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The law of large numbers for weakly correlated random elements in Hilbert spaces

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Abstract

The law of large numbers for weakly correlated random elements with values in a separable Hilbert space is proved.

Key words: *law of large numbers, Hilbert space, covariance operator, coefficient of correlation.*

1. Introduction

In the paper [1 the law of large numbers is proved for weakly correlated random elements with values in spaces $l_p, 1 \leq p < \infty$. In the present paper we prove this result for the case of separable Hilbert space taking into account the specifics of the Hilbert space.

Let us consider a sequence of random variables $\xi_1, \xi_2, \dots, \xi_n, \dots$ defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and suppose that the given random variables have finite expectations. Denote $S_n = \sum_{i=1}^{n} \xi_i$, $n = 1,2, \dots$. We say that the sequence of random variables satisfies the law of large number (LLN) if the sequence $\left(\frac{S_n - \mathbb{E}S_n}{n}\right)$ converges to zero in probability as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{S_n - \mathbb{E}S_n}{n}\right| > \varepsilon\right] = 0$$

for every $\varepsilon > 0$, where $\mathbb{E}$ is the symbol of the expectation.

The LLN is extensively investigated in the literature especially for the case of a sequence of independent (or uncorrelated) random variables. Introduction of the concepts of dependence stimulated the study of LLN with this condition, e.g. for the random variables which have non-zero correlations. In this direction one of the first result was obtained in 1928 by Khinchine [2] (see also in [3], p. 62).

The purpose of this note is to prove an analogue of the abovementioned Khinchine's result for the case of an infinite-dimensional separable Hilbert space. For the concepts and auxiliary facts about the probability distributions in infinite-dimensional spaces considered below, we refer to the monograph [4].

2. Auxiliary concepts and facts

Let H be a real separable Hilbert space with an inner product (x, y), $x, y \in H$. As we know the norm in the space H is given by the inner product by the way $\|x\| = \sqrt{(x,x)}$, $x \in H$.

Denote by $\mathfrak{B}(H)$ the Borel σ-algebra in H. The map $\xi: \Omega \to H$ is called a random element with values in H, if $\xi^{-1}(\mathfrak{B}(H)) \subset \mathfrak{F}$. Hence ξ is a random element if it is a measurable map $\xi: \Omega \to H$.

Recall some important characteristics of random elements with values in a Hilbert space. We say that a random element ξ with values in H has a weak second order if $\mathbb{E}(h, \xi)^2 < \infty$ for every $h \in H$. It is well-known that if a random element ξ has a weak second order, then the expectation $\mathbb{E}\xi$ exists and is defined as the Pettis integral of the random element ξ (see [4], p. 116). In the sequel, without loss of generality we assume that $\mathbb{E}\xi = 0$ (otherwise instead of ξ we consider the random element $\xi - \mathbb{E}\xi$). For every random element with a weak second order the covariance operator R_ξ, $R_\xi: H \to H$, is defined by the following equation

$$(R_\xi h, h) = \mathbb{E}(h, \xi)^2, \qquad h \in H.$$

It is easy to see that R_ξ is positive ($(R_\xi h, h) \geq 0$ for every $h \in H$), symmetric ($(R_\xi h, g) = (R_\xi g, h)$ for every $h, g \in H$) and linear bounded operator. Note that the covariance operator is a natural analogue of the variance of random variables. For a covariance operator the following factorization is valid (see [4], Factorization lemma, p.149):

Lemma 2.1. *If $R: H \to H$ is a covariance operator, then there exists a Hilbert space H_1 and a linear* bounded *operator $A: H \to H_1$ such that $R = A^*A$ and $A(H)$ is dense in H_1.This representation is unique in the following sense: if $R = A_1^*A_1$, where A_1 maps H onto a dense subset of a Hilbert space H_2, then there exists an isometry U of space H_1 onto H_2 such that $A_1 = UA$.*

The Hilbert space H_1 appearing in the formulation of Lemma 2.1 obviously is not unique: if H_2 is an arbitrary Hilbert space and $U: H_1 \to H_2$ is an isometry then for the operator $A_1 = UA$ we also have $A_1^*A_1 = R$. Therefore, in Lemma 2.1 as H_1, in particular, we can take the space H itself and $R^{1/2}$ – the positive square root of R – as the operator $A: H \to H$.

An operator $T: H \to H$ is said to be nuclear if it admits the representation

$$Th = \sum_{i=1}^{\infty} (a_i, h)\, b_i\,, \quad \text{for all } h \in H,$$

where the two sequences $\{a_i\}$ and $\{b_i\}$ in H are such that $\sum_{i=1}^{\infty} \|a_i\|\, \|b_i\| < \infty$.

Let $\{\varphi_k\}$ be an orthonormal basis of H. Then for an operator $T: H \to H$ we define

$$tr(T) \equiv \sum_{k=1}^{\infty} (T\varphi_k, \varphi_k)$$

if the series is convergent.

This definition could depend on the choice of the orthonormal basis. However, note the following result concerning the nuclear operators: if $T: H \to H$ is a nuclear operator then $tr(T)$ is well defined independently of the choice of the orthonormal basis $\{\varphi_k\}$ and is called the trace of the operator T. If ξ is a random element with a strong second order ($\mathbb{E}\|\xi\|^2 < \infty$) and R_ξ is its covariance operator, then obviously $\mathbb{E}\|\xi\|^2 = tr(R_\xi)$.

Let ξ and η be a weak second order random elements with values in a separable Hilbert space H. Without loss of generality we assume that $\mathbb{E}\xi = \mathbb{E}\eta = 0$. Its cross-covariance operator $R_{\xi\eta}$ is defined by the equation [4]:

$$(R_{\xi\eta} h, g) = \mathbb{E}(h, \xi)(g, \eta), \qquad h, g \in H.$$

The cross-covariance operator $R_{\xi\eta}$ is a linear bounded operator, which maps the Hilbert space H in itself. Cross-covariance operator is an analog of the covariation of random variables for the multidimensional case. It is proved that the cross-covariance operator admits the representation

$$R_{\xi\eta} = A_\xi^* V_{\xi\eta} A_\eta, \qquad (2.1)$$

where A_ξ (resp. A_η) is a linear bounded operator from H in some Hilbert space H_ξ (resp. H_η) such that $R_\xi = A_\xi A_\xi^*$ (resp $R_\eta = A_\eta A_\eta^*$) and $A_\xi(H)$ (resp. $A_\eta(H)$) is dense in H_ξ (resp. H_η), $V_{\xi\eta}: H \to H$ is a linear bounded operator and $\|V_{\xi\eta}\| \le 1$.

Operator $V_{\xi\eta}$ defined by the equation (2.1) is called a coefficient of correlation. Like in one-dimensional case, the correlation coefficient is a measure of linear dependence of the random elements [5].

3. Main result

The following theorem is the generalization of the Khinchin's result mentioned above.

Theorem. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of random elements with values in a separable Hilbert space H. Let us further assume that each ξ_n has a strong second order, $\mathbb{E}\xi_n = 0$ and the covariance operator R_n exists, $n = 1,2, \dots$. Moreover, let there exists a non-negative real function c, defined on the set of non-negative integers, such that for the coefficient of correlation V_{mn} of the random variables ξ_m and ξ_n the following inequality $\|V_{mn}\| \le c(|m-n|)$ holds for any $m, n = 1,2, \dots$. Then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN if*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} c(i) \cdot \sum_{i=1}^{n} tr(R_i)}{n^2} = 0. \qquad (3.1)$$

Proof. Denote by $\{\varphi_k\}$ an orthonormal basis of a Hilbert space H, by R_i – covariance operator of ξ_i and by A_i – square root from the operator R_i, $i = 1,2, \cdots$. Keeping in mind (2.1), for any i and j we have

$$(R_{ij}\varphi_k, \varphi_k) = (V_{ij}A_j\varphi_k, A_i\varphi_k) \le \|V_{ij}\| \|A_i\varphi_k\| \|A_j\varphi_k\|.$$

Hence,

$$\mathbb{E}\left\|\sum_{i=1}^{n} \xi_i/n\right\|^2 = \frac{1}{n^2}\sum_{k=1}^{\infty} \mathbb{E}\left(\varphi_k, \sum_{i=1}^{n} \xi_i\right)^2 = \frac{1}{n^2}\sum_{k=1}^{\infty}\sum_{i,j=1}^{n} (R_{ij}\varphi_k, \varphi_k) \le$$

$$\le \frac{1}{n^2}\sum_{k=1}^{\infty}\sum_{i,j=1}^{n} \|V_{ij}\| \|A_i\varphi_k\| \|A_j\varphi_k\| \le \frac{2}{n^2}\sum_{k=1}^{\infty}\sum_{i=0}^{n-1} c(i) \sum_{i=1}^{n} (R_i\varphi_k, \varphi_k) =$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} c(i) \cdot \sum_{i=1}^{n}\left[\sum_{k=1}^{\infty} (R_i\varphi_k, \varphi_k)\right] = \frac{2}{n^2}\sum_{i=1}^{n} c(i) \cdot \sum_{i=1}^{n} tr(R_i).$$

By Chebyshev's inequality

$$\mathbb{P}\left[\left\|\sum_{i=1}^{n} \xi_i/n\right\| > \varepsilon\right] \le \mathbb{E}\left\|\sum_{i=1}^{n} \xi_i/n\right\|^2 / \varepsilon^2.$$

Therefore

$$\lim_{n\to\infty} \mathbb{P}\left[\left\|\sum_{i=1}^{n} \xi_i/n\right\| > \varepsilon\right] = 0,$$

and the theorem is proved.

Corollary. *If*

$$lim_{n\to\infty}\left(\sum_{i=1}^{n} tr(R_i)/n\right) = 0, \tag{3.2}$$

Then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ *satisfies the LLN.*

Proof. Indeed, since the correlation coefficients satisfy conditions $\|V_{mn}\| \le 1$, then as a function c we can take the constant function $c(k) \equiv 1$ for any $k = 0,1,\dots$. Hence (3.2) immediately follows from (3.1).

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