,t;y\right) = \left\langle \exp\left[\int_0^t U\left(W\left(s\right)\right)ds\right]\delta\left(x-W\left(t\right)\right)\right\rangle$$

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$$= E_t\left\{\exp\left[\int_0^t U\left(W(s)\right)ds\right]\delta\left(x - W(t)\right)\right\} \text{ with } W(0) = y$$
$$= E_{xt}\left(\exp\left(\int_0^t U\left(W(s)\right)ds\right)\right) \qquad (21)$$

where $E_{xt}(\zeta) = E_t\left(\zeta\delta\left(x - W(t)\right)\right)$. This the Feynman Kac formula.

The expectation in eq. (21) is with respect to continuous time Brownian motion with $(dW)^2 = Ddt$.

5. Stochastic Process of Stock Prices

It is conventional to model the stock price movements over an infinitesimal time interval $(t, t+dt)$ by the stochastic differential equation representing geometric Brownian motion [34],

$$dS(t) = \mu S(t)dt + \sigma S(t)dW \qquad (22)$$

where μ is the expected drift rate (return) and σ is the volatility of the stock price at time t and dW is the standard Brownian motion increment over the interval $(t, t+dt)$. This increment is normally distributed with mean 0 and variance dt and can, therefore, be expressed as $dW = z\sqrt{dt}$ where z is the standard normal variate. This equation is an Ito process where the coefficients of dt and dW are proportional to the instantaneous stock price S. it is emphasized that this model of stock prices holds over infinitesimal time intervals $(t, t+dt)$.

This model assumes that the instantaneous percentage return $dS(t)/S(t)$ on a stock is a function of the drift rate μ and volatility σ of the stock price. The drift rate and volatility are, themselves, constant over the infinitesimal time interval $(t, t+dt)$. Thus, the expected percentage return (μ) required by investors over this infinitesimal interval from a stock is independent of the stock's price. If investors require a certain expected return over $(t, t+dt)$ when the stock price is S_1, then, ceteris paribus, they will also require the same expected return when it is S_2. It follows that the expected drift rate of the stock price over $(t, t+dt)$ is $\mu S(t)$ and the corresponding expected change in stock price over this infinitesimal interval is $\mu S(t)dt$. As to the variance of the process, the model assumes that the variability (σ^2) of the percentage return in $(t, t+dt)$ is constant and independent of the stock price i.e. an investor is just as uncertain of the percentage return when

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the stock price is S_1 as when it is S_2. It follows that the standard deviation of the change in stock price in $(t, t+dt)$ is $\sigma S(t)$.

The corresponding price process over finite time intervals can be easily obtained by the Ito equation. We have, by setting, $G \equiv \xi = \ln S$, $\frac{\partial \xi}{\partial S} = \frac{1}{S}, \frac{\partial^2 \xi}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial \xi}{\partial t} = 0$, $dS(t) = \mu S(t)dt + \sigma S(t)dW$ in the Ito equation [34]

$dG = \left(a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 G}{\partial x^2} \right)dt + b\frac{\partial G}{\partial x}dW$ for $dx = a(x,t)dt + b(x,t)dW$ that

$$d\xi = \left(\mu - \frac{\sigma^2}{2} \right)dt + \sigma dW \tag{23}$$

whence $d\xi \sim \mathbf{N}\left[\left(\mu - \frac{\sigma^2}{2} \right)dt, \sigma^2 dt \right]$.

Equivalently, $\ln S_T \sim \mathbf{N}\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right)T, \sigma^2 T \right]$ thereby showing that the stock prices follow a lognormal distribution with $\ln S_T$ being normally distributed with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right)T$ and variance $\sigma^2 T$.

The probability density function of a lognormal distribution is given by,

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma} \right)^2 \right], x > 0 \tag{24}$$

where $\ln x \sim \mathbf{N}(\mu, \sigma^2)$.

6. Fokker Planck Equation & Transition Probabilities

We, now, derive the Fokker Planck equation [35-37] satisfied by the transition probabilities $p(S,t|S',t')$ corresponding to the stochastic process of eq. (22). In line with the prescription of the Efficient Market Hypothesis [38-40], we assume that the stock prices follow a Markov process, so that, for $t''' > t'' > t'$,

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$$P\left(S''',t'''\middle|S'',t''\wedge S',t'\right)=P\left(S''',t'''\middle|S'',t''\right) \tag{25}$$

Now, by the Chapman Kolmogorov eq., that holds for Markov processes,

$$P\left(S'',t''\middle|S',t'\right)=\int P\left(S'',t''\middle|S,t\right)P\left(S,t\middle|S',t'\right)dS \tag{26}$$

Since, $\frac{\partial P\left(S^*,t\middle|S,0\right)}{\partial t}=\lim_{\Delta t\to 0}\frac{P\left(S^*,t+\Delta t\middle|S,0\right)-P\left(S^*,t\middle|S,0\right)}{\Delta t}$, we have, for any differentiable function $\rho\left(S^*\right)$,

$$\int_{-\infty}^{\infty}\rho\left(S^*\right)\frac{\partial P\left(S^*,t\middle|S,0\right)}{\partial t}dS^* = \int_{-\infty}^{\infty}\rho\left(S^*\right)\left[\lim_{\Delta t\to 0}\frac{P\left(S^*,t+\Delta t\middle|S,0\right)-P\left(S^*,t\middle|S,0\right)}{\Delta t}\right]dS^*$$

$$=\lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\rho\left(S^*\right)\left[\frac{P\left(S^*,t+\Delta t\middle|S,0\right)-P\left(S^*,t\middle|S,0\right)}{\Delta t}\right]dS^*$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\begin{array}{l}\int_{-\infty}^{\infty}\rho\left(S^*\right)\int_{-\infty}^{\infty}P\left(S^*,t+\Delta t\middle|S^{**},t\right)P\left(S^{**},t\middle|S,0\right)dS^{**}dS^*\\ -\int_{-\infty}^{\infty}\rho\left(S^*\right)P\left(S^*,t\middle|S,0\right)dS^*\end{array}\right]$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{-\infty}^{\infty}P\left(S^{**},t\middle|S,0\right)\int_{-\infty}^{\infty}P\left(S^*,t+\Delta t\middle|S^{**},t\right)\left(\rho\left(S^*\right)-\rho\left(S^{**}\right)\right)dS^{**}dS^*\right] \tag{27}$$

where we have performed the following operations in arriving at eq. (27),

(i) used the Chapman Kolmogorov eq. to write

$$P\left(S^*,t+\Delta t\middle|S,0\right)=\int_{-\infty}^{\infty}P\left(S^*,t+\Delta t\middle|S^{**},t\right)P\left(S^{**},t\middle|S,0\right)dS^{**}$$

(ii) by using the fact that when $\Delta t\to 0$, $S^*\to S^{**}$ whence

$$\int_{-\infty}^{\infty}\rho\left(S^*\right)P\left(S^*,t\middle|S,0\right)dS^*=\int_{-\infty}^{\infty}\rho\left(S^{**}\right)P\left(S^{**},t\middle|S,0\right)dS^{**}$$

(iii) used the fact that $\int_{-\infty}^{\infty}P\left(S^*,t+\Delta t\middle|S^{**},t\right)dS^*=1$

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$$\int_{-\infty}^{\infty}\rho\left(S^{**}\right)P\left(S^{**},t\middle|S,0\right)dS^{**}=\int_{-\infty}^{\infty}P\left(S^{**},t\middle|S,0\right)\int_{-\infty}^{\infty}P\left(S^{*},t+\Delta t\middle|S^{**},t\right)\rho\left(S^{**}\right)dS^{*}dS^{**}$$

By a Taylor's expansion, we can write, $\rho\left(S^{*}\right)=\rho\left(S^{**}\right)+\sum_{n=1}^{\infty}\rho^{n}\left(S^{**}\right)\frac{\left(S^{*}-S^{**}\right)^{n}}{n!}$ so that eq. (27) takes the form,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{-\infty}^{\infty}P\left(S^{**},t\middle|S,0\right)\int_{-\infty}^{\infty}P\left(S^{*},t+\Delta t\middle|S^{**},t\right)\left(\sum_{n=1}^{\infty}\rho^{n}\left(S^{**}\right)\frac{\left(S^{*}-S^{**}\right)^{n}}{n!}\right)dS^{**}dS^{*}\right]$$

$$=\int_{-\infty}^{\infty}P\left(S^{**},t\middle|S,0\right)\sum_{n=1}^{\infty}D^{(n)}\left(S^{**}\right)\rho^{(n)}\left(S^{**}\right)dS^{**} \tag{28}$$

where $D^{(n)}\left(S^{**}\right)=\frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}\left(S^{*}-S^{**}\right)^{n}P\left(S^{*},t+\Delta t\middle|S^{**},t\right)dS^{*}$ whence

$D^{(n)}\left(S\right)=\frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[S\left(t+\Delta t\right)-S\right]^{n}\right\rangle$ and $D^{(n)}\left(S_{0}\right)=\frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[S\left(t+\Delta t\right)-S\right]^{n}\right\rangle\Big|_{t=0}$

From eqs. (27) & (28), we get

$$\int_{-\infty}^{\infty}\rho\left(S^{*}\right)\frac{\partial P\left(S^{*},t\middle|S,0\right)}{\partial t}dS^{*}=\int_{-\infty}^{\infty}P\left(S^{**},t\middle|S,0\right)\sum_{n=1}^{\infty}D^{(n)}\left(S^{**}\right)\rho^{(n)}\left(S^{**}\right)dS^{**} \tag{29}$$

For the stock price model of eq. (22), we have $D^{(1)}=\mu S$, $D^{(2)}=\frac{1}{2}\sigma^{2}S^{2}$ and $D^{(n>2)}=0$. Eq. (29), therefore, reduces to,

$$\begin{aligned}\int_{-\infty}^{\infty}\rho\left(S^{*}\right)\frac{\partial P\left(S^{*},t\middle|S,0\right)}{\partial t}dS^{*}&=\int_{-\infty}^{\infty}\left(\mu S^{**}\right)P\left(S^{**},t\middle|S,0\right)\rho^{(1)}\left(S^{**}\right)dS^{**}+\\ &\int_{-\infty}^{\infty}\left(\frac{1}{2}\sigma^{2}S^{**2}\right)P\left(S^{**},t\middle|S,0\right)\rho^{(2)}\left(S^{**}\right)dS^{**}\end{aligned} \tag{30}$$

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Integrating by parts, the terms on the RHS, making use of the appropriate boundary conditions on $\rho(S)$, we obtain,

$$\int_{-\infty}^{\infty}\left(\mu S^{**}\right)P\left(S^{**},t\middle|S,0\right)\rho^{(1)}\left(S^{**}\right)dS^{**}=-\int_{-\infty}^{\infty}\frac{\partial}{\partial S^{**}}\left[\left(\mu S^{**}\right)P\left(S^{**},t\middle|S,0\right)\right]\rho\left(S^{**}\right)dS^{**}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{1}{2}\sigma^2 S^{**2}\right)P\left(S^{**},t\middle|S,0\right)\rho^{(2)}\left(S^{**}\right)dS^{**}\\&=\int_{-\infty}^{\infty}\frac{\partial^2}{\partial S^{**2}}\left[\left(\frac{1}{2}\sigma^2 S^{**2}\right)P\left(S^{**},t\middle|S,0\right)\right]\rho\left(S^{**}\right)dS^{**}\end{aligned} \qquad (31)$$

Since the function $\rho(S)$ is arbitrary, it follows from eq. (30) & (31) that,

$$\frac{\partial P\left(S^{**},t\middle|S,0\right)}{\partial t}=-\frac{\partial}{\partial S^{**}}\left[\left(\mu S^{**}\right)P\left(S^{**},t\middle|S,0\right)\right]+\frac{\partial^2}{\partial S^{**2}}\left[\left(\frac{1}{2}\sigma^2 S^{**2}\right)P\left(S^{**},t\middle|S,0\right)\right] \qquad (32)$$

using $\int_{-\infty}^{\infty}\rho\left(S^{*}\right)\frac{\partial P\left(S^{*},t\middle|S,0\right)}{\partial t}dS^{*}=\int_{-\infty}^{\infty}\rho\left(S^{**}\right)\frac{\partial P\left(S^{**},t\middle|S,0\right)}{\partial t}dS^{**}$.

This is the Fokker Planck equation for the stochastic process (22). In general, eq. (32) has the form,

$$\frac{\partial}{\partial t}p\left(S,t\middle|S',t'\right)=-\frac{\partial}{\partial S}\left[\mu S(t)\,p\left(S,t\middle|S',t'\right)\right]+\frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2(t)\,p\left(S,t\middle|S',t'\right)\right] \text{ or}$$

$$\frac{\partial}{\partial t}p=\left(\sigma^2-\mu\right)p+\left(2\sigma^2-\mu\right)S\frac{\partial p}{\partial S}+\frac{1}{2}\left(\sigma S\right)^2\frac{\partial^2 p}{\partial S^2} \qquad (33)$$

with the boundary conditions,

$$t=t':\ p\left(S,t\middle|S',t'\right)=\delta\left(S-S'\right) \qquad (34a)$$

$$S=0:\ p\left(0,t\middle|S',t'\right)=0 \qquad (34b)$$

$$S\to\infty:\ p\left(S,t\middle|S',t'\right)\to 0 \qquad (34c)$$

Justification of the boundary conditions follows from (i) at $t=t'$, stock price $S=S'$, (ii) if the stock price vanishes at any point in time, it stays zero thereafter & vice versa and, and on the other hand, if $S(0)>0$, by assumption, it can never become zero at any later time so that

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$p(0,t|S',t')=0$ essentially for $S=0$ and (iii) the stock price cannot increase unboundedly in a finite time interval.

To solve eq. (33) subject to the boundary conditions (34), we make the following substitutions,

$$p(S,t|S',t')=\frac{1}{S'}f(x,\tau) \tag{35a}$$
$$S=S'e^{x} \tag{35b}$$
$$t=t'+\frac{\tau}{(\sigma^2/2)} \tag{35c}$$

whence we get

$$\frac{\partial f}{\partial\tau}=\frac{\partial^2 f}{\partial x^2}+[3-k]\frac{\partial f}{\partial x}+[2-k]f \text{ with } k=\frac{2\mu}{\sigma^2} \tag{36}$$

and boundary conditions

$$\tau=0: f(x,0)=\delta(e^{x}-1) \tag{37a}$$
$$x\to-\infty: f(x,\tau)\to 0 \tag{37b}$$
$$x\to+\infty: f(x,\tau)\to 0 \tag{37c}$$

To convert eq. (36) to a diffusion eq. we make a second substitution,

$$f(x,\tau)=e^{\alpha x+\beta\tau}g(x,\tau) \tag{38a}$$
$$\alpha=\frac{1}{2}(k-3) \tag{38b}$$
$$\beta=\alpha^2+(3-k)\alpha+2-k=-\frac{1}{4}(k-1)^2 \tag{38c}$$

Substitution from eq. (38) in eq. (36) and simplification yields the diffusion eq.

$$Lg(x,\tau)\equiv\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)g(x,\tau)=0 \tag{39}$$

with the boundary conditions,

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$$\tau = 0 : g(x,0) = e^{-(k-3)x/2}\delta(e^x - 1) \tag{40a}$$

$$\tau = 0 : g(x,0)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0(\alpha > 0) \tag{40b}$$

$$\tau > 0 : g(x,\tau)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0(\alpha > 0) \tag{40c}$$

To solve eq. (40), we consider the general diffusion eq.

$$Lg(x,\tau) = f(x,\tau) \tag{41}$$

Let the Green function for the problem be

$$G(x,\tau;x',\tau') \tag{42}$$

satisfying

$$LG(x,\tau;x',\tau') = \delta(x-x')\delta(\tau-\tau') \tag{43}$$

We, then, have, from the theory of Green's functions,

$$\begin{aligned} Lg(x,\tau) &= f(x,\tau) = \iint dx'dt'\delta(x-x')\delta(t-t')f(x',t') \\ &= \iint dx'dt' LG(x,\tau;x',\tau')f(x',t') = L\iint dx'dt' G(x,\tau;x',\tau')f(x',t') \\ g(x,\tau) &= \iint dx'dt' G(x,\tau;x',\tau')f(x',t') \end{aligned} \tag{44}$$

where the penultimate step follows from the linearity of L and the fact that its domain of action is on (x,τ) and not (x',τ'). The final step follows from the completeness of the basis states. From eq. (43), we have

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial x^2}\right)G(x,\tau;x',0) = \delta(x-x')\delta(\tau) \tag{45}$$

Taking Laplace transform in $t\,space$ and writing $\mathrm{L}\left[G(x,\tau;x',0)\right] = \tilde{G}(x,s;x',0)$, we have $\mathrm{L}\left[\frac{\partial G(x,\tau;x',0)}{\partial t}\right] = s\tilde{G}(x,s;x',0)$ so that eq. (45) reduces to,

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$$\left(s-\frac{\partial^2}{\partial x^2}\right)\tilde{G}\left(x,s;x',0\right)=\delta\left(x-x'\right) \text{ since } \mathrm{L}\left[\delta\left(\tau\right)\right]=1. \quad (46)$$

Now, taking the Fourier transform in $x\,space$, we get

$$\tilde{G}\left(x,s;x',0\right)=\frac{1}{2\pi}\int dk\tilde{F}\left(k,s;x',0\right)e^{ikx} \quad (47)$$

with the inversion

$$\tilde{F}\left(k,s;x',0\right)=\int dx\tilde{G}\left(x,s;x',0\right)e^{-ikx} \quad (48)$$

whence eq. (46) reduces to

$$\left(s+k^2\right)\tilde{F}\left(k,s;x',0\right)=e^{-ikx'} \quad (49)$$

as $\mathrm{F}\left[\frac{\partial^2\tilde{G}\left(x,s;x',0\right)}{\partial x^2}\right]=-k^2\tilde{F}\left(k,s;x',0\right)$ and $\mathrm{F}\left[\delta\left(x-x'\right)\right]=e^{-ikx'}$.

From eq. (49) $\tilde{F}\left(k,s;x',0\right)=\frac{1}{\left(s+k^2\right)}e^{-ikx'}$ whence, inverting the Laplace transform, we get,

$$F\left(k,\tau;x',0\right)=e^{-k^2\tau}e^{-ikx'} \quad (50)$$

Inversion of the Fourier transform yields,

$$G\left(x,\tau;x',0\right)=\frac{1}{2\pi}\int dkF\left(k,\tau;x',0\right)e^{ikx}$$

$$=\frac{1}{2\pi}\int dke^{ik\left(x-x'\right)-k^2\tau}=\frac{1}{2\pi}\int dke^{-\tau\left[k^2-ik\frac{\left(x-x'\right)}{\tau}\right]}=\frac{e^{-\frac{\left(x-x'\right)^2}{4\tau}}}{2\pi}\int dke^{-\tau\left[k-i\frac{\left(x-x'\right)}{2\tau}\right]^2}=\frac{e^{-\frac{\left(x-x'\right)^2}{4\tau}}}{2\pi}\left(\frac{\pi}{\tau}\right)^{1/2}$$

$$=\frac{e^{-\frac{\left(x-x'\right)^2}{4\tau}}}{\sqrt{4\pi\tau}} \quad (51)$$

Using the Green function obtained above, we can obtain the final solution as,

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$$g(x,\tau)=\int_{-\infty}^{+\infty}dx'G(x,\tau;x',0)g(x',0)=\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{+\infty}dx'e^{-\frac{(x-x')^2}{4\tau}}g(x',0)$$

$$\overset{u=e^y}{=}\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{+\infty}du\delta(u-1)\frac{e^{-\frac{1}{2}(k-3)\ln u}}{u}\exp\left[-\frac{(x-\ln u)^2}{4\tau}\right]=\frac{1}{\sqrt{4\pi\tau}}\exp\left(-\frac{x^2}{4\tau}\right) \tag{52}$$

Substituting for $g(x,\tau)$ in eq. (38), we obtain,

$$f(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\exp\left[\frac{1}{2}(k-3)x-\frac{1}{4}(k-1)^2\tau\right]\exp\left(-\frac{x^2}{4\tau}\right)$$

$$=\frac{e^{-x}}{\sqrt{4\pi\tau}}\exp\left\{-\frac{\left[x-(k-1)\tau\right]^2}{4\tau}\right\} \tag{53}$$

Again, from eq.(35), using $k=\frac{2\mu}{\sigma^2}$, so that, $(k-1)\tau=\left(\mu-\sigma^2/2\right)(t-t')$ and, on simplification, we obtain,

$$p(S,t|S',t')=\frac{1}{\sqrt{2\pi(\sigma S)^2(t-t')}}\exp\left\{-\frac{\left[\ln\left(\frac{S}{S'}\right)-\left(\mu-\frac{\sigma^2}{2}\right)(t-t')\right]^2}{2\sigma^2(t-t')}\right\} \tag{54}$$

In terms of $\xi=\ln S$, the transition probability takes a particularly simple form as,

$$p(\xi,t|\xi',t')=\frac{1}{\sqrt{2\pi\sigma^2(t-t')}}\exp\left\{-\frac{\left[(\xi-\xi')-\left(\mu-\frac{\sigma^2}{2}\right)(t-t')\right]^2}{2\sigma^2(t-t')}\right\} \tag{55}$$

7. Finite-time Transition Probabilities and the Path Integral

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Using the expression for the transition probability given by eq. (55), we can write the probability of a finite transition of stock price from (S',t') to (S'',t'') as,

$$p(\xi'',t''|\xi',t') = \int d\xi \, p(\xi'',t''|\xi,t)\, p(\xi,t|\xi',t') \tag{56}$$

with $\xi'' = \ln S''$ and $\xi' = \ln S'$. The integral represents summation of contributions of probabilities of the transition from (S',t') to (S'',t'') by all possible intermediate paths from (S',t') to (S'',t'') i.e. $(S',t') \to (S,t) \to (S'',t'')$.

Using eq. (55), we can write eq. (56) as,

$$p(\xi'',t''|\xi',t') = \frac{1}{(2\pi\sigma^2)\sqrt{(t''-t)(t-t')}} \times \int d\xi \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\left((\xi''-\xi)-\left(\mu-\frac{\sigma^2}{2}\right)(t''-t)\right)^2}{(t''-t)} + \frac{\left((\xi-\xi')-\left(\mu-\frac{\sigma^2}{2}\right)(t-t')\right)^2}{(t-t')}\right]\right\} \tag{57}$$

Partitioning the time interval (t',t'') into $(n+1)$ sub-intervals each of length $\Delta t = \frac{t''-t'}{n+1}$, we can write, in a manner analogous to eq. (57),

$$\begin{aligned} p(\xi'',t''|\xi',t') &= \int_{-\infty}^{+\infty} d\xi_1 \ldots d\xi_n \, p(\xi'',t''|\xi_n,t_n)\, p(\xi_n,t_n|\xi_{n-1},t_{n-1}) \ldots p(\xi_1,t_1|\xi',t') \\ &= \frac{1}{(2\pi\sigma^2\Delta t)^{(n+1)/2}} \int_{-\infty}^{+\infty} d\xi_1 \ldots d\xi_n \exp\left\{-\frac{1}{2\sigma^2\Delta t}\sum_{k=1}^{n+1}\left[(\xi_k-\xi_{k-1})-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right]^2\right\} \\ &= \frac{1}{(2\pi\sigma^2\Delta t)^{1/2}} \int_{-\infty}^{+\infty} d\left[(2\pi\sigma^2\Delta t)^{-1/2}\xi_1\right] \ldots d\left[(2\pi\sigma^2\Delta t)^{-1/2}\xi_n\right] \times \\ &\quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{n+1}\left[\frac{(\xi_k-\xi_{k-1})}{\Delta t}-\left(\mu-\frac{\sigma^2}{2}\right)\right]^2 \Delta t\right\} \end{aligned} \tag{58}$$

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Now, in the limit $n \to \infty$, $\Delta t \to 0$ with $(n+1)\Delta t = (t''-t')$, we can write $\frac{(\xi_k - \xi_{k-1})}{\Delta t}$ as the differential $\frac{\partial \tilde{\xi}}{\partial t} = \dot{\tilde{\xi}}$. This enables us to identify the equivalence $\frac{1}{2\sigma^2}\dot{\tilde{\xi}}$ with the kinetic energy term of a Lagrangian,

$$\mathbf{L}\left(\dot{\tilde{\xi}}, \tilde{\xi}, t\right) = \frac{1}{2\sigma^2}\left[\dot{\tilde{\xi}} - \left(\mu - \frac{\sigma^2}{2}\right)\right]^2 \tag{59}$$

In terms of this Lagrangian, the finite time transition probability takes the form,

$$p\left(\xi'', t'' \middle| \xi', t'\right) = \frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right] \times \exp\left\{-\int_{t'}^{t''} \mathbf{L}\left(\dot{\tilde{\xi}}(\tau), \tilde{\xi}(\tau), \tau\right) d\tau\right\} \tag{60}$$

It may be noted that the coupled term in the Lagrangian viz. $\frac{1}{\sigma^2}\dot{\tilde{\xi}}\left(\mu - \frac{\sigma^2}{2}\right)$ is independent of the paths and hence, can be taken outside the path integral. This is shown below:

$$\begin{aligned} &\frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\int_{t'}^{t''} \dot{\tilde{\xi}}\, d\tau = \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\int_{t'}^{t''} d\tilde{\xi} = \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)(\xi'' - \xi') \\ &-\frac{1}{2}\left(\mu - \frac{\sigma^2}{2}\right)\int_{t'}^{t''} \frac{d^2}{d\xi^2}\xi(\tau)\, d\tau = \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)(\xi'' - \xi') \end{aligned} \tag{61}$$

We can, thus, write eq. (60) as,

$$\begin{aligned} p\left(\xi'', t'' \middle| \xi', t'\right) = &\frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \exp\left[\frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)(\xi'' - \xi')\right] \times \\ &\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right] \exp\left\{-\frac{1}{2\sigma^2}\int_{t'}^{t''}\left[\dot{\tilde{\xi}}^2 + \left(\mu - \frac{\sigma^2}{2}\right)^2\right] d\tau\right\} \end{aligned} \tag{62}$$

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$$
\begin{aligned}
&= \frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \exp\left[\frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\left(\xi'' - \xi'\right)\right] \times \\
&\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right] \exp\left\{-\frac{1}{2\sigma^2}\left[\left(\int_{t'}^{t''}\dot{\tilde{\xi}}^2 d\tau\right) + \left(\mu - \frac{\sigma^2}{2}\right)^2 \left(t'' - t'\right)\right]\right\} \\
&= \frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \exp\left\{\left[\frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)\left(\xi'' - \xi'\right)\right] + \left[-\frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2 \left(t'' - t'\right)\right]\right\} \times \\
&\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right] \exp\left[-\frac{1}{2\sigma^2}\left(\int_{t'}^{t''}\dot{\tilde{\xi}}^2 d\tau\right)\right]
\end{aligned}
\tag{63}
$$

8. Evaluation of the Path Integral

To evaluate the path integral (63), we consider the path integral

$$
\begin{aligned}
&\frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right] \exp\left[-\frac{1}{2\sigma^2}\int_{t'}^{t''}\dot{\tilde{\xi}}^2 d\tau\right] \\
&= \lim_{N\to\infty} \frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \prod_{n=1}^{N} d\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}_n\right] \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N}\left[\frac{\left(\tilde{\xi}_{n+1} - \tilde{\xi}_n\right)}{\Delta t}\right]^2 \Delta t\right\} \\
&= \lim_{N\to\infty} \frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \prod_{n=1}^{N} d\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}_n\right] \exp\left[-\frac{1}{2\sigma^2\Delta t}\sum_{n=0}^{N}\left(\tilde{\xi}_{n+1} - \tilde{\xi}_n\right)^2\right]
\end{aligned}
\tag{64}
$$

where the time interval $\left(t'' - t'\right) = \left(N+1\right)\Delta t$. The next step in the simplification involves the use of the following identity,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\xi_n \exp\left[-\alpha\left(\xi_{n+a} - \xi_n\right)^2\right] \exp\left[-\beta\left(\xi_n - \xi_{n-b}\right)^2\right] \\
&= \int_{-\infty}^{\infty} d\xi_n \exp\left[-\left(\alpha+\beta\right)\left(\xi_n - \frac{\alpha\xi_{n+a} + \beta\xi_{n-b}}{\alpha+\beta}\right)^2 - \frac{\alpha\beta}{\alpha+\beta}\left(\xi_{n+a} - \xi_{n-b}\right)^2\right] \\
&= \exp\left[-\frac{\alpha\beta}{\alpha+\beta}\left(\xi_{n+a} - \xi_{n-b}\right)^2\right] \sqrt{\frac{\pi}{\alpha+\beta}}
\end{aligned}
\tag{65}
$$

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For $n=1$, using $\alpha=\beta=\dfrac{1}{2\sigma^2\Delta t}$, $a=b=1$, we get,

$$\int_{-\infty}^{\infty} d\tilde{\xi}_1 \exp\left[-\frac{1}{2\sigma^2\Delta t}\left(\tilde{\xi}_1-\tilde{\xi}_0\right)^2\right]\exp\left[-\frac{1}{2\sigma^2\Delta t}\left(\tilde{\xi}_2-\tilde{\xi}_1\right)^2\right]$$
$$=\frac{1}{\sqrt{2}}\left(\frac{1}{2\pi\sigma^2\Delta t}\right)^{-1/2}\exp\left[\frac{1}{2\Delta t}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_2-\tilde{\xi}_0\right)^2\right] \quad (66)$$

so that,

$$\int_{-\infty}^{\infty} d\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right]\exp\left[-\frac{1}{2\sigma^2\Delta t}\left(\tilde{\xi}_1-\tilde{\xi}_0\right)^2\right]\exp\left[-\frac{1}{2\sigma^2\Delta t}\left(\tilde{\xi}_2-\tilde{\xi}_1\right)^2\right]$$
$$=\sqrt{\frac{1}{2}}\exp\left[\frac{1}{2\Delta t}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_2-\tilde{\xi}_0\right)^2\right] \quad (67)$$

Proceeding iteratively,

$$\sqrt{\frac{1}{2}}\int_{-\infty}^{\infty} d\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}_2\right]\exp\left[\frac{1}{2\Delta t}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_2-\tilde{\xi}_0\right)^2\right]\exp\left[-\frac{1}{2\sigma^2\Delta t}\left(\tilde{\xi}_3-\tilde{\xi}_2\right)^2\right]$$

$$=\frac{1}{\sqrt{3}}\exp\left[\frac{1}{3\Delta t}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_3-\tilde{\xi}_0\right)^2\right] \quad (68)$$

Performing all the N integrals, we obtain,

$$\frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}}\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right]\exp\left[-\frac{1}{2\sigma^2}\int_{t'}^{t''}\dot{\tilde{\xi}}^2 d\tau\right]$$
$$=\frac{1}{\left[2\pi\sigma^2\left(N+1\right)\Delta t\right]^{1/2}}\exp\left[\frac{1}{\left(N+1\right)\Delta t}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_{N+1}-\tilde{\xi}_0\right)^2\right] \quad (69)$$

which on taking the limit $N\to\infty$, $\left(N+1\right)\Delta t=t''-t'$ gives

$$\frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}}\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \boldsymbol{D}\left[\left(2\pi\sigma^2\Delta t\right)^{-1/2}\tilde{\xi}\right]\exp\left[-\frac{1}{2\sigma^2}\int_{t'}^{t''}\dot{\tilde{\xi}}^2 d\tau\right]$$

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$$=\frac{1}{\left[2\pi\sigma^2\left(t''-t'\right)\right]^{1/2}}\exp\left[\frac{1}{\left(t''-t'\right)}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}_{N+1}-\tilde{\xi}_0\right)^2\right]$$

$$=\frac{1}{\left[2\pi\sigma^2\left(t''-t'\right)\right]^{1/2}}\exp\left[\frac{1}{\left(t''-t'\right)}\left(-\frac{1}{2\sigma^2}\right)\left(\tilde{\xi}''-\tilde{\xi}'\right)^2\right] \tag{70}$$

Substituting from eq. (70) in eq. (66), we obtain the expression for the finite time transition probability as,

$$p\left(\xi'',t''\middle|\xi',t'\right)=\frac{1}{\left(2\pi\sigma^2\Delta t\right)^{1/2}}\exp\left\{-\frac{1}{2\sigma^2\left(t''-t'\right)}\left[\left(\xi''-\xi'\right)-\left(\mu-\frac{\sigma^2}{2}\right)\left(t''-t'\right)\right]^2\right\} \tag{71}$$

9. The Ito Lemma

Let $\psi\left(\xi\right)$ be a continuous, twice differentiable function of ξ where ξ follows the stochastic differential eq. (23) i.e. $d\xi=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma dW$. Then, the transformation eqs. are given by,

$$\sigma_\psi=\frac{\partial\psi}{\partial\xi}\sigma_\xi=\frac{\partial\psi}{\partial\xi}\sigma$$

$$\left(\mu-\frac{\sigma^2}{2}\right)_\psi=\frac{\partial\psi}{\partial\xi}\left(\mu-\frac{\sigma^2}{2}\right)_\xi+\frac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{\sigma_\psi{}^2}{2}\right)$$

$$\left(\mu-\frac{\sigma^2}{2}\right)_\psi=\frac{\partial\psi}{\partial\xi}\left(\mu-\frac{\sigma^2}{2}\right)_\xi+\frac{1}{2}\sigma_\psi\frac{\partial\sigma_\psi}{\partial\psi}$$

$$=\frac{\partial\psi}{\partial\xi}\left(\mu-\frac{\sigma^2}{2}\right)+\frac{1}{2}\frac{\partial\xi}{\partial\psi}\left(\frac{\partial\psi}{\partial\xi}\sigma\right)\frac{\partial}{\partial\xi}\left(\frac{\partial\psi}{\partial\xi}\sigma\right)$$

$$=\left[\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial\psi}{\partial\xi}+\frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial\xi^2}\right]\text{ since }\frac{\partial\sigma}{\partial\xi}=0.$$

ISSN 1512-1461

Thus, the variable ψ has the drift rate $\left[\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial\psi}{\partial\xi}+\frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial\xi^2}\right]$ and the variance rate $\sigma_\psi=\frac{\partial\psi}{\partial\xi}\sigma$ and, therefore follows the stochastic process given by the stochastic differential eq.

$$d\psi=\left[\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial\psi}{\partial\xi}+\frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial\xi^2}\right]dt+\sigma\frac{\partial\psi}{\partial\xi}dW \qquad (72)$$

This is, precisely, Ito's Lemma.

10. The Path Integral Solution Satisfies Fokker Planck Equation

The transition probability density of eq. (71) can be abbreviated by setting $t''-t=\Delta t$ and writing it as,

$$p\left(\xi'',t+\Delta t\middle|\xi,t\right)=\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\exp\left\{-\frac{\left[\left(\xi-\xi'\right)-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right]^2}{2\sigma^2\Delta t}\right\} \qquad (73)$$

Also, from eq. (56), we obtain

$$p\left(\xi'',t''+\Delta t\middle|\xi',t'\right)=\int d\xi\, p\left(\xi'',t''+\Delta t\middle|\xi,t''\right)p\left(\xi,t''\middle|\xi',t'\right) \qquad (74)$$

Writing $\xi-\xi''=\theta$, $\xi''=\xi-\theta$ and using eqs. (73) & (74), we get,

$$p\left(\xi'',t''+\Delta t\middle|\xi,t\right)=\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\int d\theta\exp\left\{-\frac{\left[-\theta-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right]^2}{2\sigma^2\Delta t}\right\}p\left(\xi''+\theta,t''\middle|\xi',t'\right) \qquad (75)$$

Expanding the exponential in the integral to first order in Δt , we get,

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$$p\left(\xi'',t''+\Delta t\middle|\xi,t\right)=\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\int d\theta\exp\left\{-\frac{\left[-\theta-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right]^2}{2\sigma^2\Delta t}\right\}p\left(\xi''+\theta,t''\middle|\xi',t'\right)$$

$$\approx\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\int_{-\infty}^{+\infty}\begin{matrix} d\theta\exp\left(-\frac{\theta^2}{2\sigma^2\Delta t}\right)\times\left[1-\frac{\theta}{\sigma^2}\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\theta^2}{\sigma^4}\left(\mu-\frac{\sigma^2}{2}\right)^2....\right]\times \\ \left[1-\frac{1}{2\sigma^2}\left(\mu-\frac{\sigma^2}{2}\right)^2\Delta t+....\right]\times \\ \left\{p\left(\xi'',t''\middle|\xi',t'\right)+\theta\frac{\partial}{\partial\xi}p\left(\xi'',t''\middle|\xi',t'\right)+\frac{\theta^2}{2}\frac{\partial^2}{\partial\xi^2}p\left(\xi'',t''\middle|\xi',t'\right)\right\}\end{matrix} \tag{76}$$

Simplification of eq. (76) gives,

$$p\left(\xi'',t''+\Delta t\middle|\xi',t'\right)\approx p\left(\xi'',t''\middle|\xi',t'\right)-\left[\begin{matrix}\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial p\left(\xi'',t''\middle|\xi',t'\right)}{\partial\xi}- \\ \frac{1}{2}\sigma^2\frac{\partial^2 p\left(\xi'',t''\middle|\xi',t'\right)}{\partial\xi^2}\end{matrix}\right]\Delta t \tag{77}$$

Expanding the LHS of eq. (77) in powers of Δt, we get,

$$p\left(\xi'',t''+\Delta t\middle|\xi',t'\right)\approx p\left(\xi'',t''\middle|\xi',t'\right)+\Delta t\frac{\partial}{\partial t''}p\left(\xi'',t''\middle|\xi',t'\right) \tag{78}$$

From eqs. (77) & (78), we obtain,

$$\frac{\partial}{\partial t}p\left(\xi'',t''\middle|\xi',t'\right)+\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial p\left(\xi'',t''\middle|\xi',t'\right)}{\partial\xi}-\frac{1}{2}\sigma^2\frac{\partial^2 p\left(\xi'',t''\middle|\xi',t'\right)}{\partial\xi^2}=0 \tag{79}$$

which is the Fokker Planck eq. for the given process. Transforming to the original variables, we get the Fokker Planck equation (33).

11. The Black Scholes Equation

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The celebrated Black Scholes equation for the pricing of contingent claims is given by,

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 C(S,t)}{\partial S^2}+rS\frac{\partial C(S,t)}{\partial S}-rC(S,t)=-\frac{\partial C(S,t)}{\partial t} \tag{80}$$

with the terminal condition $C(S'',t'')=\max(0,S''-K)$ where the symbols have their usual meaning. In terms of $\xi=\ln S$, eq. (80) takes the form,

$$\frac{\sigma^2}{2}\frac{\partial^2 C(\xi_t,t)}{\partial \xi^2}+\left(r-\frac{\sigma^2}{2}\right)\frac{\partial C(\xi_t,t)}{\partial \xi}-rC(\xi_t,t)=-\frac{\partial C(\xi_t,t)}{\partial t} \tag{81}$$

The solution to the above eq. takes the form,

$$C(\xi',t')=e^{-r(t''-t')}E_{[t',\xi']}\left[C(\xi'',t'')\right] \tag{82}$$

where the probability measure for the expectation value is the risk neutral measure conditioned upon the initial state $\xi'=\ln S'$ at time $t=t'$.

In the path integral formalism, we can write the above solution as,

$$C(\xi',t')=e^{-r(t''-t')}\int_{-\infty}^{+\infty}\left(\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}C(\xi'',t'')\exp\left(-\int_{t'}^{t''}\mathbf{L}dt\right)\boldsymbol{D}\xi(t)\right)d\xi'' \tag{83}$$

where the Lagrangian is given by,

$$\mathbf{L}=\frac{1}{2\sigma^2}\left[\frac{d\xi}{dt}-\left(r-\frac{\sigma^2}{2}\right)\right]^2$$

In terms of the transition probabilities $p(\xi'',t''|\xi',t')$, we can write the solution for $C(\xi',t')$ as,

$$C(\xi',t')=e^{-r(t''-t')}\int_{-\infty}^{+\infty}C(\xi'',t'')\,p(\xi'',t''|\xi',t')\,d\xi''$$

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$$= \frac{e^{-r(t''-t')}}{\sqrt{2\pi\sigma^2 (t''-t')}} \int_{-\infty}^{+\infty} C(\xi'',t'') \exp\left\{ -\frac{\left[(\xi''-\xi') - \left(r - \frac{\sigma^2}{2} \right)(t''-t') \right]^2}{2\sigma^2 (t''-t')} \right\} d\xi''$$

$$= \frac{e^{-r(t''-t')}}{\sqrt{2\pi\sigma^2 (t''-t')}} \int_{-\infty}^{+\infty} \max\left(0, e^{\xi''} - K\right) \exp\left\{ -\frac{\left[(\xi''-\xi') - \left(r - \frac{\sigma^2}{2} \right)(t''-t') \right]^2}{2\sigma^2 (t''-t')} \right\} d\xi'' \quad (84)$$

which can be performed as a Gaussian integral to obtain the Black Scholes formula for a European call option.

12. Conclusion

A comprehensive analysis of stock market price patterns has been empirically examined in [41-45]. A phenomenological study [45] reported that the tails of probability distributions of returns arising from stock price fluctuations of individual stocks over timescales that varied over periods of 5 min. to 16 days exhibited power law decay. However, for larger holding periods, a gradual shift towards Gaussian behavior was perceptible. For the purposes of this study, data encompassing three US stock markets extracted from two databases was considered. Similar patterns of returns were observed from price data obtained from the NIKKEI & Hang Sang indices [41].

Empirical research of stock price also evidences price patterns akin to the physical phenomena of anomalous diffusion. Super-diffusion with time dependent variance according as a power law t^{α} with $\alpha > 1.0$ is observed in some studies. In fact, many stock market indices are empirically shown to temporal evolution with variances undergoing anomalous super-diffusion [46-49]. It is pertinent to point out that numerous well-examined physical systems show properties of anomalous diffusion. Some instances of relevance include fluid motion in rapidly rotating annulus exhibiting chaotic dynamics [50], particle moving in periodic potential [51], mass transfer of fluid through porous medium [52-53], dynamics of thin films, crystal growth [54,55], heat transfer by radiation [56] and many others. An immediate approach to modeling of such systems is facilitated by the Fokker Planck equation [35] that is a convenient formalism for describing anomalous diffusion under time evolution. There is no doubt that these empirical properties of price patterns lay strong ground for the adoption of the techniques of contemporary physics for the analysis and further development of this intriguing field.

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