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MAGNETIC RELAXATION SWITCHING IS ONE OF THE METHODS OF BIO-NANO DIAGNOSTICS

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Abstract. *The presence of magnetic nanoparticles in the area of a malignant tumor significantly reduces the spin-spin relaxation time of water protons in the body. The phenomenon of reducing the relaxation time of the transverse component of magnetization when introducing magnetic nanoparticles into the body called "magnetic relaxation switching" (MRSw), makes it possible to detect the presence of a malignant tumor in the human body. In this work, we obtained the criteria that the parameters of magnetic nanoparticles (diameter, concentration, and magnetic moment) must meet so that the change in the relaxation time would be noticeable during switching.*

Keywords: magnetic relaxation switching, magnetic nanoparticles, bio-nano diagnostics.

1. Introduction

At present, the phenomenon of magnetic resonance is used not only for the study of various medical and biological objects, but is also widely used in clinical practice [1-8. The method of medical diagnostics based on the use of magnetic nanoparticles as non-contact biosensors is a rapidly developing area of medicine. The presence of magnetic nanoparticles in the area of diseased tissue significantly reduces the spin-spin relaxation time of water protons in the body. The phenomenon of reducing the relaxation time of the transverse component of magnetization called "magnetic relaxation switching" (MRSw), makes it possible to detect the presence of a malignant tumor in the human body.

Consequently, the mechanisms and theoretical models of relaxation of the magnetization of water protons $\boldsymbol{M}$ of biological fluids containing magnetic nanoparticles are discussed in detail. Here is a brief description of the phenomenon of a decrease in the time of transverse relaxation T_2 due to the presence of magnetic nanoparticles in the biological fluid.

The process of transverse relaxation can be represented as follows [9]. In a strong magnetic field of induction $\boldsymbol{B}_0$, directed along the axis $\boldsymbol{z}$, the macroscopic paramagnetic magnetization of the proton $\boldsymbol{M}$ system performs precession with a Larmor frequency around the axis $\boldsymbol{z}$, $\omega_I = \gamma B_0$, where is the gyromagnetic ratio $\gamma = 2{,}67 \cdot 10^8 \, rad/sT$ (Fig.1a). As a result of the action of a resonant $\omega = \omega_I$ radiofrequency pulse with amplitude h and duration $\tau_0 = (\gamma h/2\pi)(\pi/2)$, the magnetization $\boldsymbol{M}$ will deviate by an angle $\pi/2$ and be directed along the axis x (Fig.1.b). The interaction between the magnetic moments of individual particles during time T_2 leads to a complete dephasing of the precession of individual moments (Fig.1.c) and during time T_2 - to relaxation of the transverse component of magnetization (Fig.1.d). The introduction of magnetic nano-particles *in vivo* into the system that marked diseased cells can lead to a noticeable decrease in relaxation time $T_2^S < T_2$ (Fig. 1.e).
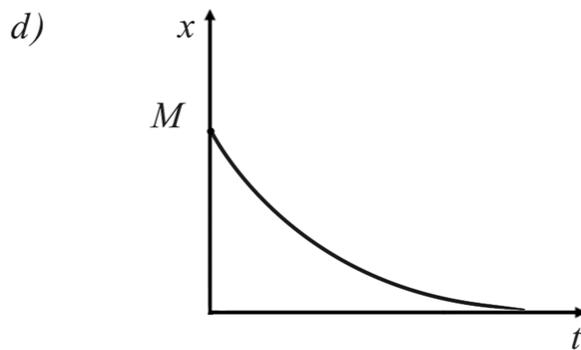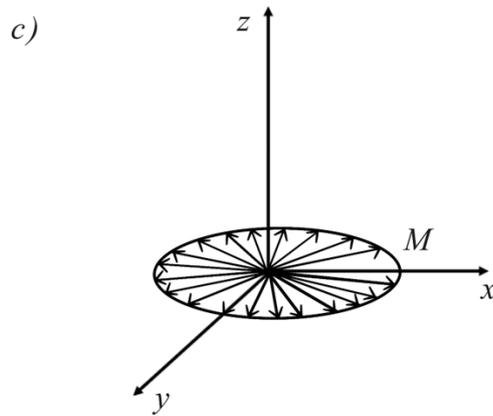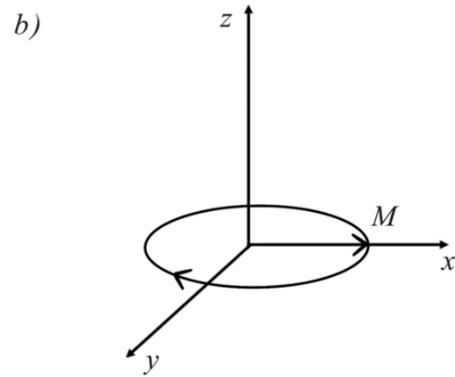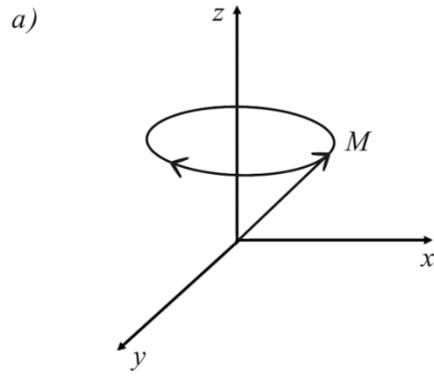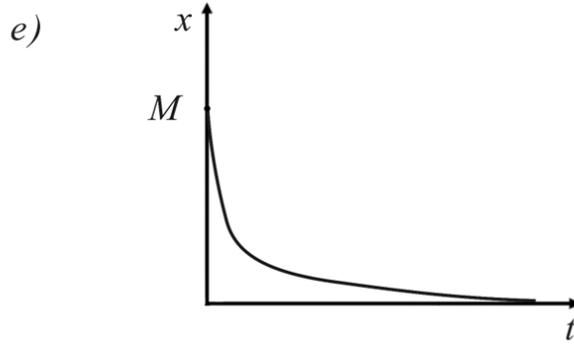
Fig.1. Relaxation of the transverse component of magnetization. a) Larmor precession of magnetization (synchronous precession of individual magnetic moments); b) Deviation of magnetization by angle $\pi/2$; c) Disphasing of the precession of the moments of individual particles; d) Relaxation of the transverse component; e) Relaxation of the transverse component in the presence of magnetic nanoparticles in the system.

In this paper, we present a quantum-mechanical consideration of the theory of transverse relaxation of protons in a biological fluid, both in the presence of magnetic nanoparticles in it and in its absence. The criteria for detecting the phenomenon of "magnetic relaxation switching" are established.

2. Master Equation for Density Matrix

Motion of a System Subject to Random Perturbation

In this section, we consider the density matrix theory for the case where the perturbation is a random function of time [10].

The equations of motion for the density matrix ρ have the form

$$\frac{1}{i}\frac{d\rho}{dt} = -[\mathcal{H}_0 + \mathcal{H}_1(t), \rho] \tag{1}$$

where perturbation Hamiltonian $\mathcal{H}_1(t)$ is a stationary random operator. In the interaction representation, given that

$$\tilde{\rho} = e^{i\mathcal{H}_0 t}\mathcal{H}_1(t)e^{-i\mathcal{H}_0 t}, \qquad \widetilde{\mathcal{H}}_1(t) = e^{i\mathcal{H}_0 t}\mathcal{H}_1(t)e^{-i\mathcal{H}_0 t} \tag{2}$$

equation (1) is reduced to the form

$$\frac{1}{i}\frac{d\tilde{\rho}}{dt} = -[\widetilde{\mathcal{H}}_1(t), \tilde{\rho}]. \tag{3}$$

Integrating the last equation by the method of successive approximations up to the second order, we get

$$\tilde{\rho}(t) = \tilde{\rho}(0) - i\int_0^t [\widetilde{\mathcal{H}}_1(t'), \tilde{\rho}(0)]dt' - \int_0^t dt' \int_0^{t'} dt'' \, [\widetilde{\mathcal{H}}_1(t'), [\widetilde{\mathcal{H}}_1(t''), \tilde{\rho}(0)]],$$

$$\tag{4}$$

From where, taking the derivative with respect to time, we obtain

$$\frac{d\tilde{\rho}}{dt} = -i[\widetilde{\mathcal{H}}_1(t), \tilde{\rho}(0)] - \int_0^t dt'[\widetilde{\mathcal{H}}_1(t), [\widetilde{\mathcal{H}}_1(t'), \tilde{\rho}(0)]]. \qquad (5)$$

Or by introducing a new variable into the integral, $\tau = t - t'$

$$\frac{d\tilde{\rho}}{dt} = -i[\widetilde{\mathcal{H}}_1(t), \rho(0)] - \int_0^t d\tau[\widetilde{\mathcal{H}}_1(t), [\widetilde{\mathcal{H}}_1(t-\tau), \tilde{\rho}(0)]] \qquad (6)$$

Since $\mathcal{H}_1(t)$- is a random operator, then according to equation (6), the random operator $\rho(t)$ is a random operator as well. Then the observed behaviour of the statistical ensemble of the system S will be described by the mean operator $\bar{\tilde{\rho}}$, which satisfies the equation obtained by averaging equation (6) over an ensemble of all random Hamiltonians $\mathcal{H}_1(t)$. It can always be assumed that $\overline{\mathcal{H}_1}(t) = 0$, in addition, the following assumptions can be made:

a) The correlation between $\widetilde{\mathcal{H}}_1(t)$ and $\tilde{\rho}(0)$ can be neglected;

b) Replacing $\tilde{\rho}(0)$ with $\tilde{\rho}(t)$, apply a quasi-static approximation;

c) It is possible to extend the upper limit of integral (6) to ∞;

d) All members of a higher order can be neglected.

Using these assumptions and omitting the dash $\bar{\tilde{\rho}}$, which means averaging, we write equation (6) in the form

$$\frac{d\tilde{\rho}}{dt} = -\int_0^\infty d\tau \overline{[\widetilde{\mathcal{H}}_1(t), [\widetilde{\mathcal{H}}_1(t-\tau), \tilde{\rho}(t)]]} \qquad (7)$$

Operator form of the basic equation

The random Hamiltonian $\mathcal{H}_1(t)$ can be presented as the following row:

$$\mathcal{H}_1(t) = \sum_q F^{(q)}(t) A^{(q)} \qquad (8)$$

where $F^{(q)}-$ are the random functions of time, $A^{(q)}-$ operators acting on the variables of the spin system.

We introduce the correlation function

$$\mathcal{G}_{qq'}(\tau) = \overline{F^{(q)}(t) F^{(-q')}(t+\tau)} \qquad (9)$$

and spectral density

$$\mathcal{J}_{qq'}(\omega) = \int_{-\infty}^\infty \mathcal{G}_{qq'}(\tau) e^{-i\omega\tau} d\tau . \qquad (10)$$

If $F^{(q)}-$ complex functions and $A^{(q)}-$non-Hermitian operators, then in order $\mathcal{H}_1$ to be a Hermitian operator, each member $F^{(q)} A^{(q)}$ must be mapped to the member $\widetilde{F^{(q)}} A^{(q)\dagger}$. Let's agree that $F^{(-q)} = F^{q'}, A^{(-q)} = A^{(q)\dagger}$. In the interaction representation for these operators we get

$$\tilde{A}^{(\pm q)}(t) = e^{i\mathcal{H}_0 t} A^{(\pm q)} e^{i\mathcal{H}_0 t} = \sum_p A_p^{(\pm q)} e^{i\omega_p^{(\pm q)} t}, \quad (11)$$

where $\omega_p^{(\pm q)}$ − are the natural frequencies of the respective operators. Then, taking into account (8), the Hamiltonians of the perturbation takes the form:

$$\widetilde{\mathcal{H}}_1(t) = \sum_{p,q} F^q A_p^{(q)} e^{i\omega_p^{(q)} t} \quad (12)$$

where $A_p^{(q)}$ - the operators acting on the variables of the system S.

Replacing $\widetilde{\mathcal{H}}_1(t)$ with expansion (12) in equation (7), neglecting non-secular terms, and assuming, for simplicity's sake, that random jumps of different molecules are not correlated,

$$\mathcal{G}_{qq'}(\tau) = \delta_{qq'}\mathcal{G}_q(\tau) \quad (13)$$

From equation (7) we get

$$\frac{d\tilde{\rho}}{dt} = -\frac{1}{2}\sum_{q,p} \mathcal{J}_q(\omega_p^{(q)})[A_p^{(-q)},[A_p^{(q)},\tilde{\rho}]]. \quad (14)$$

Equation (14) is the master equation written in operator form.

The case in which the correlation time τ_c is so short that all products $\omega_p^{(q)}\tau_c \ll 1$ are so small that all spectral densities are independent of frequency and equal to $\mathcal{J}_q(0)$. This case is known as the case of the "white" spectrum. It is also known as a case of strong narrowing of the shape of spectral density. In this case, returning to the original representation by converting the basic equation (14) from the general form, we obtain

$$\rho = e^{-i\mathcal{H}_0 t}\tilde{\rho}e^{i\mathcal{H}t}, \quad (15)$$

From the general form of the basic equation (14) we obtain

$$\frac{d\rho}{dt} = -i[\mathcal{H}_0,\rho] - \frac{1}{2}\sum_q \mathcal{J}_q(0)\left[A^{(-q)},[A^q,\rho]\right]. \quad (16)$$

3. Equations for macroscopic quantities

If the operator Q acts on the variables of the system S, then the macroscopic quantity $q(t)$ observed in the experiment containing the set of spin systems corresponding to this operator is equal to $q(t) =< Q > = Sp\{\rho(t)Q\}$. In most cases, in order to detect slow changes in the quantity $q(t)$ due to the interaction $\mathcal{H}_1$ (rather than the rapid motions described by the basic Hamiltonian $\mathcal{H}_0$), it is more convenient to perform calculations in the interaction representation

$$\tilde{q}(t) =\widetilde{< Q >}= Sp\{\tilde{\rho}(t)Q\}. \quad (17)$$

Sometimes it is possible to obtain the equation of motion for a quantity $\tilde{q}(t)$ directly without solving the basic equation. Multiplying both sides of the operator equation (7) by Q and taking the trace, we get

$$\frac{d\tilde{q}}{dt} = -Sp(\mathcal{A}\tilde{\rho}), \tag{18}$$

and the operator $\mathcal{A}$ is defined by the expression

$$\mathcal{A} = \int_0^\infty d\tau \overline{[\widetilde{\mathcal{H}}_1(t-\tau), [\widetilde{\mathcal{H}}_1(t), Q]]} . \tag{19}$$

We used the perturbation expansion (12) and the simplifying assumption $\mathcal{J}_{q,q'} = \delta_{q,q'}\mathcal{J}_q$. After simple transformations from (20) we get

$$\mathcal{A} = \frac{1}{2}\sum_{q,p}\mathcal{J}_q\left(\omega_p^{(q)}\right)[A_p^{(q)}, [A_p^{(-q)}, Q]] . \tag{20}$$

4. Relaxation due to the dipole-dipole interaction

Dipole-dipole interaction operator

Hamiltonian of interaction $\mathcal{H}_1$ between two spins usually depend on the magnitude and orientation of their magnetic moments, as well as on the length and direction of the vector describing their relative location. One of the types of interaction between two spins is the dipole-dipole interaction, the energy of which is described by the well-known expression [9,10]

$$(\mathcal{H}_1)_{IS} = \left(\frac{\mu_0}{4\pi}\right)^2 \frac{\gamma_I\gamma_S\hbar^2}{r_{IS}^3}\left\{\boldsymbol{I}_I\boldsymbol{I}_S - 3\frac{(\boldsymbol{I}_I\boldsymbol{r}_{IS})(\boldsymbol{I}_S\boldsymbol{r}_{Is})}{r_{IS}^3}\right\} \tag{21}$$

Here $\boldsymbol{I}_{I,S}$ are spin operators of two types, $\gamma_{I,S}$ − the gyromagnetic ratios of the spins of the two types, r_{IS} − the distance between the spins I and S, $\mu_0 = 4\pi \cdot 10^{-7}\ H/m$ −the magnetic constant.

Let us write the dipole-dipole interaction (21) in the form of expansion (8), where $F^{(q)}$ − random functions of the relative positions of the two spins, $A^{(q)}$ − operators acting on spin variables. In the case under consideration

$$F^{(1)} = \frac{\sin\theta\cos\theta e^{-i\varphi}}{r^3}, \quad F^{(2)} = \frac{\sin^2\theta e^{-2i\varphi}}{r^3}, \quad F^{(3)} = \frac{1-3\cos^2\theta}{r^3}, \tag{22}$$

$$A^{(0)} = \alpha\{-\frac{2}{3}I_zS_z + \frac{1}{6}(I_+S_- + I_-S_+)\},$$
$$A^{(1)} = \alpha\{I_zS_+ + I_+S_z\}, \qquad A^{(2)} = \frac{1}{2}\alpha I_+S_+ \tag{23}$$

$$\alpha = -\frac{3}{2}\left(\frac{\mu_0}{4\pi}\right)^2\gamma_I\gamma_S\hbar^2. \tag{24}$$

Let us assume that the random change in the orientation of the vector is isotropic, so that

$$\overline{F^{(q)}(t)F^{(q')}(t+\tau)} = \delta_{qq'}G^{(q)}(\tau), \qquad (25)$$

$$\mathcal{J}^{(q)} = \int_{-\infty}^{\infty} G^{(q)}(\tau)e^{-i\omega\tau}d\tau.$$

Relaxation of the transverse magnetization component

The time dependence of the amplitude of precessing magnetization in a plane perpendicular to the direction of the applied constant inductance field B_0 can be studied by the method described above. Suppose that at the moment $t = 0$ the magnetization is directed along the x axis of the laboratory coordinate system. At the moment t, its amplitude is represented by the operator

$$I_x \cos \omega_I t + I_y \sin \omega_I t = e^{-i\mathcal{H}_0 t}I_x e^{i\mathcal{H}_0 t} \qquad (26)$$

Its average value is

$$Sp\{e^{-i\mathcal{H}_0 t}I_x e^{i\mathcal{H}_0 t}\rho\} = Sp\{I_x\tilde{\rho}\} = \widetilde{<I_x>}, \qquad (27)$$

and motion is described by the equation

$$\frac{d}{dt} < \widetilde{I_x + I_x'} > = -Sp\{\mathcal{A}_x\sigma^*\} = -<\widetilde{\mathcal{A}_x}>, \qquad (28)$$

where $\mathcal{A}_x$ is determined by the ratio

$$\mathcal{A}_x = \frac{1}{2}\mathcal{J}^{(1)}(\omega_I)\left[A^{(-1)},\left[A^{(1)}, I_x + I_x'\right]\right] + \text{H. C.} \qquad (29)$$

$$+ \frac{1}{2}\mathcal{J}^{(2)}(2\omega_I)\left[A^{(-1)},\left[A^{(1)}, I_x + I_x'\right]\right] + \text{H. C.}$$

$$+ \frac{1}{2}\mathcal{J}^{(0)}(0)\left[A^{(-1)},\left[A^{(1)}, I_x + I_x'\right]\right] +$$

Here, H.C. stands for Hermitian conjugation.

As a result of simple calculations, taking into account the switching relations of spin operators and taking into account that $\omega_p^{(q)} \to n\omega_I$ ($n = 0,1,2$) from (28,29) we get

$$\frac{d}{dt} < \widetilde{I_x + I_x'} > = -\frac{1}{T_2} < \widetilde{I_x + I_x'} >$$

where

$$\frac{1}{T_2} = \gamma_I^4\hbar^2 I(I+1)\left\{\frac{3}{8}\mathcal{J}^{(2)}(2\omega_I) + \frac{15}{4}\mathcal{J}^{(1)}(\omega_I) + \frac{3}{8}\mathcal{J}^{(0)}(0)\right\} \qquad (30)$$

The obtained result (30) is easily generalized to the case where each spin interacts with several identical spins, provided that their movements are not correlated. To do this, it is enough to do the summation in (30)

$$\{\cdots\} \to \sum_k\{\frac{3}{8}\mathcal{J}_{ik}^{(2)}(2\omega_I) + \frac{15}{4}\mathcal{J}_{ik}^{(1)}(\omega_I) + \frac{3}{8}\mathcal{J}_{ik}^{(0)}(0)\} . \qquad (31)$$

As a result of summing from (31) for the transverse relaxation rate, we get

$$\frac{1}{T_2} = \gamma_I{}^2 < \mu_I^2(p_I) > \left\{ \frac{3}{8} \mathcal{J}^{(2)}(2\omega_I) + \frac{15}{4} \mathcal{J}^{(1)}(\omega_I) + \frac{3}{8} \mathcal{J}^{(0)}(0) \right\}. \quad (32)$$

here

$$< \mu_I^2(p_I) >= \mu_I^2 \mathcal{M}(p_I) \quad (33)$$

is the statistical average of the square of the magnetic moments of the nuclear spins, $\mu_I = \gamma_I \hbar \sqrt{I(I+1)} \approx 2.3 \cdot 10^{-26} a \cdot m^2$- the magnetic moment of the proton, $I = 1/2 -$ the spin of the proton,

$$\mathcal{M}(p_I) = \frac{\int_0^\pi \exp(-p_I \cos\theta) \cos^2\theta \, d(\cos\theta)}{\int_0^\pi \exp(-p_I \cos\theta) d(\cos\theta)} = 1 + \frac{2}{p_I^2} - \frac{2}{p_I} \cot(p_I), \quad (34)$$

where

$$p_I = \mu_I B_0 / kT$$

is a parameter of the Boltzmann factor.

For small values of the argument $p_I \ll 1$, expanding the function $\mathcal{M}(p_I)$ into the series $\cot(p_I) \approx 1/p_I + p_I/3 + \cdots$, we obtain $\mathcal{M}(p_I \ll 1) \approx 1/3$, and for large values argument $p_I \gg 1$- saturation effect $\mathcal{M}(p_I \gg 1) \to 1$. In Fig. 2, which shows the function $\mathcal{M}(p)$, the asymptotic values of the function are connected by a dotted curve. At body temperatures in vivo in a magnetic field with induction $B_0 = 1 \, T$, a high-temperature approximation takes place - $p_I = \mu_I B_0 / kT \ll 1$. Therefore, for the mean squared moment of protons from (33) we obtain $< \mu_I^2(p_I) > \cong \mu_I^2/3$.

In the case of small correlation times $n\omega_I \tau_c \ll 1$, where $n = 0,1,2$, (or in the case of a "white" spectrum), as noted above, the spectral density is independent of the frequency $\mathcal{J}(\omega) = \mathcal{J}_I$. In this case, for the expression in curly bracket in (31), we get $\{\cdots\} = 13.5 \, \mathcal{J}_I$.. Then, from (30), taking into account these approximations for the rate of transverse relaxation, we obtain

$$\frac{1}{T_2} = \frac{9}{2} \gamma^2 \mu_I^2 \mathcal{J}_I \quad (35)$$

The dependence of the spectral density $\mathcal{J}_I$ on the parameters of the system is considered below.

Transverse relaxation of nuclear spins due to magnetic nanoparticles

Suppose that the system contains S spins along with nuclear spins I. The basic Hamiltonian for unequal spins is of the form $\mathcal{H}_0 = \hbar(\omega_I I + \omega_S S)$, where $\omega_I \neq \omega_S$.

In the magnetic-relaxation switching problem, the spins of water protons act as I spins, and the spins of magnetic nanoparticles act as S spins. In bio-nano-medicine, nano-particles of magnetite (Fe_3O_4) with a ferrimagnetic order[11], with a magnetic diameter $d_m = 10 \, nm$ and with a magnetic volume - $V_m \approx 5.2 \cdot 10^{-25} \, m^3 (V_m = \frac{\pi}{6} d_m^3)$.

The magnetic moment of a magnetite nano-particle is calculated by the formula $\mu_S = M_S V_m$, where $M_S -$is the magnetization of saturation. For magnetite nanoparticles with a spinel crystallographic structure [10] $M_S = 4.8 \cdot 10^5 \, a/m$. Then, for the magnetic moment of nanoparticles, we obtain $\mu_S \approx 2.5 \cdot 10^{-19} a \cdot m^2$.

We calculate the cyclic precession frequency by the formula $\omega_S = \mu_S B_0/\hbar$ of the magnetic moment of the nanoparticles noted above in the magnetite of sizes noted above in a magnetic field with induction $B_0 = 1T$, and we obtain $\omega_S/2\pi \approx 8 \cdot 10^8 \, MHz$. In the same magnetic fields for the Larmor frequency of protons we have $\omega_I/2\pi = 42.57 \, MHz$. Thus, the frequencies of S and I spins satisfy the condition $\omega_S \gg \omega_I$.

To describe slow relaxation processes occurring in the system for times greater than $\tau_S = 2\pi/\omega_S$, let us use the Hamiltonian of interaction $\mathcal{H}_1(t)$ averaged over small times τ_S,

$$\overline{\mathcal{H}_1} = \frac{1}{T}\int_0^T \widetilde{\mathcal{H}}_1(t)dt \,, \qquad T \gg \tau_S. \qquad (36)$$

Writing with the help of (11,12) the Hamiltonians of the dipole-dipole interaction $\widetilde{\mathcal{H}}_1$ (23,25) in the interaction representation and averaging according to rule (34), we get

$$\overline{\mathcal{H}_1} = \alpha\{-\frac{2}{3}I_z S_z + I_+ S_z e^{i\omega_I t}\} \,. \qquad (37)$$

Comparing the interaction expression (37) with the expression (24), it can be seen that as a result of averaging (36), most of the terms of the dipole-dipole interaction are averaged to zero.

The relaxation equation of transverse magnetization $\widetilde{<I_x>} = Sp\{I_x \rho\}$ for this case can be obtained in the same way as it was obtained in the case of equation (30)

$$\frac{d}{dt}\widetilde{<I_x>} = -\widetilde{<\mathcal{A}_x^I>}, \qquad (38)$$

where

$$2\mathcal{A}_x^I = \frac{4}{9}\alpha^2 \mathcal{J}_S^{(0)}(0)\big[I_z S_z, [I_z S_z, I_x]\big] +$$

$$+\alpha^2 \mathcal{J}_S^{(1)}(\omega_I)\big[I_- S_z, [I_+ S_z, I_x]\big] + H.\,C. \qquad (39)$$

After simple calculations similar to those made in deriving the ratio (30), we obtain the relaxation equation

$$\frac{d}{dt}\widetilde{<I_x>} = -\frac{\widetilde{<I_x>}}{T_2^S} \qquad (40)$$

where

$$\frac{1}{T_2^S} = \gamma_I^2 < \mu_S^2(p_S) > \{\frac{1}{6}\mathcal{J}_S^{(0)}(0) + \frac{3}{4}\mathcal{J}_S^{(1)}(\omega_I)\} \qquad (41)$$

is the transverse relaxation rate caused in the system by the presence of magnetic nano-particles, $< \mu_S^2(p_S) >$- the root mean square value of the magnetic moments of nano-particles.

Super-paramagnetism of magnetic nano-particles

The physical properties of magnetic nanoparticles are mainly due to two features. The first of them is associated with a large value of the magnetic moment (on the order of several tens of thousands of Bohr magnetons), the second is associated with the presence of magnetic anisotropy energy $A = KV_m$ inherited from a massive sample. Here K is the energy density of magnetic anisotropy. Magnetite nano-particles ($Fe_3 O_4$), which is often used in magnetic resonance bio-nano-diagnostics, is a ferrimagnetic with the crystalline structure of the mineral spinel. The anisotropy density of magnetite nano-particles, as is known [12], is of the order $K \approx 10^2 \, {}^J/_{m^3}$. Then for the anisotropy energy of magnetite nano-particles with the volume noted above $V_m \approx 5.2 \cdot 10^{-25} \, m^3$, we obtain a value $A \approx 5.2 \cdot 10^{-23} \, m^3$ that is significantly less than the thermal energy of the body at normal temperature. In this case, thermal energy can provide a reversal of the magnetic moment of the nanoparticle, that is, overcoming the energy barrier formed by magnetic anisotropy. In this case, the system of nanoparticles is in a super-paramagnetic state, and the root-mean-square value in (33) can be taken from the Boltzmann distribution similarly to (34). As a result of simple transformations from (33) we obtain

$$< \mu_S^2(p_S) >= \mu_S^2 \mathcal{M}(p_S) \qquad (42)$$

where

$$\mathcal{M}(p_S) = [1 + \frac{2}{p_S^2} - \frac{2}{p_S}\cot(p_S)]$$

-statistical mean of the square of the magnetic moment of the nano-particle (Fig.2), $p_S = \mu_S B_0/kT$ is the Boltzmann factor, which for nano-particles in the case under consideration ($B_0 = 1T, \mu_S = 2.5 \cdot 10^{-19} a \cdot m^2, T = 310 \, grad$), has the value of $p_S \approx 58$. Then, for the mean square, calculating with (42) $\mathcal{M}(58) \approx 1$ and for the mean square, we get $< \mu_S^2(p_S) >\approx \mu_S^2$.
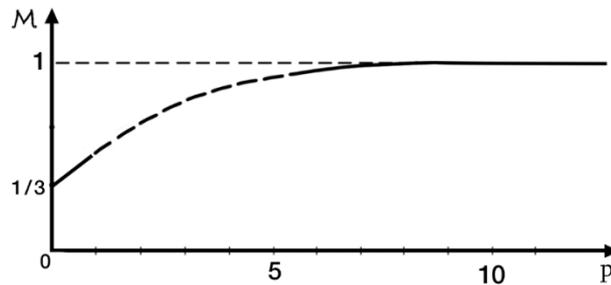
Fig.2. Graph of the dependence of the mean square of the magnetic moment of nanoparticles in the superparamagnetic state $.- < \mu^2(p) >$ on the parameter $p = \frac{\mu B_0}{kT}$. For $p \ll 1$ we have $\mathcal{M} = 1/3$. For larger values of the parameter $p \gg 1$, saturation occurs ($\mathcal{M} \to 1$) .

In the case of small correlation times $\omega_I \tau_c \ll 1$, the spectral density does not depend on frequency $\mathcal{J}_S(\omega) = \mathcal{J}_S$ and from (41) we obtain

$$\frac{1}{T_2^S} = \frac{11}{12}\gamma_I^2\mu_S^2\mathcal{M}(p_S)\mathcal{J}_S \qquad (43)$$

Correlation function determined by random molecular translational motion

Random changes in the dipole moment inside the molecule are due to the rotation of the molecules. However, it is more efficient to take into account the interaction between the spins of different molecules. And when considering the interaction between the spins of different molecules, their relative motions are usually more significant than their characteristic rotations. Using the results presented in the monograph [10], if random movements of liquid molecules are dominant, then the spectral density can be represented as

$$,\mathcal{J}(\omega) = \frac{N}{dD}\mathcal{F}(\omega\tau_c) \qquad (44)$$

where

$$\mathcal{F}(\omega\tau_c) = \int_0^\infty [\mathcal{J}_{3/2}(x)]^2\frac{xdx}{x^4+\omega^2\tau_c^2},$$

$\mathcal{J}_{3/2}(x)$– Bessel function, D – diffusion coefficient, d – total diameter of the nano-particle, τ_c– correlation time.

Let us consider the two limiting cases of integral (44): the case of short correlation times $\omega\tau_c \ll 1$ and long times $\omega\tau_c \gg 1$. To do this, we will use the known [13] values of integrals,

$$\mathcal{F}(\omega\tau_c \ll 1) = \int_0^\infty \left[\mathcal{J}_{3/2}(x)\right]^2\frac{dx}{x^3} = \frac{15}{2} \approx 0.13,$$

$$\mathcal{F}(\omega\tau_c \gg 1) = \frac{1}{(\omega\tau_c)^2}\lim_{\alpha\to\infty}\int_0^\alpha \left[\mathcal{J}_{3/2}(x)\right]^2 xdx = \frac{1}{(\omega\tau_c)^2}\lim_{\alpha\to\infty}\frac{\alpha^2}{2}\left[\mathcal{J}_{5/2}(\alpha)\right]^2 =$$
$$\frac{1}{(\omega\tau_c)^2}\lim_{\alpha\to\infty}\frac{\alpha}{\pi}.$$

If we assume that $\alpha/\omega\tau_c \to 1$, then we get

$$.\mathcal{F}(\omega\tau_c \gg 1) \to {}^1\!/\!{\omega\tau_c} \to 0$$

A graph of the function $\mathcal{F}(\omega\tau_c)$, plotted by taking into account the asymptotic formulas obtained above is shown in Fig.3.
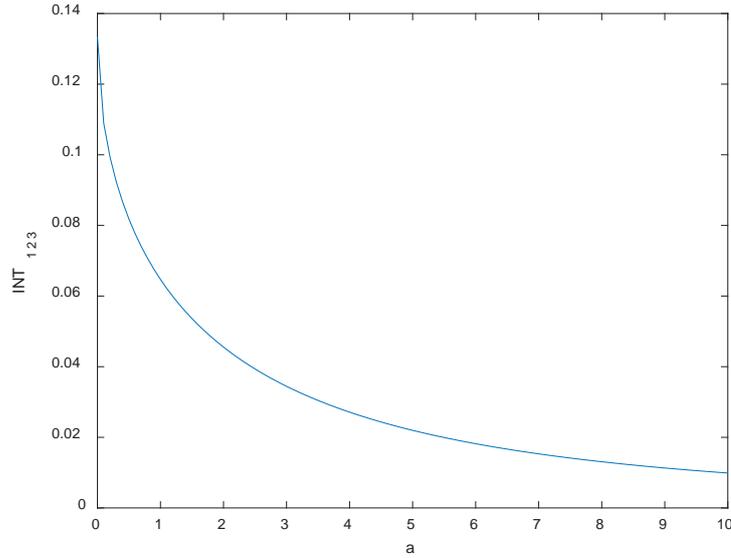
Fig.3. Graph of the function $\mathcal{F}(a)$, where $a = \omega\tau_\mathrm{c}$.

The correlation time τ_c can be expressed in terms of the diffusion coefficient D and the total diameter of the nano-particle d in the form

$$\tau_c = \frac{d^2}{2D},$$

and the diffusion coefficient is used using the Stokes formula $D = kT/3\pi d\eta$, where η is the viscosity coefficient. Then, in the case in which the random movements of the liquid molecules are dominant, by representing the nano-particles as rigid spherical particles, we get,

$$\tau_c = \frac{3\pi d^3 \eta}{2kT} \tag{45}$$

For the parameter values used above from (45), we obtain the value $\tau_\mathrm{c} \approx 10^{-9}\, s$ at which the approximation of the "white" spectrum occurs: $\omega_I \tau_c \approx 10^{-3}$. Then, in the case of the "white" spectrum, taking into account the limiting value of the integral $\mathcal{F}(\omega\tau_\mathrm{c} \ll 1) = \frac{15}{2}$, from (44), we obtain

$$\mathcal{J}_i = \frac{2\pi}{5} \frac{N_i \eta}{kT}, \tag{46}$$

where $i = I, S$. Another limiting case $\omega\tau_c \gg 1$ of spectral density (44) is not of interest.

For the relaxation times (35.43), taking into account the ratios (42.46), we get

$$\frac{1}{T_2} = \frac{36}{5} \pi^2 \gamma^2 \mu_I^2 \frac{N_I \eta}{kT}$$

$$\frac{1}{T_2^S} = \frac{28}{75}\pi^2\gamma^2\mu_S^2\mathcal{M}(p_S)\frac{N_S\eta}{kT}. \tag{47}$$

6. Conclusion

In order for the change in the relaxation rate caused by the presence of magnetic nanoparticles to be effective, the condition $T_2^S \le T_2$ must be met. The main physical parameters for nano-particles are volume V_m and concentration N_S. For these parameters, using ratios (47), the criterion "MRI switching" can be represented as

$$\frac{1/T_2^S}{1/T_2^I} = \frac{7}{135}\,\mathcal{M}\left(\frac{M_S V B_0}{kT}\right)\left(\frac{M_S V}{\mu_I}\right)^2\frac{N_S}{N_I} \gtrsim 1$$

For magnetite nano-particles, with the parameter values noted above, as well as taking into account the numerical values - $\mathcal{M}(58) \approx 1$, $T \approx 310\,grad$, the criteria for the concentration of nano-particles from (48) are obtained in the form $N_I \approx 6.7\cdot10^{28}\ 1/m^3\ N_S \gtrsim 1.2\cdot10^{16}\ 1/m^3$.

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