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ON APPROXIMATE SOLUTION OF THE DIRICHLET HARMONIC PROBLEM FOR AN INFINITE PLANE WITH A CUT-TYPE HOLE

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Abstract

The possibility of application of the modified version of MFS to approximate solution of the Dirichlet problem for harmonic function in the case of infinite plane with cut-type hole is shown. Numerical examples are considered to illustrate effectiveness and simplicity of the proposed method.

Key words*: Dirichlet harmonic problem, the method of fundamental solution (MFS), the modified MFS.*

Let a domain *D* be an infinite plane $z = x + iy \equiv (x, y)$ with a cut-type hole *B*, bounded with closed piecewise smooth contour $S(S = \sum_{j=1}^{m} S^j)$, without multiple points. Under a cut-type hole we mean that on each piece of the hole its width *d* is positive, and the length of the whole hole *l* >>*d*. We will mean that parametric equations of the smooth lines S^j are known: $x = x^j(t),\ y = y^j(t)$, where $a_j \le t \le \beta_j$.

We will consider the following Dirichlet exterior boundary problem for the Laplace equation. We need to find a function $u(z) \equiv u(x, y) \in C^2(D) \cap C(\overline{D})$, satisfying the following conditions:

$$\Delta u(z) = 0, \quad \forall z \equiv (x, y) \in D, \tag{1}$$

$$u(z) = g(z), \quad \forall z \in S, \tag{2}$$

$$u(z) = c + O\left(\frac{1}{|z|}\right), \ |z| \to \infty, \tag{3}$$

where $g(z) \equiv g(x, y)$ is a given continuous function on the contour *S* and *c* is a real constant, provided $|c| < \infty$. Problems of type (1), (2), (3) arise in hydromechanics, elasticity theory and other areas of mathematical physics. It is known [1,2] that the problem (1), (2), (3), is correct, i.e. a solution exists, is unique and depends continuously on data. The request of boundedness of the solution at infinity is essential for uniqueness of the solution [1,2]. It can be easily shown that if we fix in advance the value of the constant *c*, this will be a rather strong restriction. In fact, since under the conditions (1), (2), (3) for the function $u(x, y)$ the minimax principle [1,2] is fulfilled, the problem (1), (2), (3) with a fixed in advance *c* may turn out generally unsolvable. Hence, the constant *c* should be defined from the boundary condition (2) when solving the problem (1), (2), (3).

It should be noted that when dealing with the problem of type (1), (2), (3), finite difference methods and variational methods are not applicable directly. For this we must pass to a finite domain (e.g. a disk) by means of conformal mapping and then use these methods (see [3,4,5]). However, the problem of conformal mapping is more difficult and therefore it is not reasonable to solve the stated problem in this way (except for some cases).

It should be also noted that the possibility of solving the problems of type (1), (2), (3) is studied theoretically using singular integral equations (Cauchy type integral) in works of N. I. Muskhelishvili and his successors (e.g. see [6]).

In the paper [7] a method of approximate solution of problem (1), (2), (3) for general form of a hole is described, which represents a synthesis of the method of fundamental solutions (MFS) (Kupradze-Aleksidze [8,9,10]) and the method of conformal mapping. By use of these methods, a modified system of fundamental solutions of a Laplace operator

$$\{\psi_k(z)\}_{k=1}^{\infty} \equiv \{\psi(z,\tilde{z}_k\}_{k=1}^{\infty} = \left\{ \ln\left| \frac{R^2(\tilde{z}_k - z)}{(z - z_0)(\tilde{z}_k - z_0)} \right| \right\}_{k=1}^{\infty} \tag{4}$$

is constructed. Here z_0 is the "center" of the domain (hole) B; R is a real constant, $(0 < R < \infty)$, a radius of some disk G with center at the point z_0; $\{\tilde{z}_k\}_{k=1}^{\infty}$ is a countable everywhere-dense set of points, located on the auxiliary closed Liapunov contour $\tilde{S}$, which lies inside the domain (hole) B, bounded by the contour S and $\min \rho(S,\tilde{S}) > 0$, where ρ is a distance. It is easy to see that the following theorem is valid ([7]).

Theorem. *The system of functions* $\{\psi_k(z)\}_{k=1}^{\infty}$ *is linearly independent and complete not only in the space* $L_2(S)$ *but also in* $C(S)$.

On the basis of the stated theorem the approximate solution $u^{(N)}(x,y)$ to the problem (1), (2), (3) is sought in the form

$$u^{(N)}(z) \equiv u^{(N)}(x,y) = \sum_{k=1}^{N} a_k^{(N)} \ln\left| \frac{R^2(\tilde{z}_k - z)}{(z - z_0)(\tilde{z}_k - z_0)} \right|, \tag{5}$$

where the auxiliary points (simulation sources) $\tilde{z}_k$ $(k = 1, 2, \cdots, N)$ are located "uniformly" on the contour $\tilde{S}$. In the numerical experiments given below, coefficients $a_k^{(N)}$ are defined by means of a collocation method [9,10], i, e. a system of linear algebraic equations

$$\sum_{k=1}^{N} a_k^{(N)} \psi(z_j, \tilde{z}_k) = g(z_j) \tag{6}$$

is solved, where the collocation points $z_j\,(j = 1,2,\cdots,N)$ are located "uniformly" on S. From (5), for the approximate value of constant *c* we have

$$c^{(N)} = \lim_{z\to\infty} u^{(N)}(z) \equiv u^{(N)}(\infty) = \sum_{k=1}^{N} a_k^{(N)} \ln\left| \frac{R^2}{\tilde{z}_k - z_0} \right| \quad or \quad \left|c^{(N)}\right| < \infty . \tag{7}$$

It is known [9,10] that in a general case, while solving boundary problems using MFS the conditionality matrix of the system (6) improves when the contour $\tilde{S}$ tends to *S*.

Since $\tilde{S}$ is close to *S* in our case, this condition is fulfilled automatically.

In the tables below, on the basis of experiments the dynamics of the accuracy of a solution of the boundary value problems is given in dependence on *N*, when auxiliary points $\tilde{z}_k$ and collocation points $z_k (k=1,2,\cdots,N)$ are located "uniformly" respectively on the contours $\tilde{S}$ and S (in the examples an "optimal" contour in the sense of approximation of the boundary function is taken as the contour $\tilde{S}$). In the Examples 1 – 4 and 5 – 8, the functions

$$g_1(z)=\frac{|z|-1}{|z|+1}, z\in S,\ \ g_2(z)=\ln|z-z_o^*|-\ln|z|+1,\ z\in S$$

are considered respectively as the boundary functions. In all examples the origin z=0 was inside the domain B, z_o=0, z_o^*=1 and R=1. Evidently, for $g(z)=g_2(z)$, the exact solution to the problem (1), (2), (3) for $u(\infty)=1$, is the function

$$u(z)=\ln|z-z_o^*|-\ln|z|+1, z\in\overline{D}.$$

In the tables below, N is the number of "uniformly" located points on the contours S and $\tilde{S}$; ε_1 and ε_2 are the *a posteriori* estimates of maximal and mean square errors of the solution of the problem (1), (2), (3) for $g(z)=g_1(z); c^{(N)}=u^{(N)}(\infty)$; ε_3 and ε_4 are *a posteriori* estimates of the maximal error of the solution of the problem (1), (2), (3) on the contour S for $g(z)=g_2(z)$ respectively on the contour S and a circle $\Gamma(O,300)$ with center at the origin and radius 300, B is located inside this circle and

$$\gamma=|u(\infty)-u^{(N)}(\infty)|=|1-c^{(N)}|.$$

In *numerical* experiments the values $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ are found by use of M=10000 "uniformly" located points respectively on the contours S and Γ, and the calculations are carried out with double precision.

Example 1. As a domain D consider the exterior of an ellipse $S: x=a\cos t,\ y=b\sin t,\ 0\le t\le 2\pi$, and let an ellipse $\tilde{S}:\tilde{x}=a_1\cos t,\ \tilde{y}=b_1\sin t$ be auxiliary.

In this case, in numerical experiments the auxiliary points $\tilde{z}_k$, collocation points $z_k\ (k=1,2,\cdots,N)$ and the points $z_j\ (j=1,2,\cdots,M)$ to calculate ε_1 and ε_2, are "uniformly" located on $\tilde{S}$ and S with respect to the abscissa x and the ordinates of these points are found from the canonical equation of the ellipse. In the Table 1 the results of numerical experiments for the various values of a, b, a_1, b_1 and N are given.

Table 1

a=20; b=1; a_1=19.95; b_1=0.1				a=100; b=1; a_1=99.95; b_1=0.05		
N	ε_1	ε_2	C(N)	ε_1	ε_2	C(N)
100	0.69E-02	0.94E-03	0.7837	0.54E-01	0.77E-02	0.9338
500	0.42E-03	0.18E-04	0.7843	0.13E-02	0.88E-04	0.9364
1000	0.42E-03	0.18E-04	0.7843	0.76E-03	0.24E-04	0.9364
1500	0.29E-03	0.11E-04	0.7843	0.75E-03	0.18E-04	0.9364
2000	0.20E-03	0.67E-05	0.7843	0.74E-03	0.18E-04	0.9364
a=200; b=1; a_1=199.999; b_1=0.01				a=200; b=0.2; a_1=199.999; b_1=0.0001		
100	0.14E+00	0.20E-01	0.9586	0.24E+00	0.41E-01	0.9487
500	0.25E-02	0.24E-03	0.9636	0.48E-01	0.39E-02	0.9612
1000	0.13E-02	0.63E-04	0.9637	0.40E-01	0.19E-02	0.9618
1500	0.95E-03	0.37E-04	0.9638	0.24E-01	0.86E-03	0.9619
2000	0.78E-03	0.25E-04	0.9638	0.12E-01	0.39E-03	0.9620

Example 2. The domain D is an exterior of a rectangle with center at the origin, with the sides parallel to the axes and equal to $a^*=2a$ and $b^*=2b$. As $\tilde{S}$ a rectangle of the same type is considered with $a_1^* = 2a_1$ and $b_1^* = 2b_1$. In the Table 2 the results of the numerical experiments for the various values of a, b, a_1, b_1 and N are given.

Table 2

a=20; b=1; a_1=19.99; b_1=0.99				a=100; b=1; a_1=99.9; b_1=0.9		
N	ε_1	ε_2	$C^{(N)}$	ε_1	ε_2	$C^{(N)}$
100	0.32E-01	0.13E-01	0.7833	0.74E-01	0.15E-01	0.9321
500	0.44E-02	0.14E-02	0.7925	0.13E-02	0.16E-02	0.9370
1000	0.13E-02	0.41E-03	0.7932	0.33E-02	0.37E-03	0.9373
1500	0.77E-03	0.17E-03	0.7934	0.11E-02	0.12E-03	0.9373
2000	0.76E-03	0.89E-04	0.7934	0.57E-03	0.71E-04	0.9374
a=200; b=1; a_1=199.999; b_1=0.9				a=200; b=0.1; a_1=199.999; b_1=0.09		
100	0.94E-01	0.20E-01	0.9577	0.60E-01	0.10E-01	0.9613
500	0.31E-01	0.34E-02	0.9634	0.28E-01	0.15E-02	0.9641
1000	0.12E-01	0.11E-02	0.9639	0.50E-02	0.87E-04	0.9642
1500	0.59E-02	0.52E-03	0.9640	0.43E-02	0.70E-04	0.9642
2000	0.31E-02	0.26E-03	0.9641	0.37E-02	0.60E-04	0.9642

Example 3. As the cut the interior of "crescent" with the center at the origin and semi-axes a and b is taken. We mean a cut with the shape formed by the intersection of two circles of the same radii. As $\tilde{S}$ a "crescent" of the same kind with semi-axes a_1 and b_1 is taken. In the Table 3 the results of numerical experiments for various a, b, a_1, b_1 and N are given.

Table 3

a=20; b=1; a_1=19.999; b_1=0.01				a=100; b=1; a_1=99.99; b_1=0.01		
N	ε_1	ε_2	$C^{(N)}$	ε_1	ε_2	$C^{(N)}$
100	0.22E-01	0.34E-02	0.7780	0.54E-01	0.79E-02	0.9345
500	0.76E-02	0.54E-03	0.7806	0.30E-02	0.29E-03	0.9359
1000	0.48E-02	0.24E-03	0.7808	0.18E-02	0.12E-03	0.9361
1500	0.35E-02	0.14E-03	0.7808	0.13E-02	0.70E-04	0.9361
2000	0.28E-02	0.10E-03	0.7808	0.93E-03	0.45E-04	0.9361
a=200; b=1; a_1=199.999; b_1=0.001				a=200; b=0.2; a_1=199.9999; b_1=0.0001		
100	0.14E+00	0.20E-01	0.9580	0.24E+00	0.41E-01	0.9680
500	0.26E-02	0.32E-03	0.9634	0.44E-01	0.50E-02	0.9605
1000	0.17E-02	0.12E-03	0.9636	0.40E-01	0.19E-02	0.9617
1500	0.14E-02	0.84E-04	0.9636	0.24E-01	0.87E-03	0.9619
2000	0.11E-02	0.58E-04	0.9637	0.12E-01	0.40E-03	0.9619

Example 4. A domain B with boundary $S = S^1 \cup S^2$ is considered as a cut, where the equations of the lines S^1 and S^2 have the following form:

$$S^1 : y = \frac{b}{1+x^2} - \frac{b}{1+a^2}, \qquad x \in [-a, a];$$

$$S^2 : y = -\frac{b}{1+x^2} + \frac{b}{1+a^2}, \qquad x \in [-a, a].$$

In the numerical experiments a contour $\tilde{S} = \tilde{S}^1 \cup \tilde{S}^2$ is taken as an auxiliary contour, where

$$\tilde{S}^1 : \tilde{y} = \frac{b_1}{1+\tilde{x}^2} - \frac{b_1}{1+a_1^2}, \qquad \tilde{x} \in [-a_1, a_1];$$

$$\tilde{S}^2 : \tilde{y} = -\frac{b_1}{1+\tilde{x}^2} + \frac{b_1}{1+a_1^2}, \qquad \tilde{x} \in [-a_1, a_1].$$

Actually, the values a and b represent the semi-axes of the figure B. In the Table 4 the results of numerical experiments for the various of a, b, a_1, b_1 and N are given.

Table 4

a=20; b=1; a_1=19.999; b_1=0.9				a=100; b=1; a_1=99.9999; b_1=0.7		
N	ε_1	ε_2	$C^{(N)}$	ε_1	ε_2	$C^{(N)}$
100	0.44E-01	0.23E-01	0.7417	0.11E+00	0.40E-01	0.9038
500	0.14E-01	0.31E-02	0.7624	0.21E-01	0.64E-01	0.9280
1000	0.81E-02	0.12E-02	0.7645	0.77E-02	0.25E-02	0.9306
1500	0.58E-02	0.70E-03	0.7652	0.41E-02	0.14E-02	0.9314
2000	0.46E-02	0.46E-03	0.7654	0.32E-02	0.95E-03	0.9318
a=200; b=1; a_1=199.99999; $b_1=10^{-5}$				a=200; b=0.2; a_1=199.999999; $b_1=10^{-6}$		
100	0.21E+00	0.56E-01	0.9280	0.30E+00	0.84E-01	0.9113
500	0.44E-01	0.92E-02	0.9570	0.11E+00	0.17E-01	0.9545
1000	0.18E-01	0.37E-02	0.9598	0.10E+00	0.86E-02	0.9585
1500	0.99E-02	0.22E-02	0.9606	0.80E-01	0.55E-02	0.9596
2000	0.84E-02	0.15E-02	0.9609	0.56E-01	0.37E-02	0.9602

Example 5. Take $g(z)=g_2(z)$ as the boundary function $g(z)$. If for the values a and b we consider only two cases: 1) a=200, b =0.01; 2) a=200, b=0.001 then for the above indicated four holes we get the following numerical results.

For the Example 1:

Table 5

1) a_1=199.999999; b_1=0.00001				2) a_1=199.999999; b_1=0.00001		
N	ε_3	ε_4	γ	ε_3	ε_4	γ
800	0.5E-06	0.17E-08	0.146E-08	0.3E-04	0.18E-06	0.147E-06
1600	0.4E-06	0.52E-09	0.435E-09	0.2E-04	0.52E-07	0.437E-07

For the Example 2:

Table 6

1) a_1=199.999; b_1=0.00001				2) a_1=199.9999; b_1=0.000001		
N	ε_3	ε_4	γ	ε_3	ε_4	γ
802	0.61E-04	0.83E-09	0.672E-09	0.96E-04	0.79E-09	0.664E-09
1602	0.60E-04	0.41E-09	0.334E-09	0.95E-04	0.40E-09	0.329E-09

For the Example3:

Table 7

1) a_1=199.999999; b_1=0.00001				2) a_1=199.999999; b_1=0.00001		
N	ε_3	ε_4	γ	ε_3	ε_4	γ
800	0.2E-05	0.11E-08	0.883E-09	0.3E-04	0.11E-06	0.888E-07
1600	0.1E-05	0.52E-09	0.437E-09	0.2E-04	0.53E-07	0.439E-07

For the Example 4:

Table 8

1) a_1=199.999999; b_1=0.00001				2) a_1=199.999999; b_1=0.00001		
N	ε_3	ε_4	γ	ε_3	ε_4	γ
800 1600	0.5E-06 0.4E-06	0.13E-08 0.61E-09	0.113E-08 0.509E-09	0.3E-04 0.2E-04	0.13E-06 0.60E-07	0.109E-06 0.500E-07

The models of the holes in the Example 5 are more complex than in the Examples 1-4, however, the accuracy of the solution of the boundary problem (1), (2), (3) is higher. This is stipulated by the fact that in the Example 5 the boundary function $g(z) = g_2(z)$, $g(\infty) = 1$, is analytic on the boundary S and thus the rate of convergence is higher (see [10]).

In the Tables 5-8 the numbers $N =$800, 802, 1600, 1602 are chosen specially for the function $g(z) = g_2(z)$. The point is that the function $g_2(z)$, $z \in S$ has peaks in the neighbourhood of the points $z = 0$ and $z_0^* = 1$, i.e., the function $g_2(z)$ increases when $z \in S$ and $z \to 0$ or $z \to 1$. As the numerical experiments show, for such boundary functions the use of MFS or the modified MFS may give good results. Therefore, in our case, it is sufficient for the four auxiliary points (two upper and two lower) to be sufficiently close to the points $z = 0$ and $z_0^* = 1$ respectively. Evidently, this closeness depends on the rate of increase of the function $g_2(z)$ in the neighborhood of the points $z = 0$ and $z_0^* = 1$ and on the specific form of the fundamental solutions.

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