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Orbitization of Quantum Mechanics

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Abstract

In this article some results are collected about finite orbits and orbits of observable operators at the states of quantum mechanical systems, orbital Hilbert spaces of finite orbits and Frechet-Hilbert spaces of all orbits, orbital operators acting in the Hilbert space of finite orbits and in the Frechet-Hilbert space of all orbits. Moreover, the problem of the approximate solution of equations containing orbital operators in the Hilbert space of finite orbits and in the Frechet-Hilbert space of all orbits is considered. The creation of operators orbits, orbital spaces, orbital operators, we call as orbitization of quantum mechanics or quantum mechanics with orbital operators and the totality of the results obtained as orbital quantum mechanics.

Keywords: *position operator, momentum operator, orbit of operator, Frechet-Hilbert spaces.*

Introduction

One of the axioms of quantum mechanics states, "To each real-valued function f on the classical phase space there is associated a self-adjoint operator $\hat{f}$ on the quantum Hilbert space". The operator $\hat{f}$ is called the quantization of f. There is considered the quantization's of a few very special classical observables, such as position, momentum, and energy ([1], Sect. 13, p.255). For a particle moving in $\mathbb{R}$ the classical phase space is $\mathbb{R}^2$ with the pairs (x, p), x being the particle's position and p being its momentum. In that case if the function f is the position function, $f(x, p) = x$, then the associated operator $\hat{f}$ is the position operator X, given by multiplication by x, i.e. quantization of position function is position operator X, defined by equality

$$X\psi(x) = x\psi(x)\,. \tag{1}$$

If f is the momentum function $f(x, p) = p$, then $\hat{f}$ is the momentum operator P, defined by the equality

$$P\psi(x) = -i\hbar \frac{d\psi}{dx}(x)\,, \tag{2}$$

where $\hbar$ is the Plank's constant. Note that quantization of xp, i.e. $(xp)^{\wedge}$ is neither XP nor PX, they are not self-adjoint and $XP \neq PX$. In this case a reasonable candidate for the quantization would be $\widehat{xp} = \frac{1}{2}(XP + PX)$.

One of the important model systems in quantum mechanics is the harmonic oscillator. This is a system capable of performing harmonic oscillations. In physics, the model of a harmonic oscillator plays an important role, especially in the study of small oscillations of systems around a position of stable equilibrium. An example of such oscillations in quantum mechanics is the oscillations of atoms in solids, molecules, etc. The harmonic oscillator in quantum mechanics is the quantum analogue of the simple harmonic oscillator. However, here we consider not the forces acting on the particle, but the Hamiltonian, that is the total energy for a harmonic oscillator, in which there is a parabolic potential energy. For the Hamiltonian $\mathcal{H}$ of the quantum harmonic oscillator the following representation is valid

$$\mathcal{H}\psi = \frac{P^2\psi}{2m} + m$$

$$\frac{\omega^2 X^2\psi}{2} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m}{2}\omega^2 X^2\psi = aP^2\psi + bX^2, \quad (3)$$

where m is the mass of the particle, ω is the frequency of oscillator, $a = \frac{1}{2m}$, $b = m\frac{\omega^2}{2}$, $P^2\psi = -\hbar^2\frac{d^2\psi(x)}{dx^2}$ and $X^2\psi = x^2\psi(x)$. According to ([1], Sect. 13.1) the Hamiltonian $\mathcal{H}$ is a quantization of classical Hamiltonian $\mathcal{H}(x,p)$ $=ap^2 + bx^2$, since each term contains only x or only p. The first term in the Hamiltonian represents the kinetic energy of the particle, and the second term represents its potential energy.

The mathematical model of quantum mechanics, created in the 30s of the XX century, describe quantum-mechanical systems by vectors of separable complex quantum Hilbert space ([1], Section 13.1, p.255) H and with unbounded self-adjoint operators defined on them. The quantum Hilbert space in this case is as usual the Hilbert space $L^2(\mathbb{R})$, the elements of which are called the states of quantum-mechanical systems. To each observable physical quantity it corresponds a self-adjoint operator on H. Such classical observables are above mentioned Hamiltonian $\mathcal{H}$ of the quantum harmonic oscillator, which corresponds to the observable "energy", the position operator X and the momentum operator P .

In the case of particle moving in real line R, the operators $\mathcal{H}$, X and P are described by unbounded self-adjoint operators in $H = L^2(\mathbb{R})$. Neither the position nor the momentum operator are defined as mappings the entire Hilbert space $L^2(\mathbb{R})$ into itself. After all, for $\psi \in L^2(\mathbb{R})$ the function $x\psi(x)$ may fail to be in $L^2(\mathbb{R})$. Similarly, a function ψ in $L^2(\mathbb{R})$ may fail to be differentiable, and even if it is differentiable, the derivative may fail to be in $L^2(\mathbb{R})$. The operators X and P are unbounded operators in the space $L^2(\mathbb{R})$.

Later, in the 50s of the XX century, the basic concepts of quantum mechanics were represented by the methods of the theory of generalized functions. It is very important that in the space of generalized functions observable operators became continuous. But the application of the basic and generalized functions spaces are difficult because of the non-metrizability of their topologies. In [2] the topologies of basic and generalized functions are presented as projective and inductive limits of the family of Frechet-Hilbert spaces and their strong duals, which simplifies the use of these spaces. The Frechet-Hilbert spaces were originally defined by us as a strict projective limits of the sequence of Hilbert spaces [2], but now this concept has been extended by European mathematicians and it is widely used without this requirement. This definition of Frechet-Hilbert spaces contains the nuclear Frechet and countable Hilbert spaces. For the strict projective limit of the sequence of Hilbert spaces, we retained the name “strict Frechet-Hilbert spaces”.

In this situation, it became necessary to replace the quantum Hilbert space by the Frechet-Hilbert spaces, and to extend there the theories of self-adjoint operators and computational methods. For this purpose, we have developed the best approximation theory in Frechet spaces [3] and studied topological and geometric properties of strict Frechet-Hilbert spaces in [4]. The extention of selfadjoint operators theory in strict Frechet-Hilbert spaces was first investigated in [5], continued in [6] and, for the Frechet-Hilbert spaces, in [7]. It was extended the Ritz method ([8], see also [7]), the least squares method [9], the theories of spline [10] and central [11] algorithms.

While strengthening the quantum Hilbert space topology, for the Hamiltonian $\mathcal{H}$ of quantum harmonic oscillator the Hilbert spaces of finite orbits $D(\mathcal{H}^n)$, $n \in \mathbb{N}$ are obtained. This is the space of the states on which the operator $\mathcal{H}$ acts n-times. $D(\mathcal{H}^n)$ is identified with the space of n-orbits

$\mathrm{orb}_n(\mathcal{H},\psi) = (\psi, \mathcal{H}\psi, \cdots, \mathcal{H}^n \psi)$ ([12], see also [13]). In this case the particle that is in the state ψ is subjected to potential energy and the observer gives us $\mathcal{H}\psi$, which is still the state because $\mathcal{H}\psi \in D(\mathcal{H}) \subset H$. It is still instantly acted upon by the potential energy of $\mathcal{H}$ and the observer gives us the state $\mathcal{H}^2\psi$. After n-action, the particle enters the $H^n\psi$ state. Totally these states can be described by an n-orbit $\mathrm{orb}_n(\mathcal{H},\psi) = (\psi, \mathcal{H}\psi, \mathcal{H}^2\psi, \ldots, \mathcal{H}^n\psi)$.

Continuing this process infinitely we get the infinite sequence

$$\mathrm{orb}(\mathcal{H},\psi) = (\psi, \mathcal{H}\psi, \mathcal{H}^2\psi, \ldots, \mathcal{H}^n\psi, \ldots),$$

which we call the orbit of operator $\mathcal{H}$ at the point ψ [14].
The unbounded self-adjoint operator $\mathcal{H}$ forms the self-adjoint orbital operators

$$\mathcal{H}_n\colon\ D(\mathcal{H}_n) \subset (L^2(\mathbb{R}))^{n+1} \to \mathrm{Im}\,\mathcal{H}_n \subset (L^2(R))^{n+1}$$

defined by the equality

$$\mathcal{H}_0(\psi) = \psi,\ \ \mathcal{H}_n(\mathrm{orb}_n(\mathcal{H},\psi)) = \mathrm{orb}_n(\mathcal{H},\mathcal{H}\psi),\ \ n \geq 1\,.$$

The self-adjoint orbital operator $\mathcal{H}^\infty\colon D(\mathcal{H}^\infty) \to D(\mathcal{H}^\infty)$ is defined by the equality

$$\mathcal{H}^\infty \mathrm{orb}(\mathcal{H},\psi) = \mathrm{orb}(\mathcal{H},\mathcal{H}\psi),$$

i.e. action of orbital operator $\mathcal{H}^\infty$ on $\mathrm{orb}(\mathcal{H},\psi)$ means the action of $\mathcal{H}$ on all coordinates of the orbit in the space of all orbits $D(\mathcal{H}^\infty)$. $D(\mathcal{H}^\infty)$ is a well-known space and after the introduction of orbital operator $\mathcal{H}^\infty$, the space $D(\mathcal{H}^\infty)$ acquired new content. This Frechet-Hilbert space of all orbits coincides with the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$. Significance of this space for quantum mechanics is also noted in [15]. $D(\mathcal{H}^\infty)$ is the projective limit of the sequence of spaces $\{D(\mathcal{H}^n)\}$, i.e. the study of computational processes in the space $D(\mathcal{H}^\infty)$ can be reduced to the study of computational processes in $D(\mathcal{H}^n)$ [14]. In the problems of computational mathematics, this means that the equation given in the Frechet-Hilbert space $D(\mathcal{H}^\infty)$ is projected onto the spaces $D(\mathcal{H}^n)$ and calculation of the ε-complexity in the Frechet space of all orbits is reduced to calculate the ε-complexity in some n-orbit Hilbert space. Note that the self-adjoint operator $\mathcal{H}^\infty$ is a topological isomorphism onto the space $D(\mathcal{H}^\infty)$. That is, the flaw of von Neumann's theory was somewhat corrected. This orbital operator $\mathcal{H}^\infty$ has also recently appeared in the paper [16].

The equation $\mathcal{H}u = f$ containing the operator $\mathcal{H}$, which in the space $D(\mathcal{H}^n)$ (resp. in the space $D(\mathcal{H}^\infty)$) has the form $\mathcal{H}_n(\mathrm{orb}_n(\mathcal{H},u)) = \mathrm{orb}_n(\mathcal{H},f)$ (resp. $\mathcal{H}^\infty \mathrm{orb}(\mathcal{H},u) = \mathrm{orb}(\mathcal{H},f)$), is considered. For the obtained equations, a linear spline central (strongly optimal) algorithm is constructed in the Hilbert space $D(\mathcal{H}^n)$ (resp. in the Frechet space $D(\mathcal{H}^\infty)$) [14]. Construction of the spline algorithms for the ill-posed problem of computerized tomography in the spaces of orbits $D(R^*R)^{-n}$) and $D(R^*R)^{-\infty}$), where R is Radon transform, is given in [17] and [11]. Similarly, the ε-complexity will be calculated for the computed tomography task according to finite orbits and spline algorithms built into the space of all orbits. The spaces $D(X^n)$ and $D(X^\infty)$, operators X_n and X^∞ for the position operator X are defined analogously. The spaces $D(P^n)$ and $D(P^\infty)$, operators P_n and P^∞ for the momentum operator P are defined as well.

Generalization of *canonical commutation relations* between X_n and P_n in the space of orbits has the following form

$$X_n P_n \mathrm{orb}_n\,(X,\psi)\text{-}P_n X_n\,\mathrm{orb}_n\,(X,\psi)= i\hbar\ \ \mathrm{orb}_n\,(X,\psi)$$

and is given in ([18], see also [19]). In the paper [19] the generalization of Heisenberg uncertainty principle for the orbital operators is also given. The norms of orbital spaces $D(X^n)$ and $D(P^n)$ are strengthening the topology of the space $L^2(\mathbb{R})$. Creation of orbits of operators, orbital spaces and orbital operators, we call *orbitization of quantum mechanics or quantum mechanics with orbital operators* and the results obtained *orbital quantum mechanics*.

Thus, the represented orbits $\mathrm{orb}_n(\mathcal{H},\psi)$, $\mathrm{orb}(\mathcal{H},\psi)$ and the orbital operators $\mathcal{H}_n$ and $\mathcal{H}^\infty$ describe the state of the particle more adequately because we have the whole infinite sequence of observer data on the particle. For the required modeling accuracy, the study of computational processes associated with an infinite sequence of observer data is reduced to the study of computational processes with a finite data sequence. This was considered in [14] for calculation of the inverse of the harmonic oscillator in the spaces of orbits and in [11] for computerized tomography problem. This process is coordinated by a functional (quasinorm of metric) built specifically by us in [9]. That is, it is a matter of bringing an infinite coordinate computational process to a finite coordinate computational process based on certain requirements or other considerations for accuracy. Orbital quantum mechanics will similarly study orbits, orbital operators, orbital spaces and orbital equations for the position and momentum observables X and P. As well as the orbits, orbital operators, orbital spaces and orbital equations for operators of creation C, annihilation A and numerical N are studied. Each of the considered operators produce n-finite orbits $\mathrm{orb}_n(\mathcal{H},\psi)$, $\mathrm{orb}_n(X,\psi)$, $\mathrm{orb}_n(P,\psi)$, $\mathrm{orb}_n(C,\psi)$, $\mathrm{orb}_n(A,\psi)$, $\mathrm{orb}_n(N,\psi)$, $(n\in\mathbb{N}_0)$ and orbits $\mathrm{orb}(\mathcal{H},\psi)$, $\mathrm{orb}(X,\psi)$, $\mathrm{orb}(P,\psi)$, $\mathrm{orb}(C,\psi)$, $\mathrm{orb}(A,\psi)$, $\mathrm{orb}(N,\psi)$ in the state ψ of quantum Hilbert space. They also generate n-finite orbital operators $\mathcal{H}_n$, X_n, P_n, C_n, A_n, N_n, which act accordingly on the Hilbert space of finite n-orbits

$$D(\mathcal{H}_n), D(X_n), D(P_n), D(C_n), D(A_n), D(N_n).$$

These operators also generate $\mathcal{H}^\infty$, X^∞, P^∞, C^∞, A^∞, N^∞ orbital operators that operate accordingly in the Frechet spaces of all orbits $D(\mathcal{H}^\infty)$, $D(X^\infty), D(P^\infty), D(C^\infty), D(A^\infty), D(N^\infty)$. When $n=0$ a classical case is obtained.

This new mathematical model - orbital quantum mechanics essentially improved the possibilities of computations and gives possibility to consider new computational processes that not contained in the frames of Hilbert spaces and was not considered up to now.

While weakening the topology of quantum Hilbert space $L^2(\mathbb{R})$, the quantum strict Frechet-Hilbert spaces $L^2_{loc}(\mathbb{R})$ [20] are obtained as well, that essentially extend the space of states of quantum mechanical systems. The space $L^2_{loc}(\mathbb{R})$ contains the functions e^{ikx}, which avoid the problems described in ([1], Sect. 3.4, p. 52). $L^2_{loc}(\mathbb{R})$ contains the algebraic polynomials and many other simple functions. Extension of the quantum Hilbert space is also driven by the demands of machine learning and with increase of the memory of continuous-variable quantum computers [21]. In [20] the position and momentum operator in the quantum Frechet-Hilbert space $L^2_{loc}(\mathbb{R})$ are extended. The general problem of extension of self-adjoint operator from Hilbert space to strict

Frechet-Hilbert space is considered in [5]. The selfadjointness and continuity of the extension of position and momentum operators in this space is proved due to the generalization of the Hellinger-Teoplitz theorem for Frechet-Hilbert spaces [7]. Generalization of canonical commutation relation for extended operators in this space is proved as well [20]. Geometric and topological properties of strict Frechet-Hilbert spaces that are represented as strict projective limit of the sequence of Hilbert subspaces are given in more details in [4].

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