ISSN 1512-1461

UDC: 537 Electromagnetism

THE MOTION OF THE POINT CHARGED PARTICLE WITH ZERO RADIATION REACTION FORCE

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Abstract:
We show the existence of the unique motion regime for a point charged particle that corresponds to zero radiation reaction force. The explicit expressions for the motion law, for the trajectory and for the external electromagnetic field providing this motion are find out. It is proved that there exists the reference frame where this motion is rectilinear and has zero initial velocity.

Keywords:
Hyperbolic motion; radiation reaction force – Lorentz-Abraham-Dirac force.

INTRODUCTION

The equation of motion of a radiating charged particle and problems relates to it is discussing during long time (the nonrelativistic version of it was discussed by Lorentz [1], more than a hundred years ago). The relativistic generalization of the equation was originally found by Abraham in 1905 [2]. A new deduction of the Lorentz covariant equation of motion was given by Dirac in 1938 [3]. This equation is therefore called the Lorentz-Abraham-Dirac equation, or for short, the LAD equation. Gal'tsov and Spirin [4] have reviewed and compared two diﬀerent approaches to radiation reaction in the classical electrodynamics of point charges: a local calculation of the self-force, using the equation of motion and a global calculation consisting in integration of the electromagnetic energy-momentum flux through a hypersurface encircling the world-line. With reference to Dirac [3] and Teitelboim [5], they interpreted the so called Schott term [6] in radiation reaction (RR) force physically in the following way: the Schott force is the finite part of the derivative of the momentum of the electromagnetic field, which is bound to the charge.

It should be noted that many authors (see [7]-[8] and the sources cited there) define uniformly accelerated motion in flat space-time as $\ddot{u}_k=0$. In the present article is shown that this definition is not completely correct: the condition $\ddot{u}_k=0$ corresponds to freely moving particle that does not radiate. Nevertheless, we show that there exists another unique regime of point charge motion when the LAD force is zero. Just this regime corresponds to a hyperbolic motion as it was introduced by M. Born in 1909 [9] – the relativistic generalization of a uniformly accelerated motion. In the Sec. 1 we derive the general motion law of the point charge that moves without radiation friction force and prove that this law is unique. We present some useful mathematical transformation formulas for the motion in such regime. In the Sec. 2 we find out the equation of the trajectory which corresponds to zero LAD force and prove that in general it is a hyperbole. There is shown that there exists the inertial Reference Frame (RF) such that the trajectory in this RF becomes rectilinear. The explicit expression for the velocity of this RF respect to the Laboratory RF is find out. In the Sec. 3 we find the configuration of the external electromagnetic field providing such motion. In the Sec. 4 we compare the previous results with the case when the charge moves in a uniform electrostatic field.

ISSN 1512-1461

1. Motion Law

Let the radiation reaction force known as the Lorentz- Abraham-Dirac (LAD) force equals to zero:

$$F_e^i = (2/3)e^2(g^{ik} - u^i u^k)\, d^2 u_k / d\tau^2 = 0. \qquad (1.1)$$
$$(i = 0,1,2,3)$$

The equations (1.1) is linear respect to $d^2 u_k / d\tau^2 = \dot\omega_k$ and obviously has a trivial (formal) solution

$$\dot\omega_k = 0, \quad k = 0,1,2,3. \qquad (1.2)$$

Then, taking into account the well-known restrictions

$$u^k u_k = 1 \quad \Rightarrow \quad u^k \omega_k = 0 \qquad (1.3)$$

we find

$$\omega^k \omega_k = -u^k \dot\omega_k, \qquad (1.4)$$

which shows that (1.2) leads to the condition

$$\omega^2 = 0, \;\; \omega_0^{(0)} = \pm |\vec\omega^{(0)}|. \qquad (1.5)$$

It is well known that a 4-acceleration ω^k is a space-like vector (see, e.g. [10] §7):

$$\omega_0 = \gamma^4 \vec a \cdot \vec\upsilon, \quad \vec\omega = \gamma^2 \vec a + \gamma^4 (\vec a \cdot \vec\upsilon)\vec\upsilon; \quad \omega^2 = -\gamma^4\left(\vec a^{\,2} + \gamma^2 (\vec a \cdot \vec\upsilon)^2\right) = -a_0{}^2 \le 0 \qquad (1.6)$$

(here and bellow $\vec a$ denotes a 3-acceleration vector of the particle, moving with a 3-velosity $\vec\upsilon$; $\gamma = (1-\upsilon^2)^{-1/2}$ stands for a relativistic factor; a_0 denotes a magnitude of a 3-acceleration vector in the particle's Proper Frame, PF). So, the conditions (1.4)-(1.6) mean

$$\dot\omega_k = 0 \quad \Rightarrow \quad \omega^2 = 0 \quad \Rightarrow \quad \vec a = \vec a_0 = 0,$$

and we have to conclude that the following statements are true:

Statement 1. *A trivial solution of the system of equations (1.1) corresponds to uniform motion* $\vec a = 0$, *so a charged particle does not radiate in this case.*

Statement 2. *Except a trivial solution* $\dot\omega_k = 0,\ k = 0,1,2,3,$ *the system of equations (1.1) has nontrivial solutions as well, when*

$$\dot\omega_k \ne 0, \quad k = 0,1,2,3.$$

Proof. The system (1.1) consists of linear homogeneous equations respect to $\dot\omega_k$, $k = 0,1,2,3,$ and the determinant of the system is zero. In order to calculate it let us calculate previously a characteristic polynomial of the system

$$P(\lambda) = \det(\lambda g_k^i - u^i u_k) = \lambda^4 - \lambda^3 \mathrm{Tr}(u^i u_k).$$

All other summands of the characteristic polynomial $P(\lambda)$ are zero. Taking into account that

$$\mathrm{Tr}(u^i u_k) = u^k u_k = (u_0)^2 - u_\alpha u_\alpha = 1,$$

one concludes that

$$\det(g_k^i - u^i u_k) = P(1) = 0. \qquad (1.7)$$

It is easy to check that the rank of the system (1.1) is 3:

$$\det(g_\beta^\alpha - u^\alpha u_\beta) = \det\begin{bmatrix} 1+(u_1)^2 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & 1+(u_2)^2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & 1+(u_3)^2 \end{bmatrix} = 1 + u_\alpha u_\alpha = (u_0)^2 > 0. \qquad (1.8)$$

Therefore, general solutions of the system of linear algebraic equations (1.1) form a 1-dimensional linear space [11], (that contains the solution (1.2) as well as nontrivial solutions $\dot\omega_k \ne 0$, $k = 0,1,2,3$).

Rewriting the system of equations (1.1) as

$$(g_\beta^\alpha - u^\alpha u_\beta)\dot\omega_\alpha = u_\beta u^0 \dot\omega_0 \quad (\beta = 1,2,3)$$

one can present general solution of the homogeneous linear system (1.1) as follows:

$$\dot\omega_k = (\dot\omega_0 / u_0) u_k, \quad k = 0,1,2,3 \quad (\forall \dot\omega_0 \in \ ). \qquad (1.9)$$

ISSN 1512-1461

Then, taking here into account the conditions (1.4) and (1.3) one finds

$$\omega^2 = \omega^k\omega_k = -u^k\dot{\omega}_k = -(\dot{\omega}_0/u_0)u^k u_k = -\dot{\omega}_0/u_0$$

and therefore (1.9) (and equivalent to it the condition (1.1)) can be presented as follows:

$$\dot{\omega}_k = -\omega^2 u_k, \quad k = 0,1,2,3. \qquad (1.10)$$

Multiplying the equations (1.10) by ω^k and summarizing one gets:

$$\omega^k\dot{\omega}_k = \frac{1}{2}\frac{d}{d\tau}\omega^2 = -\omega^2\omega^k u_k = 0.$$

Hence, according to the (1.6),

$$\omega^2 = -a_0^{\,2} = \text{const}. \qquad (1.11)$$

Taking into account that

$$u_k = \dot{x}_k, \quad \omega_k = \ddot{x}_k, \quad k = 0,1,2,3$$

and integrating the equations (1.10) one can find

$$\ddot{x}_k = a_0^{\,2}x_k + \omega_k^{(0)}, \quad k = 0,1,2,3, \qquad (1.12)$$

where (1.11) is used and we denoted

$$\omega_k^{(0)} = \ddot{x}_k(\tau = 0), \quad k = 0,1,2,3.$$

Then, assuming that the initial conditions are

$$x_k(\tau = 0) = 0, \quad u_k(\tau = 0) = \dot{x}_k(\tau = 0) = u_k^{(0)} \equiv \beta_k, \quad k = 0,1,2,3,$$

general solutions of the system (1.12)

$$x_k = A_k\cosh(a_0\tau) + B_k\sinh(a_0\tau) - \omega_k^{(0)}/a_0^{\,2}, \quad k = 0,1,2,3, \qquad (1.13)$$

can be presented as follows:

$$x_k = \omega_k^{(0)}a_0^{-2}\big(\cosh(a_0\tau) - 1\big) + \beta_k a_0^{-1}\sinh(a_0\tau). \qquad (1.14)$$

It should be emphasized that in the notations used bellow through this article $\beta_0 = \gamma_{|\tau=0}$, $\vec{\beta} = (\gamma\vec{\upsilon})_{|\tau=0}$, so, we have

$$\beta_k\beta^k = \beta_0^{\,2} - \vec{\beta}^2 = 1. \qquad (1.15)$$

Thus,

$$u_k = \dot{x}_k = \omega_k^{(0)}a_0^{-1}\sinh(a_0\tau) + \beta_k\cosh(a_0\tau), \qquad (1.16)$$

$$\omega_k = \dot{u}_k = \ddot{x}_k = \omega_k^{(0)}\cosh(a_0\tau) + \beta_k a_0\sinh(a_0\tau) \;\; (= a_0^{\,2}x_k + \omega_k^{(0)}), \qquad (1.17)$$

$$\dot{\omega}_k = a_0^{\,2}\left[\omega_k^{(0)}a_0^{-1}\sinh(a_0\tau) + \beta_k\cosh(a_0\tau)\right] \quad (= a_0^{\,2}u_k), \qquad (1.18)$$

which proves the Statement 2 is proved.

Note 1. It is easy to check that the solutions (1.16)-(1.17) satisfy all necessary conditions (1.3)-(1.4). In the particle's PF one has to assume in (1.16)-(1.17) the next: $\beta_k = (1,\vec{0})$, $\omega_0^{(0)} = 0$, $\vec{\omega}^{(0)} = \vec{a}_0$.

Note 2. The formulas (1.14) and (1.16)-(1.18) are derived for the case $a_0 \neq 0$. Nevertheless, they have limit when $a_0 \to 0$. Namely, taking into account that in this case $\omega_k^{(0)} \to 0$ and using (1.14) and (1.16)-(1.18) we can find

$$x_k = \beta_k\tau, \quad u_k = \beta_k, \quad \omega_k = \dot{\omega}_k = 0, \quad k = 0,1,2,3.$$

Hence, *a motion which satisfies the condition (1.1) is possible only if the initial value of acceleration* $a_0 \neq 0$ (except the trivial case of uniform motion without any radiation).

Below we assume $a_0 \neq 0$ and use a notation

$$\omega_k^{(0)}/a_0 = \alpha_k, \quad k = 0,1,2,3. \qquad (1.19)$$

Taking into consideration the relations (1.3) and (1.6), obviously, one gets

ISSN 1512-1461

$$\alpha^k \alpha_k = \alpha_0{}^2 - \vec{\alpha}^2 = -1, \tag{1.20}$$

$$\alpha^k \beta_k = \alpha_0 \beta_0 - \vec{\alpha} \cdot \vec{\beta} = 0. \tag{1.21}$$

Therefore, based on the Cauchy-Bunyakovsky inequality [12] one should conclude that

$$\alpha_0{}^2 \beta_0{}^2 = (\vec{\alpha} \cdot \vec{\beta})^2 \le \vec{\alpha}^2 \vec{\beta}^2 = (\alpha_0{}^2 + 1)(\beta_0{}^2 - 1) = \alpha_0{}^2 \beta_0{}^2 + \beta_0{}^2 - \alpha_0{}^2 - 1 \;\Rightarrow\; \beta_0{}^2 - \alpha_0{}^2 - 1 \ge 0. \tag{1.22}$$

Note 3. The motion laws (1.16) and (1.17) can be expressed in matrix form as follows:

$$\begin{pmatrix} a_0{}^{-1}\omega_k(\tau) \\ u_k(\tau) \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}, \quad k = 0,1,2,3. \;\; (\vartheta = a_0 \tau) \tag{1.23}$$

So, one can consider these motion laws as an orthogonal transformation – rotation on the angle $\vartheta = a_0 \tau$ in 2-dimensional pseudo-Euclidean space $\mathsf{P}^2_{1,1}$. Then one has to conclude that the following relations must be true (taking into account that metric is pseudo-Euclidean)

$$\begin{pmatrix} a_0{}^{-1}\omega_k(\tau), & u_k(\tau) \end{pmatrix} \begin{pmatrix} a_0{}^{-1}\omega_l(\tau) \\ u_l(\tau) \end{pmatrix} =$$

$$= a_0{}^{-2}\omega_k(\tau)\omega_l(\tau) - u_k(\tau)u_l(\tau) = \mathrm{inv} = \alpha_k \alpha_l - \beta_k \beta_l, \quad k,l = 0,1,2,3. \tag{1.24}$$

Particularly, assuming here $k=0,\ l=1,2,3,$ one obtains the important relation that we shall use bellow

$$a_0{}^{-2}\omega_0(\tau)\vec{\omega}(\tau) - u_0(\tau)\vec{u}(\tau) = \alpha_0 \vec{\alpha} - \beta_0 \vec{\beta}. \tag{1.25}$$

Assuming in (1.24) $k = l$ we obtain

$$a_0{}^{-2}\omega_k(\tau)\omega_k(\tau) - u_k(\tau)u_k(\tau) = \alpha_k \alpha_k - \beta_k \beta_k, \quad k = 0,1,2,3, \tag{1.26}$$

which, after using a pseudo-Euclidean metric and summarizing by $k,$ gives us

$$a_0{}^{-2}\omega_k(\tau)\omega^k(\tau) - u_k(\tau)u^k(\tau) = \alpha_k \alpha^k - \beta_k \beta^k.$$

Taking here into account (1.3), (1.15) and (1.21) we easily obtain the known expression (1.6):

$$a_0{}^{-2}\omega^2(\tau) - 1 = -2 \quad \Rightarrow \quad \omega^2(\tau) = -a_0{}^2.$$

Obviously, the converse is not true: the general relation (1.6) does not give us neither 10 relations (1.24) nor 4 relations (1.26) which are not valid in general case but for the charge's motion in accordance to the condition (1.1).

The next important relation is referred to the 6 skew-symmetric tensors $A_{kl} = -A_{lk} \in \mathsf{P}^2_{1,1}$, $k,l = 0,1,2,3,$ that must remain invariant (as a skew-symmetric rank $= 2$ tensor in a 2-dimensional space) under transformations induced by (1.23):

$$A_{kl} = a_0{}^{-1}\omega_k(\tau)u_l(\tau) - a_0{}^{-1}u_k(\tau)\omega_l(\tau) = \mathrm{inv} = \alpha_k \beta_l - \beta_k \alpha_l, \quad k,l = 0,1,2,3. \tag{1.27}$$

In particular, one has

$$a_0{}^{-1}\big(\omega_0(\tau)\vec{u}(\tau) - u_0(\tau)\vec{\omega}(\tau)\big) = \alpha_0 \vec{\beta} - \beta_0 \vec{\alpha}. \tag{1.28}$$

Obviously, the modulus of the invariant Euclidean 3-vectors remain invariant under transformation (1.23) as well. In particular, taking into account (1.15), (1.21) and (1.22), for the squares of the vectors (1.25) and (1.28) one gets

$$[a_0{}^{-2}\omega_0(\tau)\vec{\omega}(\tau) - u_0(\tau)\vec{u}(\tau)]^2 = (\alpha_0 \vec{\alpha} - \beta_0 \vec{\beta})^2 =$$

$$= \alpha_0^2(\alpha_0^2 + 1) - 2\alpha_0^2 \beta_0^2 + \beta_0^2(\beta_0^2 - 1)^2 = (\beta_0^2 - \alpha_0^2)(\beta_0^2 - \alpha_0^2 - 1), \tag{1.29}$$

$$a_0{}^{-2}[\omega_0(\tau)\vec{u}(\tau) - u_0(\tau)\vec{\omega}(\tau)]^2 = (\alpha_0 \vec{\beta} - \beta_0 \vec{\alpha})^2 =$$

$$= \alpha_0^2(\beta_0^2 - 1)^2 - 2\alpha_0^2 \beta_0^2 + \beta_0^2(\alpha_0^2 + 1) = \beta_0^2 - \alpha_0^2. \tag{1.30}$$

As well, we will use these relations below.

2. The trajectory.

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Using the motion laws (1.14) one can eliminate proper time and obtain an equation for a trajectory which corresponds to the condition (1.1).

First of all, let us prove the next statement:

Statement 3. *The trajectory of the motion corresponding to the condition (1.1) is a flat curve.*

In order to proof the Statement 3, the next Lemma can be used:

Lemma 1. *For the motion corresponding to the condition (1.1) the next relation is true:*

$$(d/d\tau)\left(\vec{u}(\tau)\times\vec{\omega}(\tau)\right)=0. \tag{2.1}$$

Proof of the Lemma. Based on the formulas (1.25) and (1.28) one can conclude that

$$\left(a_0^{-2}\omega_0(\tau)\vec{\omega}(\tau)-u_0(\tau)\vec{u}(\tau)\right)\times a_0^{-1}\left(\omega_0(\tau)\vec{u}(\tau)-u_0(\tau)\vec{\omega}(\tau)\right)=(\alpha_0\vec{\alpha}-\beta_0\vec{\beta})\times(\alpha_0\vec{\beta}-\beta_0\vec{\alpha})$$

$$\Rightarrow\ a_0^{-1}\left(u_0^2(\tau)-a_0^{-2}\omega_0^2(\tau)\right)\vec{u}(\tau)\times\vec{\omega}(\tau)=(\beta_0^2-\alpha_0^2)\vec{\beta}\times\vec{\alpha}=\text{inv}.$$

Besides, using (1.26) and (1.22) one has

$$u_0^2(\tau)-a_0^{-2}\omega_0^2(\tau)=\beta_0^2-\alpha_0^2\geq 1. \tag{2.2}$$

Therefore we get

$$\vec{u}(\tau)\times\vec{\omega}(\tau)=a_0(\vec{\beta}\times\vec{\alpha})=\text{inv}. \tag{2.3}$$

So, the statement of the Lemma 1 is proved.

Proof of the Statement. Using expressions [13] for unit tangent $\vec{t}(\tau)$, and for main normal $\vec{n}(\tau)$ of a curve in 3-dimensional Euclidean space we get for the trajectory under consideration

$$\vec{t}(\tau)=|\vec{\upsilon}|^{-1}\vec{\upsilon}(\tau)=|\vec{u}|^{-1}\vec{u}(\tau)\quad\Rightarrow\quad\vec{t}^{\,2}=1\Rightarrow\vec{t}(\tau)\cdot\dot{\vec{t}}(\tau)=0\quad\Rightarrow$$

$$\dot{\vec{t}}(\tau)=|\dot{\vec{t}}|\,\vec{n}(\tau),\quad\vec{n}^{\,2}=1.$$

Then, calculating the derivative we find

$$\dot{\vec{t}}(\tau)=\frac{d}{d\tau}\vec{t}(\tau)=\frac{d}{d\tau}\frac{\vec{u}(\tau)}{|\vec{u}(\tau)|}=\frac{\dot{\vec{u}}(\tau)|\vec{u}(\tau)|-\vec{u}(\tau)\left(\dot{\vec{u}}(\tau)\cdot\vec{u}(\tau)\right)|\vec{u}(\tau)|^{-1}}{\vec{u}^{\,2}(\tau)}=$$

$$=\frac{\vec{\omega}(\tau)\vec{u}^{\,2}(\tau)-\vec{u}(\tau)\left(\vec{\omega}(\tau)\cdot\vec{u}(\tau)\right)}{|\vec{u}(\tau)|^3}=\frac{\vec{u}(\tau)\times\left(\vec{\omega}(\tau)\times\vec{u}(\tau)\right)}{|\vec{u}(\tau)|^3}. \tag{2.4}$$

Let us assume firstly that $\vec{\beta}\times\vec{\alpha}\neq 0.$ Then, according to (2.3) we have

$$\vec{u}(\tau)\times\vec{\omega}(\tau)\neq 0,\quad\forall\tau.$$

Therefore $\dot{\vec{t}}(\tau)\neq 0$ and one gets

$$\vec{n}(\tau)=\frac{\dot{\vec{t}}(\tau)}{|\dot{\vec{t}}(\tau)|}=\frac{\vec{u}(\tau)\times\left(\vec{\omega}(\tau)\times\vec{u}(\tau)\right)}{|\vec{u}(\tau)\times\left(\vec{\omega}(\tau)\times\vec{u}(\tau)\right)|}=\frac{\vec{u}(\tau)\times\left(\vec{\omega}(\tau)\times\vec{u}(\tau)\right)}{|\vec{u}(\tau)||\vec{\omega}(\tau)\times\vec{u}(\tau)|}. \tag{2.5}$$

Calculating now the binormal $\vec{b}(\tau)=\vec{t}(\tau)\times\vec{n}(\tau)$ of the trajectory one gets

$$\vec{b}(\tau)=\vec{t}(\tau)\times\vec{n}(\tau)=\frac{\vec{u}(\tau)}{|\vec{u}(\tau)|}\times\frac{\vec{u}(\tau)\times\left(\vec{\omega}(\tau)\times\vec{u}(\tau)\right)}{|\vec{u}(\tau)||\vec{\omega}(\tau)\times\vec{u}(\tau)|}=$$

$$=\vec{u}(\tau)\times\frac{\vec{\omega}(\tau)\vec{u}^{\,2}(\tau)-\vec{u}(\tau)\left(\vec{\omega}(\tau)\cdot\vec{u}(\tau)\right)}{\vec{u}^{\,2}(\tau)\,|\vec{\omega}(\tau)\times\vec{u}(\tau)|}=\frac{\vec{u}(\tau)\times\vec{\omega}(\tau)}{|\vec{\omega}(\tau)\times\vec{u}(\tau)|}.$$

So, according to the Lemma

$$\vec{b}(\tau)=\frac{\vec{u}(\tau)\times\vec{\omega}(\tau)}{|\vec{\omega}(\tau)\times\vec{u}(\tau)|}=\frac{\vec{\beta}\times\vec{\alpha}}{|\vec{\beta}\times\vec{\alpha}|}=\text{inv} \tag{2.6}$$

and for the case $\vec{\beta}\times\vec{\alpha}\neq 0$ the Statement is proved.

If now $\vec{\beta}\times\vec{\alpha}=0,$ then, according to (2.3),

$$\vec{u}(\tau)\times\vec{\omega}(\tau)=0,\quad\forall\tau.$$

Therefore, according to (2.4) we have

ISSN 1512-1461

$$\vec{\beta}\times\vec{\alpha}=0 \;\Rightarrow\; \dot{\vec{t}}(\tau)=0 \;\Rightarrow\; \vec{t}(\tau)=\text{const}, \tag{2.7}$$

hence, the trajectory is rectilinear. So, the Statement is fair in this case too.

In order to find the explicit equation of the trajectory let us present (1.14) as follows:

$$\begin{aligned} x_0 &= a_0^{-1}\big(\alpha_0(\cosh\vartheta-1)+\beta_0\sinh\vartheta\big),\\ \vec{x} &= a_0^{-1}\big(\vec{\alpha}(\cosh\vartheta-1)+\vec{\beta}\,\sinh\vartheta\big),\\ \vec{x}_\perp &= a_0^{-1}\vec{\beta}_\perp\sinh\vartheta.\end{aligned} \tag{2.8}$$

$$(\vartheta=a_0\tau,\quad a_0\neq 0)$$

Here we have denoted $\vec{\beta}\equiv\vec{\alpha}(\vec{\alpha}\cdot\vec{\beta})\vec{\alpha}^{-2}$, $\vec{\beta}_\perp=\vec{\beta}-\vec{\beta}$, $\vec{x}\equiv\vec{\alpha}(\vec{\alpha}\cdot\vec{x})\vec{\alpha}^{-2}$, $\vec{x}_\perp=\vec{x}-\vec{x}$. Besides, we assume that $a_0>0$. Then, introducing new dimensionless variables

$$\xi_0=x_0a_0,\qquad \vec{\xi}=a_0\vec{x},\qquad \vec{\xi}_\perp=a_0\vec{x}_\perp,$$

and rewrite (2.8) in the coordinate system having the spatial axes $(\beta_\perp\neq 0)$ as follows:

$$Ox\;\;\vec{\alpha},\quad Oy\;\;\vec{\beta}_\perp,$$

we get

$$\begin{aligned}\xi_0 &= \alpha_0(\cosh\vartheta-1)+\beta_0\sinh\vartheta,\\ \xi &= \alpha(\cosh\vartheta-1)+\beta\;\sinh\vartheta,\\ \xi_\perp &= \beta_\perp\sinh\vartheta.\end{aligned} \tag{2.9}$$

$$(\alpha>0,\;\beta_\perp>0)$$

Now, it is obvious that

$$\xi_0=\alpha_0\Big(\sqrt{1+\xi_\perp^2/\beta_\perp^2}-1\Big)+\beta_0\xi_\perp/\beta_\perp, \tag{2.10}$$

$$\xi=\alpha\Big(\sqrt{1+\xi_\perp^2/\beta_\perp^2}-1\Big)+\beta\;\xi_\perp/\beta_\perp. \tag{2.11}$$

$$(\alpha>0,\;\beta_\perp>0;\;\;\beta_0^2=\beta^2+\beta_\perp^2+1)$$

The equation (2.11) gives us the trajectory of a charged particle which is moving in accordance to the condition (1.1). Rewriting (2.11) as

$$(\xi-\beta\;\xi_\perp/\beta_\perp+\alpha)^2-\alpha^2(\xi_\perp/\beta_\perp)^2=\alpha^2,$$

or, equivalently,

$$\big[\xi^2-2\beta\;\beta_\perp^{-1}\xi\;\xi_\perp+(\beta^2-\alpha^2)\beta_\perp^{-2}\xi_\perp^2\big]+2\alpha(\xi-\beta\;\beta_\perp^{-1}\xi_\perp)=0, \tag{2.12}$$

one can conclude that the trajectory is a hyperbole as far the matrix of the quadratic form in the square brackets (in respect with the variables ξ , $\xi_\perp$)

$$D_2=\begin{bmatrix}1 & -\beta\;/\beta_\perp\\ -\beta\;/\beta_\perp & (\beta^2-\alpha^2)\beta_\perp^{-2}\end{bmatrix}$$

has negative determinant:

$$\det D_2=(\beta^2-\alpha^2)\beta_\perp^{-2}-\beta^2\beta_\perp^{-2}=-\alpha^2\beta_\perp^{-2}<0.$$

$$(\alpha>0,\;\beta_\perp>0)$$

Hence, the eigenvalues of the matrix D_2 are of opposite signs.

Note 4. The determinant of the full quadratic form matrix in the left hand side of (2.12) is

ISSN 1512-1461

$$\det D_3 = \det\begin{bmatrix} 1 & -\beta\ /\beta_\perp & \alpha \\ -\beta\ /\beta_\perp & \left(\beta^2-\alpha^2\right)\beta_\perp^{-2} & -\alpha\beta\ /\beta_\perp \\ \alpha & -\alpha\beta\ /\beta_\perp & 0 \end{bmatrix} = \alpha^4\beta_\perp^{-2} > 0. \quad (\alpha>0,\ \beta_\perp>0)$$

Therefore under conditions $\alpha>0,\ \beta_\perp>0$ the hyperbole is non degenerated [14].

The equation (2.10) easily can be solved in respect with the variable $\eta = \xi_\perp/\beta_\perp = \sinh\vartheta$ as follows:

$$(\xi_0-\beta_0\eta)^2 + 2\alpha_0(\xi_0-\beta_0\eta) - \alpha_0^{\,2}\eta^2 = 0 \qquad \Rightarrow$$
$$\eta = (\beta_0^{\,2}-\alpha_0^{\,2})^{-1}\left[\beta_0(\alpha_0+\xi_0) \pm |\alpha_0|\sqrt{\beta_0^{\,2}-\alpha_0^{\,2}+(\alpha_0+\xi_0)^2}\,\right].$$

Then a charged particle's motion law which corresponds to the condition (1.1) can be written down in terms of lab time $x_0 = a_0^{-1}\xi_0$ as

$$\xi_\perp = \frac{\beta_\perp}{\beta_0^{\,2}-\alpha_0^{\,2}}\left[\beta_0(\xi_0+\alpha_0) \pm |\alpha_0|\sqrt{\beta_0^{\,2}-\alpha_0^{\,2}+(\xi_0+\alpha_0)^2}\,\right], \tag{2.13}$$

$$\xi\ = \frac{\alpha}{\alpha_0}\left(\xi_0 - \frac{\beta_0-\beta}{\beta_\perp}\xi_\perp\right). \qquad (\beta_\perp>0,\ a_0^{\,2}\neq 0) \tag{2.14}$$

In the case when $\beta_\perp = 0\ \ (a_0^2\neq 0)$ the equations (2.8) reduce to the following ones:

$$\begin{aligned} &\xi_0 = \alpha_0(\cosh\vartheta-1) + \beta_0\sinh\vartheta, \\ &\vec{\xi}\ = \vec{\xi} = \vec{\alpha}(\cosh\vartheta-1) + \vec{\beta}\sinh\vartheta, \\ &\xi_\perp = 0. \end{aligned} \tag{2.15}$$

So, the trajectory is rectilinear, as far in this case $\vec{\beta}\quad\vec{\alpha}$.

The correspondent motion law in terms of the lab time $x_0 = a_0^{-1}\xi_0$ is

$$(\xi_0\vec{\beta} - \beta_0\vec{\xi} + \alpha_0\vec{\beta} - \beta_0\vec{\alpha})^2 - (\alpha_0\vec{\xi} - \xi_0\vec{\alpha})^2 = (\alpha_0\vec{\beta} - \beta_0\vec{\alpha})^2,$$

which can be presented equivalently as (here, as well as above, $\xi\equiv|\vec{\xi}\,|,\ \alpha\equiv|\vec{\alpha}\,|,\ \beta\equiv|\vec{\beta}\,|$)

$$\xi^2(\beta_0^{\,2}-\alpha_0^{\,2}) + 2\xi\xi_0(\alpha_0\alpha-\beta_0\beta) + \xi_0^2(\beta^2-\alpha^2) + 2(\beta_0\xi-\xi_0\beta)(\alpha_0\beta-\beta_0\alpha)^2 = 0. \tag{2.16}$$

Taking into account (1.29), (1.30), (1.15), (1.20) and (1.22), in the case $\vec{\beta}\quad\vec{\alpha}$ one gets

$$\beta_0^{\,2}-\alpha_0^{\,2}=1,\quad \beta^2-\alpha^2=\beta_0^{\,2}-\alpha_0^{\,2}-2=-1,\quad (\alpha_0\beta-\beta_0\alpha)^2=\beta_0^{\,2}-\alpha_0^{\,2}=1,\quad \alpha_0\alpha-\beta_0\beta=0.$$

Therefore, in the case $\vec{\beta}\quad\vec{\alpha}$ the motion law (2.16) has the form as follows:

$$\xi^2 - \xi_0^{\,2} + 2(\beta_0\xi-\xi_0\beta) = 0. \tag{2.16$'$}$$

The orthogonal invariants (see, e.g. [14]) of the quadratic form (2.16) are

$$\det\begin{bmatrix} \beta_0^{\,2}-\alpha_0^{\,2} & \alpha_0\alpha-\beta_0\beta & -\beta_0(\alpha_0\beta-\beta_0\alpha) \\ \alpha_0\alpha-\beta_0\beta & \beta^2-\alpha^2 & \beta(\alpha_0\beta-\beta_0\alpha) \\ -\beta_0(\alpha_0\beta-\beta_0\alpha) & \beta(\alpha_0\beta-\beta_0\alpha) & 0 \end{bmatrix} = \beta_0^2-\beta^2 = 1 > 0,$$

$$\det\begin{bmatrix} \beta_0^{\,2}-\alpha_0^{\,2} & \alpha_0\alpha-\beta_0\beta \\ \alpha_0\alpha-\beta_0\beta & \beta^2-\alpha^2 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 < 0,$$

so, the motion law (2.16) describes the dependence $\vec{x} = \vec{x}(x_0)$ as a non-degenerated hyperbole.

Theorem 1. *If* $\vec{\alpha}\times\vec{\beta}\neq 0$, *then there exists a Lorentz transformation that makes the 3-dimensional vectors* $\vec{\alpha}$ *and* $\vec{\beta}$ *collinear.*

Before proving we need two useful Lemmas:

ISSN 1512-1461

Lemma 2. *If the 3-vectors $\vec{\alpha}$ and $\vec{\beta}$ are not collinear, so if $\vec{\alpha}\times\vec{\beta}\neq 0$ then the vectors*

$$\vec{e}_1=\frac{\alpha_0\vec{\beta}-\beta_0\vec{\alpha}}{|\alpha_0\vec{\beta}-\beta_0\vec{\alpha}|}\qquad \vec{e}_2=\frac{\alpha_0\vec{\alpha}-\beta_0\vec{\beta}}{|\alpha_0\vec{\alpha}-\beta_0\vec{\beta}|},\qquad \vec{e}_3=\frac{\vec{\alpha}\times\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|},\tag{2.17}$$

form an orthonormal basis in 3-dimensional space.

Proof of the Lemma 2. According to (1.29), (1.30) and (1.22), one has:

$$\begin{aligned}&(\alpha_0\vec{\beta}-\beta_0\vec{\alpha})^2=\beta_0^2-\alpha_0^2\geq 1>0,\\&(\alpha_0\vec{\alpha}-\beta_0\vec{\beta})^2=(\beta_0^2-\alpha_0^2)(\beta_0^2-\alpha_0^2-1)\geq 0,\\&(\vec{\alpha}\times\vec{\beta})^2=\vec{\alpha}^2\vec{\beta}^2-(\vec{\alpha}\cdot\vec{\beta})^2=\beta_0^2-\alpha_0^2-1\geq 0.\end{aligned}\tag{2.18}$$

Equalities in the formulas (2.18) are reached for collinear vectors $\vec{\alpha}$ and $\vec{\beta}$ only. So, under the condition of the Lemma 2, the definition (2.17) is correct. All three vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are unit by the definition.

According to (2.17), obviously,

$$\vec{e}_1\cdot\vec{e}_3=0,\qquad \vec{e}_2\cdot\vec{e}_3=0.$$

Let us check $\vec{e}_1\cdot\vec{e}_2$. Taking into account (1.15), (1.20) and (1.21), one gets

$$(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\cdot(\alpha_0\vec{\alpha}-\beta_0\vec{\beta})=\alpha_0\beta_0(\vec{\alpha}^2+\vec{\beta}^2)-(\alpha_0^2+\beta_0^2)\vec{\alpha}\cdot\vec{\beta}=\alpha_0\beta_0(\vec{\alpha}^2-\alpha_0^2+\vec{\beta}^2-\beta_0^2)=0.$$

So, the Lemma 2 is proved.

Note 5. It is easy to check that the basis (2.17) is right oriented:

$$\vec{e}_1\times\vec{e}_2=\vec{e}_3,\qquad \vec{e}_2\times\vec{e}_3=\vec{e}_1,\qquad \vec{e}_3\times\vec{e}_1=\vec{e}_2.\tag{2.19}$$

Lemma 3. *Let us suppose $\vec{\alpha}\times\vec{\beta}\neq 0$ and in every point of the trajectory (for any proper time τ) let define the following 3-vectors*

$$\vec{\lambda}\equiv a_0^{-1}\big(\omega_0(\tau)\vec{u}(\tau)-u_0(\tau)\vec{\omega}(\tau)\big),\quad \vec{\mu}\equiv a_0^{-2}\omega_0(\tau)\vec{\omega}(\tau)-u_0(\tau)\vec{u}(\tau),\quad \vec{\nu}=\vec{\lambda}\times\vec{\mu},\tag{2.20}$$

where $u_k(\tau)$ and $\omega_k(\tau)$, $k=0,1,2,3,$ are defined by formulas (1.16)-(1.17). Then the vectors $\vec{\lambda}/|\vec{\lambda}|$, $\vec{\mu}/|\vec{\mu}|$, $\vec{\nu}/|\vec{\nu}|$ form a uniform 3-dimensional orthonormal basis along the particle's trajectory and so form the global Cartesian coordinate system in 3-dimensional Euclidean space.

Proof of the Lemma 3. According to (1.25) and (1.28) we have

$$\vec{\lambda}=\alpha_0\vec{\beta}-\beta_0\vec{\alpha},\qquad \vec{\mu}=\alpha_0\vec{\alpha}-\beta_0\vec{\beta}.\tag{2.21}$$

Therefore, according to the lemma 2, the vectors $\vec{\lambda}/|\vec{\lambda}|$ and $\vec{\mu}/|\vec{\mu}|$ are correctly defined, orthonormal and are invariant along the trajectory. Hence, in the arbitrary point of the trajectory the vector

$$(\vec{\lambda}/|\vec{\lambda}|)\times(\vec{\mu}/|\vec{\mu}|)=\vec{\nu}/(|\vec{\lambda}||\vec{\mu}|)=\vec{\nu}/|\vec{\nu}|$$

is defined correctly and uniquely and give us (together with the vectors $\vec{\lambda}/|\vec{\lambda}|$ and $\vec{\mu}/|\vec{\mu}|$) the third vector of 3-dimensional orthonormal global basis. Thus, the Lemma 3 is proved.

Note that, according to (2.20), (2.21) and (2.18), one has

$$\begin{aligned}&\vec{\nu}=(\alpha_0\vec{\beta}-\beta_0\vec{\alpha})\times(\alpha_0\vec{\alpha}-\beta_0\vec{\beta})=(\beta_0^2-\alpha_0^2)\vec{\alpha}\times\vec{\beta},\\&\vec{\nu}^2=(\beta_0^2-\alpha_0^2)^2(\vec{\alpha}\times\vec{\beta})^2=(\beta_0^2-\alpha_0^2)^2(\beta_0^2-\alpha_0^2-1)=\vec{\lambda}^2\vec{\mu}^2.\end{aligned}$$

Proof of the Theorem. Using formulas (see, e.g. [15], §2.9) for an arbitrary oriented boost one obtains

$$\vec{\alpha}\to\vec{\alpha}'=\vec{\alpha}+(\gamma-1)\vec{\upsilon}^{-2}\vec{\upsilon}(\vec{\alpha}\cdot\vec{\upsilon})-\gamma\vec{\upsilon}\alpha_0,\qquad \vec{\beta}\to\vec{\beta}'=\vec{\beta}+(\gamma-1)\vec{\upsilon}^{-2}\vec{\upsilon}(\vec{\beta}\cdot\vec{\upsilon})-\gamma\vec{\upsilon}\beta_0.$$

Assuming now that the 3-dimensional vectors $\vec{\alpha}'$ and $\vec{\beta}'$ become collinear we get a condition on the parameter of the boost (the velocity) $\vec{\upsilon}$

$$0=\vec{\alpha}'\times\vec{\beta}'=\vec{\alpha}\times\vec{\beta}+\vec{\alpha}\times\vec{\upsilon}[(\gamma-1)]\vec{\upsilon}^{-2}\vec{\beta}\cdot\vec{\upsilon}-\gamma\beta_0]-\vec{\beta}\times\vec{\upsilon}[(\gamma-1)]\vec{\upsilon}^{-2}\vec{\alpha}\cdot\vec{\upsilon}-\gamma\alpha_0],$$

that can be rearranged as follows:

$$0=\vec{\alpha}\times\vec{\beta}+(\gamma-1)\vec{\upsilon}^{-2}[(\vec{\alpha}\times\vec{\upsilon})(\vec{\beta}\cdot\vec{\upsilon})-(\vec{\beta}\times\vec{\upsilon})(\vec{\alpha}\cdot\vec{\upsilon})]-\gamma(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\times\vec{\upsilon}.$$

Then, one can transform the expression in square brackets:

$$[\vec{\alpha}(\vec{\beta}\cdot\vec{\upsilon})-\vec{\beta}(\vec{\alpha}\cdot\vec{\upsilon})]\times\vec{\upsilon}=[\vec{\upsilon}\times(\vec{\alpha}\times\vec{\beta})]\times\vec{\upsilon}=(\vec{\alpha}\times\vec{\beta})\cdot\vec{\upsilon}^{\,2}-\vec{\upsilon}[\vec{\upsilon}\cdot(\vec{\alpha}\times\vec{\beta})].$$

Therefore, after obvious simplifications we obtain an equation

$$(\alpha_0\vec{\beta}-\beta_0\vec{\alpha})\times\vec{\upsilon}+(1-\gamma^{-1})\vec{\upsilon}^{-2}\vec{\upsilon}[\vec{\upsilon}\cdot(\vec{\alpha}\times\vec{\beta})]+\vec{\alpha}\times\vec{\beta}=0. \tag{2.22}$$

Now, expressing in the basis (2.17) the velocity we are looking for,

$$\vec{\upsilon}=\upsilon_1\vec{e}_1+\upsilon_2\vec{e}_2+\upsilon_3\vec{e}_3,$$

and substituting it in the equation (2.22) we obtain:

$$({\beta_0}^2-{\alpha_0}^2)^{1/2}(\upsilon_2\vec{e}_3-\upsilon_3\vec{e}_2)+({\beta_0}^2-{\alpha_0}^2-1)^{1/2}[(1-\gamma^{-1})\vec{\upsilon}^{-2}(\upsilon_1\vec{e}_1+\upsilon_2\vec{e}_2+\upsilon_3\vec{e}_3)\upsilon_3+\vec{e}_3]=0.$$

Therefore,

$$\upsilon_1=0,$$
$$-\upsilon_3({\beta_0}^2-{\alpha_0}^2)^{1/2}+({\beta_0}^2-{\alpha_0}^2-1)^{1/2}(1-\gamma^{-1})\vec{\upsilon}^{-2}\upsilon_2\upsilon_3=0,$$
$$\upsilon_2({\beta_0}^2-{\alpha_0}^2)^{1/2}+({\beta_0}^2-{\alpha_0}^2-1)^{1/2}[(1-\gamma^{-1})\vec{\upsilon}^{-2}{\upsilon_3}^2+1]=0.$$

These equations have unique solution (the case $\upsilon_3\neq 0$ leads to $\gamma^{-1}=\sqrt{1-\upsilon^2}=2$ and is impossible):

$$\upsilon_1=0,\;\; \upsilon_3=0,\;\; \upsilon_2=-\frac{({\beta_0}^2-{\alpha_0}^2-1)^{1/2}}{({\beta_0}^2-{\alpha_0}^2)^{1/2}}.$$

Therefore, in accordance with (2.18) we finally find

$$\vec{\upsilon}=-\frac{({\beta_0}^2-{\alpha_0}^2-1)^{1/2}}{({\beta_0}^2-{\alpha_0}^2)^{1/2}}\frac{\alpha_0\vec{\alpha}-\beta_0\vec{\beta}}{({\beta_0}^2-{\alpha_0}^2-1)^{1/2}({\beta_0}^2-{\alpha_0}^2)^{1/2}}=-\frac{\alpha_0\vec{\alpha}-\beta_0\vec{\beta}}{{\beta_0}^2-{\alpha_0}^2}. \tag{2.23}$$

Thus, the Theorem is proved.

As it was mentioned above, if $\vec{\alpha}\times\vec{\beta}=0$ then, according to (1.22) and (2.18) one has

$$\beta_0^2-\alpha_0^2=1,\quad \alpha_0\vec{\alpha}-\beta_0\vec{\beta}=0. \tag{2.24}$$

So, one has to conclude that the next statement is true:

Statement 4. *If* $\vec{\alpha}=k\vec{\beta}$ *then* $k=\beta_0/\alpha_0$:

$$\vec{\alpha}\times\vec{\beta}=0 \;\Leftrightarrow\; \vec{\alpha}=(\beta_0/\alpha_0)\vec{\beta}. \tag{2.25}$$

Corollary. If $\vec{\alpha}\times\vec{\beta}=0$ then, according to the formula (2.24) one has

$$\vec{\beta}\beta_0=\vec{\alpha}\alpha_0,$$

and the relation (1.21) gives

$$\vec{\beta}^2\beta_0=\vec{\beta}\cdot\vec{\alpha}\,\alpha_0={\alpha_0}^2\beta_0.$$

Then, taking into account (2.25), the last relation leads to the equalities as follows:

$$\alpha_0=\pm|\vec{\beta}|,\;\; \vec{\alpha}=\vec{\beta}\beta_0/\alpha_0=\pm\vec{\beta}\beta_0/|\vec{\beta}|,\;\; |\vec{\alpha}|=\beta_0>0. \tag{2.26}$$

Note 6. Obviously, collinearity of the 3-dimensional vectors $\vec{\alpha}'$ and $\vec{\beta}'$ does not break under any boost along the common direction of these vectors. But the magnitudes of the vectors $\vec{\alpha}'$ and $\vec{\beta}'$ do change. According to the conditions (1.15), the 4-vector β^k is time-like (while, according to (1.20), the 4-vector α^k is space-like). Therefore, it is clear that the boost with the parameters

$$\vec{\upsilon} = \vec{\beta}'/\beta_0', \quad \gamma' = \beta_0' \tag{2.27}$$

makes $\vec{\beta}' \to \vec{\beta}'' = 0$ (obviously, this boost turns the charged particle in its instantaneously PF). Then

$$\beta'^k \to \beta''^k = (1, \vec{0}), \tag{2.28}$$

and, based on (2.24), we find

$$\alpha'^k \to \alpha''^k = (0, \vec{\alpha}''),$$

where, according to the Lorentz transformation formulas one has

$$\vec{\alpha}'' = \gamma'(\vec{\alpha}' - \alpha_0'\vec{\beta}'/\beta_0') = \beta_0'\vec{\alpha}' - \alpha_0'\vec{\beta}'.$$

$$(\vec{\alpha}' \times \vec{\beta}' = 0, \ \ \vec{\alpha}' \neq 0, \ \ \vec{\beta}' \neq 0)$$

The condition $\vec{\alpha}' \times \vec{\beta}' = 0$ provides the necessary condition

$$\vec{\alpha}''^2 = (\beta_0'\vec{\alpha}' - \alpha_0'\vec{\beta}')^2 = 1.$$

Besides, according to the Statement 4, the relations (2.24) and (2.25) give us the following ones:

$$\vec{\alpha}'' = \beta_0'\vec{\alpha}' - \alpha_0'\vec{\beta}_0' = \frac{\vec{\beta}'}{\alpha_0'} = \frac{\vec{\alpha}'}{\beta_0'}. \tag{2.29}$$

Of course, the relations similar to (2.26) are fulfilled in this case as well.

Note 7. Obviously, the condition (1.1) is not changing under any Lorentz transformation. Therefore, the Theorem 1 provides the possibility to investigate all physically significant phenomena of such motion in the special RF – in the instantaneously PRF of the charged particle. This, according to (1.25), means that one can choose arbitrary point of the trajectory as initial one and, according to Note 6, assume that the initial velocity of the motion is zero. Therefore, according to (2.7), the trajectory (respect to such RF) must be rectilinear and the motion laws have to be described by the formulas (2.15)-(2.16') where one has to assume

$$\beta_0 = 1, \quad \vec{\beta} = \vec{0}, \quad \alpha_0 = 0, \quad \vec{\alpha} = (1, 0, 0).$$

So, in this case one gets

$$\begin{aligned} &\xi_0 = \sinh\vartheta, \quad \xi_x = \cosh\vartheta - 1, \quad \xi_y = \xi_z = 0. \\ &(\xi_x + 1)^2 - \xi_0^2 = 1. \end{aligned} \tag{2.30}$$

Just this case of motion was called by M. Born [9] a *hyperbolic motion*.

In order to find physically significant features in any inertial RF it is enough to perform the correspondent Lorentz transformation from instantaneously PF of the charge to the given RF.

3. The External Field Configuration

It seems rather interesting to find a configuration of external electromagnetic fields F_{kl} that provide the motion law (1.14). As far in this case no LAD force exists one gets the exact equation ([10], §23)

$$m\dot{u}_k = eF_{kl}u^l. \tag{3.1}$$

Theorem 2. *The only field tensor F_{kl} that provides the motion satisfying the condition (1.1) corresponds to static uniform electromagnetic fields.*

Proof. Substituting into the motion equations (3.1) the motion laws (1.16) and (1.17) we find

$$\alpha_k \cosh\vartheta + \beta_k \sinh\vartheta = \frac{e}{ma_0}F_{kl}(\alpha^l \sinh\vartheta + \beta^l \cosh\vartheta). \tag{3.2}$$

The relation (3.2) can be satisfied identically if we choose the field tensor according to the conditions

$$\frac{e}{ma_0}F_{kl}\,\beta^l=\alpha_k, \qquad \frac{e}{ma_0}F_{kl}\,\alpha^l=\beta_k. \tag{3.3}$$

Introducing dimensionless values

$$\frac{e}{ma_0}F_{kl}\equiv\mathscr{F}_{kl}, \tag{3.4}$$

the system of equations (3.3) can be rewritten in simpler form, namely,

$$\mathscr{F}_{kl}\,\beta^l=\alpha_k, \quad \mathscr{F}_{kl}\,\alpha^l=\beta_k, \tag{3.5}$$

or, in detail

$$\begin{aligned}&-\mathscr{F}_{0\nu}\,\beta_\nu=\alpha_0,\ \ \mathscr{F}_{\mu 0}\,\beta_0-\mathscr{F}_{\mu\nu}\,\beta_\nu=\alpha_\mu,\\&-\mathscr{F}_{0\nu}\,\alpha_\nu=\beta_0,\ \ \mathscr{F}_{\mu 0}\,\alpha_0-\mathscr{F}_{\mu\nu}\,\alpha_\nu=\beta_\mu.\end{aligned} \tag{3.5'}$$

As far for the field tensor $\mathscr{F}_{kl}$ one has the only equations (3.5) that contain only 4-vectors $\alpha_k,\ \beta_k,$ this tensor can to be expressed by some combinations of the (constant) 4-vectors $\alpha_k,\ \beta_k$ only. Let us look for the skew-symmetric tensor $\mathscr{F}_{kl}$ in the form

$$\mathscr{F}_{kl}=C_1(\alpha_k\beta_l-\beta_k\alpha_l)+C_2\varepsilon_{klmn}\,\alpha^m\beta^n, \tag{3.6}$$

where C_1 and C_2 are arbitrary (real) functions (of 4-coordinates) should be found. Obviously, the factor C_1 must be a Lorentz scalar, while C_2 must be Lorentz pseudoscalar. Substituting (3.6) into (3.5) and taking into account the general conditions (1.15), (1.20) and (1.21) one obtains:

$$\begin{aligned}&C_1(\alpha_k\beta_l-\beta_k\alpha_l)\beta^l+C_2\varepsilon_{klmn}\,\alpha^m\beta^n\beta^l=\alpha_k\ \Rightarrow\ C_1\alpha_k=\alpha_k,\\&C_1(\alpha_k\beta_l-\beta_k\alpha_l)\alpha^l+C_2\varepsilon_{klmn}\,\alpha^m\beta^n\alpha^l=\beta_k\ \Rightarrow\ C_1\beta_k=\beta_k.\end{aligned}$$

Hence, the conditions (3.5) lead to $C_1=1,$ while the factor $C_2=C(t,x_1,x_2,x_3)$ remains undetermined yet. So, we find (for sake of brevity bellow we omit the subindex at the C_2)

$$\mathscr{F}_{kl}=\alpha_k\beta_l-\beta_k\alpha_l+C\varepsilon_{klmn}\alpha^m\beta^n. \tag{3.7}$$

Using the notations (3.4) one gets from (3.7) an expressions for an electric and magnetic fields as follows (we assume $F_{0\nu}=E_\nu,\ F_{\mu\nu}=\varepsilon_{\mu\nu\lambda}H^\lambda=-\varepsilon_{\mu\nu\lambda}H_\lambda,\ \varepsilon^{0123}=-\varepsilon_{0123}=1,$ see [10]):

$$\vec{E}=\frac{ma_0}{e}\left(\beta_0\vec{\alpha}-\alpha_0\vec{\beta}+C\vec{\beta}\times\vec{\alpha}\right),\qquad \vec{H}=\frac{ma_0}{e}\left(\vec{\beta}\times\vec{\alpha}-C(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\right). \tag{3.8}$$

$$\left(\alpha_k=(\alpha_0,-\vec{\alpha}),\ \ \beta_k=(\beta_0,-\vec{\beta})\right)$$

Based on the Lemma 3 one can rewrite (3.8) in more convenient form:

$$\begin{aligned}&\vec{E}=(m/e)\gamma\left(\vec{\omega}-\omega_0\vec{\upsilon}+C\vec{\upsilon}\times\vec{\omega}\right)=(m/e)\gamma^3(\vec{a}+C\vec{\upsilon}\times\vec{a}),\\&\vec{H}=(m/e)\gamma\left(\vec{\upsilon}\times\vec{\omega}-C(\vec{\omega}-\omega_0\vec{\upsilon})\right)=(m/e)\gamma^3(\vec{\upsilon}\times\vec{a}-C\vec{a})\end{aligned} \tag{3.8'}$$

(recall that $\omega_0=\gamma^4\vec{a}\cdot\vec{\upsilon},\ \vec{\omega}=\gamma^2\vec{a}+\gamma^4(\vec{a}\cdot\vec{\upsilon})\vec{\upsilon}$).

Based on (3.8), (3.8') one can conclude, first of all, that if $a_0=0$ then $\vec{E}=\vec{H}=0.$ This result is physically clear and has to be expected.

The next observation refers to the Lorentz force acting on the moving charge

$$\begin{aligned}\vec{F}_L&=e(\vec{E}+\vec{\upsilon}\times\vec{H})=m\gamma\left(\vec{\omega}-\omega_0\vec{\upsilon}+C\vec{\upsilon}\times\vec{\omega}+\vec{\upsilon}\times(\vec{\upsilon}\times\vec{\omega})-C\vec{\upsilon}\times\vec{\omega}\right)=\\&=m\gamma\left(\vec{\omega}(1-\vec{\upsilon}^2)+\vec{\upsilon}(\vec{\upsilon}\cdot\vec{\omega}-\omega_0)\right)=m\gamma\vec{\omega}(1-\vec{\upsilon}^2)=m\gamma^{-1}\vec{\omega}=m\gamma\left(\vec{a}+\gamma^2(\vec{a}\cdot\vec{\upsilon})\vec{\upsilon}\right)\end{aligned} \tag{3.9}$$

ISSN 1512-1461

(recall that $\vec{F}_L$ does not coincide with spatial part of 4-force). So, in all points of the trajectory of the charge moving in accordance with the condition (1.1) the Lorentz force acting on it does not depend on undetermined (pseudo)scalar function $C = C(t, x_1, x_2, x_3)$.

In order to find restrictions on the (pseudo)scalar function C one has to use Maxwell equations for electromagnetic field [10]. Taking into consideration that we have no field sources in the region under investigation (the region where the charged particle is moving), all Maxwell equations must be uniform:

$$\varepsilon^{klmn}\partial_l F_{mn} = 0, \quad \partial_l F^{lk} = 0. \qquad (k = 0,1,2,3) \tag{3.10}$$

Using the notation (3.4) and substituting the expression (3.7) into (3.10), we find the necessary restrictions:

$$\varepsilon^{klmn}\varepsilon_{mnij}\alpha^i\beta^j\partial_l C = 0, \quad \varepsilon^{klmn}\alpha_m\beta_n\partial_l C = 0, \quad (k = 0,1,2,3)$$

that, after obvious simplifications (see, e. g., [10], §6), can be presented as follows (here and bellow we denote $(\partial/\partial x^0)C \equiv \partial_t C$):

$$(\beta_0\vec{\alpha} - \alpha_0\vec{\beta})\cdot\nabla C = 0, \quad (\beta_0\vec{\alpha} - \alpha_0\vec{\beta})\partial_t C + (\vec{\alpha}\times\vec{\beta})\times\nabla C = 0; \tag{3.11}$$

$$(\vec{\alpha}\times\vec{\beta})\cdot\nabla C = 0, \qquad (\vec{\alpha}\times\vec{\beta})\partial_t C - (\beta_0\vec{\alpha} - \alpha_0\vec{\beta})\times\nabla C = 0. \tag{3.12}$$

Let us suppose firstly that $\vec{\alpha}\times\vec{\beta} \neq 0$. According to the Lemma 2, from the firs equations of the (3.11) and (3.12) follows (we assume that the spatial coordinate axes are directed along the (2.17) basis vectors: $Ox \;\; \vec{e}_1$, $Oy \;\; \vec{e}_2$, $Oz \;\; \vec{e}_3$):

$$\nabla C = \vec{e}_2\partial_y C, \qquad \partial_x C = \partial_z C = 0 \;\Rightarrow\; C = C(t, y). \tag{3.13}$$

Substituting these results into the second equations of the (3.11), (3.12) and using the Lemma 2 again, one gets

$$\begin{aligned} &|\beta_0\vec{\alpha} - \alpha_0\vec{\beta}|\,\vec{e}_1\partial_t C + |\vec{\alpha}\times\vec{\beta}|\,(\vec{e}_3\times\vec{e}_2)\,\partial_y C = 0; \\ &|\vec{\alpha}\times\vec{\beta}|\,\vec{e}_3\partial_t C - |\beta_0\vec{\alpha} - \alpha_0\vec{\beta}|\,(\vec{e}_1\times\vec{e}_2)\partial_y C = 0. \end{aligned} \tag{3.14}$$

According to (2.19), one has $\vec{e}_1\times\vec{e}_2 = \vec{e}_3$, $\vec{e}_3\times\vec{e}_2 = -\vec{e}_2\times\vec{e}_3 = -\vec{e}_1$. Therefore the system (3.14) reduces to the homogeneous system of linear equations respect to the variables $\partial_t C$ and $\partial_y C$:

$$\begin{aligned} &|\beta_0\vec{\alpha} - \alpha_0\vec{\beta}|\,\partial_t C - |\vec{\alpha}\times\vec{\beta}|\,\partial_y C = 0, \\ &|\vec{\alpha}\times\vec{\beta}|\,\partial_t C - |\beta_0\vec{\alpha} - \alpha_0\vec{\beta}|\,\partial_y C = 0. \end{aligned} \tag{3.15}$$

According to (2.18), the system (3.15) has nonzero determinant:

$$\begin{bmatrix} |\beta_0\vec{\alpha} - \alpha_0\vec{\beta}| & -|\vec{\alpha}\times\vec{\beta}| \\ |\vec{\alpha}\times\vec{\beta}| & -|\beta_0\vec{\alpha} - \alpha_0\vec{\beta}| \end{bmatrix} = (\vec{\alpha}\times\vec{\beta})^2 - (\beta_0\vec{\alpha} - \alpha_0\vec{\beta})^2 = -1,$$

and therefore has a trivial solutions only:

$$\partial_t C = \partial_y C = 0. \tag{3.16}$$

The last result together with (3.13) shows that if $\vec{\alpha}\times\vec{\beta} \neq 0$, then the undetermined coefficient C in (3.7) is constant and therefore the tensor (3.7) (see also (3.8)) describes a static and uniform electromagnetic field.

Let us consider now the case when $\vec{\alpha}\times\vec{\beta} = 0$. As it was shown above, in this case the relations (2.24)-(2.26) are fulfilled. Applying these relations to the general formulas (3.8), one obtains

$$\begin{aligned} &\vec{E} = (m/e)a_0(\beta_0^2 - \alpha_0^2)\vec{\beta}/\alpha_0 = \pm(\vec{\beta}/|\vec{\beta}|)(ma_0/e) = \pm(\vec{\upsilon}/|\vec{\upsilon}|)(m/e)a_0, \\ &\vec{H} = -C\vec{E}. \end{aligned} \tag{3.17}$$

Thus, according to (2.7), in this case we have rectilinear motion along collinear uniform electric and magnetic fields. As far parallel to charge's velocity magnetic field does not affect on the particle, therefore, the case reduces to the charge's motion along a uniform electrostatic field. Let us check that the magnetic field in (3.17) is static as well as electric one.

Indeed, if $\vec{\alpha}\times\vec{\beta}=0$ then general equations (3.11) and (3.12) give us the restrictions on C as follows:

$$(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\cdot\nabla C=0,\ (\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\partial_t C=0 \quad (\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\times\nabla C=0.$$

As far, according to (2.18), $(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})\neq 0$, these restrictions give us

$$\nabla C=\partial_t C=0 \Leftrightarrow C=\text{const.}$$

Thus, the electromagnetic field is static and uniform in this case as well.

In order to finish the proof of the Theorem 2, one has to show that the solution constructed above is unique (no electromagnetic fields except constructed in (3.7) can exist). Indeed, let us assume that there exists an extra electromagnetic field with a field tensor F_{kl}' such that the motion equation

$$e(F_{kl}+F_{kl}')u^l=m\dot{u}_k \tag{3.18}$$

is satisfied simultaneously with the equation (3.1). Here we assume that the tensor F_{kl} is expressed according to (3.7) with arbitrary constant C. Therefore we get for the field F_{kl}' the following condition:

$$(e/m)F_{kl}'u^l=0 \quad\Leftrightarrow\quad F_{k0}'u^0+F_{k\lambda}'u^\lambda=0 \quad (k=0,1,2,3) \tag{3.19}$$

$$\Leftrightarrow \vec{E}'\cdot\vec{\upsilon}=0 \qquad (\text{for } k=0), \tag{3.20}$$

$$\vec{E}'+\vec{\upsilon}\times\vec{H}'=0 \ \ (\text{for } k=1,2,3) \tag{3.20$'$}$$

(one should note that the condition (3.20) follows to the conditions (3.20′)).

For any point P of the particle's trajectory we can assume $\vec{\upsilon}=\vec{\upsilon}_p=0$ (according to the Note 7 such RF can be chosen without loss of generality) and then the relation (3.20′) gives

$$\vec{E}'_P=-\vec{\upsilon}_p\times\vec{H}'_p=0 \qquad\Rightarrow\quad \partial_t\vec{E}'_P=-(\partial_t\vec{\upsilon}_p)\times\vec{H}'_p-\vec{\upsilon}_p\times\partial_t\vec{H}'_p. \tag{3.21}$$

Therefore, assuming

$$\partial_t\vec{E}'_P=0 \tag{3.22}$$

we obtain

$$(\partial_t\vec{\upsilon}_p)\times\vec{H}'_p=0. \tag{3.23}$$

Taking into account that, according to (1.16) and (1.17) we have

$$\partial_t\vec{\upsilon}_p=\partial_t(\gamma^{-1}\vec{u})_p=-\gamma_p(\vec{\upsilon}_p\cdot\partial_t\vec{\upsilon}_p)\vec{u}_p+\gamma^{-1}{}_p\partial_t\vec{u}_p=$$
$$=\gamma^{-1}{}_p(\partial_\theta\vec{u})_p(\partial_t\theta)_p=\gamma^{-2}{}_p\vec{\omega}^{(0)}a_0=a_0{}^2\vec{\alpha}\neq 0,$$

(3.23) leads to $\vec{H}'_p=0$. Therefore, in the chosen RF for any point P of the particle's trajectory we have

$$(F_{kl}')_P=0 \tag{3.24}$$

Being Lorentz covariant the condition (3.24) must be true in any inertial RF.

Hence, we have no extra field in the motion law (3.18) except static and uniform fields (3.8) or (3.17). Thus, the Theorem 2 is completely proved.

It is instructive to compare Lorentz invariants of the electromagnetic fields (3.8) and (3.17). As far for the invariants of the electromagnetic field we get

$$\begin{aligned}\vec{E}^2-\vec{H}^2&=(ma_0/e)^2(1-C^2)\left[(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})^2-(\vec{\alpha}\times\vec{\beta})^2\right],\\(\vec{E}\cdot\vec{H})^2&=(ma_0/e)^4(-C)^2\left[(\beta_0\vec{\alpha}-\alpha_0\vec{\beta})^2-(\vec{\alpha}\times\vec{\beta})^2\right]^2,\end{aligned} \tag{3.25}$$

ISSN 1512-1461

then, according to (2.18) one obtains

$$\vec{E}^2 - \vec{H}^2 = (ma_0/e)^2(1-C^2), \quad (\vec{E}\cdot\vec{H})^2 = (ma_0/e)^4 C^2. \tag{3.26}$$

Thus, the both Lorentz invariants of the electromagnetic fields (3.8) and (3.17) coincide; it does not matter are the vectors $\vec{\alpha}$ and $\vec{\beta}$ collinear, or not. The reason of such coincidence is that one always can transfer to the inertial RF such that in the chosen RF the (uniform) electric and the magnetic fields become collinear, and so, in order to calculate invariants, we always can assume, without loose of generality, that the fields have configuration (3.17). As it is known (see, e.g. [16], problem 10.62), the velocity $\vec{V}$ which provides such Lorentz transformation can be expressed (not uniquely) as follows:

$$\vec{V} = \frac{\vec{E}\times\vec{H}}{2(\vec{E}\times\vec{H})^2}\left[\vec{E}^2 + \vec{H}^2 - \left((\vec{E}^2 - \vec{H}^2)^2 + 4(\vec{E}\cdot\vec{H})^2\right)^{1/2}\right]. \tag{3.27}$$

Using here the results (3.26) and (3.8) and taking into account (2.18), one gets

$$\vec{V} = \frac{(\beta_0\vec{\alpha} - \alpha_0\vec{\beta})\times(\vec{\beta}\times\vec{\alpha})(C^2+1)}{2(\beta_0\vec{\alpha} - \alpha_0\vec{\beta})^2(\vec{\beta}\times\vec{\alpha})^2(C^2+1)^2}\left[(2\beta_0^2 - 2\alpha_0^2 - 1)(C^2+1) - \left((1-C^2)^2 + 4C^2\right)^{1/2}\right] =$$

$$= \frac{\beta_0\vec{\beta} - \alpha_0\vec{\alpha}}{2(\beta_0^2 - \alpha_0^2)(\beta_0^2 - \alpha_0^2 - 1)}\left[(2\beta_0^2 - 2\alpha_0^2 - 1) - 1\right] = \frac{\beta_0\vec{\beta} - \alpha_0\vec{\alpha}}{\beta_0^2 - \alpha_0^2}. \tag{3.28}$$

Hence, the Lorentz transformation which makes the (uniform) electric and the magnetic fields collinear is the same as was found in the Theorem 1 (see the formula (2.23)). Therefore, such Lorentz transformation makes collinear four 3-vectors: $\vec{\alpha}', \vec{\beta}', \vec{E}'$ and $\vec{H}'$ (see (3.17)). Moreover, as it is mentioned in the Note 6, one can perform the additional boost along the common direction of these 3-vectors to transfer in the moving charge's instantaneously PF. It must be emphasized that such boost does not change the vectors $\vec{E}'$ and $\vec{H}'$ while the vectors $\vec{\alpha}'$ and $\vec{\beta}'$ do change. According to (2.28) and (2.29) one gets:

$$\vec{\beta}' \to \beta''^k = (1, \vec{0}), \quad \vec{\alpha}' \to \alpha''^k = (0, \vec{\beta}'/\alpha_0') = (0, \vec{\alpha}'/\beta_0'), \quad |\vec{\alpha}''| = 1. \tag{3.29}$$

Obviously, one can choose this common direction as Ox axis. Then, according to (3.29), the fields (3.17) get the form as follows:

$$\vec{E}'' = \left(a_0 m/e, 0, 0\right), \quad \vec{H}'' = \left(C, 0, 0\right). \tag{3.30}$$

$$\left(\forall C \in \quad\right)$$

Hence, we have proved the following

Theorem 3. *Any motion of charged particle which corresponds to zero LAD force (hyperbolic motion) can be studied in special Reference Frame where initial speed of the particle is zero and initial acceleration is provided by the uniform electrostatic field. The permissible magnetic field is static and uniform as well and is collinear with the electric field.*

All other kinds of hyperbolic motion of a charged particle can be obtained from the previous case by appropriate Lorentz transformation. The trajectory of charged particle motion obtained by such Lorentz transformation is a hyperbole.

4. Motion in Purely Electrostatic (uniform) Field

It seems interesting to compare the results obtained above with a motion equations of point charge in uniform electrostatic field $\vec{E} = \text{const}$ (see, e. g., [10], §20):

$$\dot{u}_0 = (e/m)\vec{E}\cdot\vec{u}, \quad \dot{\vec{u}} = (e/m)\vec{E}u_0 \quad \Rightarrow$$

$$\ddot{u}_0 = (e/m)^2\vec{E}^2 u_0, \quad \ddot{\vec{u}} = (e/m)^2\vec{E}(\vec{E}\cdot\vec{u}).$$

If one supposes

$$\vec{u} = \vec{u}\ \ + \vec{u}_\perp, \qquad \vec{u}\ \ = \vec{E}(\vec{E}\cdot\vec{u})/\vec{E}^2, \qquad \vec{u}_\perp = \vec{u} - \vec{u}\ \ ,$$

$$-\omega^2 = a_0{}^2 = (e/m)^2\vec{E}^2, \qquad (\vec{a} \neq \vec{a}_0)$$

then

$$\vec{E}\cdot\vec{u} = \vec{E}\cdot\vec{u}\ \ , \quad \vec{E}\cdot\vec{u}_\perp = 0$$

and the motion equations can be rewritten as follows:

$$\begin{aligned} &\ddot{u}_0 = (e/m)^2\vec{E}^2 u_0, \\ &\ddot{\vec{u}}\ \ = (e/m)^2\vec{E}(\vec{E}\cdot\vec{u}) = (e/m)^2\vec{E}^2\vec{u}\ \ , \\ &\dot{\vec{u}}_\perp = 0. \end{aligned} \tag{4.1}$$

Thus, one can see that the known (see, e.g., [16], Chapt. IX, §2, problem № 692) motion law of a charged particle in uniform electrostatic field that has a form:

$$\begin{aligned} &x_0 = \vec{a}_0\cdot\vec{\beta}\,\frac{\cosh\vartheta - 1}{a_0^2} + \beta_0\frac{\sinh\vartheta}{a_0}, && \vec{x} = \vec{\beta}\tau + \vec{a}_0\beta_0\frac{\cosh\vartheta - 1}{a_0^2} + \vec{a}_0(\vec{a}_0\cdot\vec{\beta})\frac{\sinh\vartheta - \vartheta}{a_0^3}, \\ &u_0 = \vec{a}_0\cdot\vec{\beta}\,\frac{\sinh\vartheta}{a_0} + \beta_0\cosh\vartheta, && \vec{u} = \vec{\beta} + \vec{a}_0\beta_0\frac{\sinh\vartheta}{a_0} + \vec{a}_0(\vec{a}_0\cdot\vec{\beta})\frac{\cosh\vartheta - 1}{a_0{}^2}, \\ &\omega_0 = \vec{a}_0\cdot\vec{\beta}\cosh\vartheta + a_0\beta_0\sinh\vartheta, && \vec{\omega} = \vec{a}_0\beta_0\cosh\vartheta + \vec{a}_0(\vec{a}_0\cdot\vec{\beta})\frac{\sinh\vartheta}{a_0}, \end{aligned} \tag{4.2}$$

$$(\vartheta = a_0\tau)$$

does not coincide with the expressions (1.14), (1.16) and (1.17) completely. Namely, in contrast with the solutions (1.17) and (1.18) from the (4.2) we get the next relations:

$$\begin{aligned} &\dot{\omega}_0 = a_0(\vec{a}_0\cdot\vec{\beta}\sinh\vartheta + \beta_0 a_0\cosh\vartheta) \quad (= a_0{}^2 u_0), \\ &\dot{\vec{\omega}} = \vec{a}_0(\beta_0 a_0\sinh\vartheta + \vec{a}_0\cdot\vec{\beta}\cosh\vartheta) \quad (= a_0{}^2\vec{u} - a_0{}^2\vec{\beta}_\perp), \end{aligned} \tag{4.3}$$

where $\vec{\beta}_\perp = \vec{u}_\perp$ is defined according to (2.2). So, (4.2) satisfies the equation (1.11) if and only if

$$\begin{aligned} &\vec{\beta}_\perp = 0, \\ &\vec{a}_0(\vec{a}_0\cdot\vec{x}) = \vec{a}_0{}^2\vec{x}, \qquad (a_0 \neq 0) \\ &\vec{x}_\perp = 0. \end{aligned} \tag{4.4}$$

If the condition (4.4) does not fulfill then the radiation reaction force (LAD force) is nonzero.

Hence, we have proved the following statement:

Statement 5. *The radiation reaction force – the LAD force which acts on a point charge moving in uniform electrostatic field equals to zero if and only if the particle is moving along the field.*

Obviously, in such case (in the given reference frame) a trajectory is rectilinear and the motion law is described by the equation (2.10). This is just the case which is called by M. Born [9] "hyperbolic motion".

Note 8. As far in a general case the LAD force is

$$F_e^i = (2/3)e^2(g^{ik} - u^i u^k)\dot{\omega}_k = (2/3)e^2\left[\dot{\omega}^i - u^i\left(u_0\dot{\omega}_0 - \vec{u}\cdot\dot{\vec{\omega}}\right)\right],$$

which for the motion described by equations (4.2) (in the reference frame where the strength of an electric field is $\vec{E} = \vec{a}_0 m/e = \text{const}$) gives us

$$F_e^i = (F_e^0, \vec{F}_e),$$

$$F_e^0 = (2/3)e^2(\dot{\omega}^0 - u^0 a_0^2(1+\vec{u}_\perp^2)) = (2/3)e^2 a_0^2(u^0 - u^0(1+\vec{u}_\perp^2)) =$$
$$= -(2/3)e^2 a_0^2 u^0 \vec{u}_\perp^2, \qquad (4.5)$$

$$\vec{F}_e = (2/3)e^2\left(\dot{\vec{\omega}} - \vec{u}a_0^2(1+\vec{u}_\perp^2)\right) = (2/3)e^2 a_0^2\left(\vec{u} - \vec{u}_\perp - \vec{u}(1+\vec{u}_\perp^2)\right) =$$
$$= -(2/3)e^2 a_0^2(\vec{u}_\perp + \vec{u}\vec{u}_\perp^2). \qquad (4.5')$$

Obviously, this result satisfies the general condition

$$F_e^i u_i = -(2/3)e^2 a_0^2\left[\vec{u}_\perp^2 u_0^2 - (\vec{u}_\perp + \vec{u}_\perp^2\vec{u})\cdot\vec{u}\,\right] =$$
$$= -(2/3)e^2 a_0^2 \vec{u}_\perp^2(u_0^2 - 1 - \vec{u}^2) = 0.$$

The results (4.5)-(4.5′) show again that the following statement is true:

Statement 6. *If* $\vec{\beta}_\perp \neq 0$, *then along the component* $\vec{\beta}$ *except the uniform electric field* $\vec{E} = \text{const}$ *acts a force* $\vec{F}_e = -(2/3)e^2 a_0^2 \vec{u}\,\vec{u}_\perp^2$ *that makes the motion along the field non hyperbolic.*

5. Discussion and conclusion

The results obtained in the article show that the motion with zero radiation reaction force (the Lorentz- Abraham-Dirac force) exists in the nontrivial case if and only if the motion is hyperbolic (relativistic analogies of the uniformly accelerated motion). The trajectory of such motion is studied as well. There is shown that in general this trajectory is a hyperbole. We have proven that in general case one can study this motion in the special inertial RF where the motion of the charged particle is rectilinear and the initial velocity of it is zero. For arbitrary initial conditions there is found the explicit expression for the Lorentz transformation parameter that allows to perform transformation to such RF. There is find out the necessary and sufficient configuration of the electromagnetic field providing the correspondent motion law of the charge and explicit expressions for it in general RF and in special one mentioned above.

There is shown that in general a uniform electrostatic field does not provide the motion with zero LAD force. In order to find the exact motion law and to estimate the radiated energy-momentum there is necessary to solve the third order differential equation that describes the motion with nonzero LAD force. The relations obtained (see (4.5)-(4.5′)) indicate that there is possible to use $\vec{u}_\perp$ as a small parameter for built up the solution of that equation numerically or executing corresponding decomposition.

On our opinion, it seems rather interesting to investigate how much does the radiated energy-momentum intensity change when the charge motion slightly deviates from the hyperbolic regime.

Besides, there is interesting to study the influence of the particle's own magnetic moment on the phenomena considered above.

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Article received: 2019-12-10