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On a property of a convergent series

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Abstract

An elementary proof of a property of convergent series consisting of non-increasing non-negative real numbers is proposed.

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It is known that a series consisting of a non-increasing sequence of non-negative numbers has several properties. One of them is the statement as follows.

Proposition 1. *Let* (x_n) *be a nonincreasing sequence of nonnegative numbers and assume that* $\sum_{n=1}^{\infty} x_n < \infty$. *Then the following is true* $\lim_{n\to\infty} n \cdot x_n = 0$.

In this note, we consider one remarkable property of such series, which was proved in a more general case in **[1, Theorem 2]**. We give another, more elementary proof of this property.

Proposition 2. *Let* $0 < p_1 < p_2 < \infty$ *and* (x_n) *be a nonincreasing sequence of nonnegative numbers. Then*

$$\sum_{n=1}^{\infty} x_n^{p_1} < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2}\right)^{p_1/p_2} < \infty .$$

We need the following elementary lemma.

Lemma. (i) *For any positive integer* k *we have*

$$\sum_{n=1}^{k} n^{-p_1/p_2} \le \frac{p_2}{p_2-p_1} k^{-p_1/p_2+1} . \tag{1}$$

(ii) *For any nonincreasing sequence* $\alpha_1, \alpha_2, \cdots$ *of nonnegative numbers the following inequalities hold:*

$$\frac{1}{2}\sum_{i=m+1}^{\infty} 2^i \alpha_{2^i} \le \sum_{i=2^m}^{\infty} \alpha_i \le \sum_{i=m}^{\infty} 2^i \alpha_{2^i}, \qquad \text{for any } m = 0, 1, \cdots, \tag{2}$$

provided that all the series converge.

Proof. (i) (1) is easy consequence of the integral test.

(ii) Let us prove (2). It is clear that

$$\sum_{i=2^m}^{\infty} a_i = \sum_{i=m}^{\infty} \sum_{j=2^i}^{2^{i+1}-1} a_j \le \sum_{i=m}^{\infty} 2^i a_{2^i} \qquad \text{for any } m = 0, 1, \cdots.$$

On the other hand, for any $m = 0, 1, \cdots$ we have

$$\sum_{i=2^m}^{\infty} a_i = \sum_{i=m}^{\infty} \sum_{j=2^i}^{2^{i+1}-1} a_j \ge \sum_{i=m}^{\infty} 2^i a_{2^{i+1}-1} \ge \frac{1}{2} \sum_{i=m}^{\infty} 2^{i+1} a_{2^{i+1}} = \frac{1}{2} \sum_{i=m+1}^{\infty} 2^i a_{2^i}.$$

Proof of Proposition 2. It is easy to see that, by assumption, the sequence of nonnegative numbers

$$\left(n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} \right)_{n=1}^{\infty}$$

is nonincreasing. Then, applying (2) we can obtain

$$\infty > \sum_{n=1}^{\infty} n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} \ge \frac{1}{2} \sum_{n=1}^{\infty} 2^{n(1-p_1/p_2)} \left(\sum_{k=2^n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} \ge \frac{1}{2} \sum_{n=1}^{\infty} 2^n \, x_{2^{n+1}}^{p_1} \ge \frac{1}{2} \sum_{n=2}^{\infty} x_n^{p_1}.$$

This proves the implication

$$\sum_{n=1}^{\infty} n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} x_n^{p_1} < \infty.$$

The proof of the inverse implication

$$\sum_{n=1}^{\infty} x_n^{p_1} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} < \infty$$

is less obvious. From **Lemma** we get

$$\sum_{n=1}^{\infty} n^{-p_1/p_2} \left(\sum_{k=n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} \le \left(\sum_{k=1}^{\infty} x_k^{p_2} \right)^{p_1/p_2} + \sum_{n=1}^{\infty} 2^{n(1-p_1/p_2)} \left(\sum_{k=2^n}^{\infty} x_k^{p_2} \right)^{p_1/p_2} \le$$

$$\le \sum_{k=1}^{\infty} x_k^{p_1} + 2x_2^{p_1} + 2^{2-p_1/p_2} \sum_{k=2}^{\infty} k^{p_1/p_2-1} x_k^{p_1} + 2 \sum_{n=2}^{\infty} 2^{n(1-p_1/p_2)} \left(\sum_{k=2^{n-1}}^{\infty} k^{p_1/p_2-1} x_k^{p_1} \right) \le$$

$$\le \left(3 + 2^{2-p_1/p_2}\right) \sum_{k=1}^{\infty} x_k^{p_1} + 2^{3-p_1/p_2} \sum_{k=1}^{\infty} k^{p_1/p_2-1} x_k^{p_1} \left(\sum_{n=1}^{k} n^{-p_1/p_2} \right) \le$$

$$\le \left(3 + 2^{2-p_1/p_2}\right) \sum_{k=1}^{\infty} x_k^{p_1} + 2^{3-p_1/p_2} \sum_{k=1}^{\infty} \frac{p_2}{p_2 - p_1} k^{-p_1/p_2+1} k^{p_1/p_2-1} x_k^{p_1} =$$

$$= \left(2 + 2^{2-p_1/p_2} + \frac{p_2}{p_2 - p_1} 2^{3-p_1/p_2}\right) \sum_{k=1}^{\infty} x_k^{p_1} < \infty,$$

which completes the proof.

It should be mentioned that this assertion can be found in **[2]**, but, unfortunately, the proof contains some misprints there.

If you take $p_1 = 1$ and $p_2 = 2$ then **Proposition 2** immediately implies the following

Corololary. *Let* (x_n) *be a nonincreasing sequence of nonnegative numbers. Then the series* $\sum_{n=1}^{\infty} x_n$ *converges if and only if the following condition is valid*

$$\sum_{n=1}^{\infty} n^{-1/2} \left(\sum_{k=n}^{\infty} x_k^2\right)^{1/2} < \infty.$$

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